Our goal in these notes is to understand the long-time behavior of solutions to ODEs. For this it will be very useful to introduce the notion of \( \omega \)-limit sets. A remarkable result - the Poincaré–Bendixson theorem - is that for planar ODEs, one can have a rather good understanding of \( \omega \)-limit sets. I have been benefited a lot from the textbook *Differential equations, dynamical systems and an introduction to chaos* by Hirsch–Smale–Devaney while preparing for these notes. As always, this is a preliminary version, if you have any comments or corrections, even very minor ones, please let me know.

Consider the following ODE, i.e.,
\[
  u'(t) = F(u(t)),
\]
where \( F : \mathbb{R}^n \to \mathbb{R}^n \) is \( C^1 \). In particular, the Picard–Lindelöf theorem applies. Therefore, we know that given any initial data, there exists a unique maximal solution.

**Definition 0.1 (Flow map).** Let \( u_0 \in \mathbb{R}^n \) and \( u(t) \) be the unique maximal solution to the ODE (0.1) with initial data \( u_0 \). Suppose \( t \) is in the maximal interval of existence. Define
\[
  \varphi_t(u_0) = u(t).
\]

**Remark 0.2.** The Picard–Lindelöf theorem implies that for every \( u_0 \in \mathbb{R}^n \), there exists \( \epsilon \) such that \( \varphi_t(u_0) \) is well-defined for \( t \in (-\epsilon, \epsilon) \). Moreover, it is in fact true that \( \varphi_t : \mathbb{R}^n \to \mathbb{R}^n \) is \( C^1 \) (whenever it is defined) (Exercise: think about why this is true).

**Definition 0.3 (\( \omega \)-limit set).** Let \( u_0 \) be such that \( \varphi_t(u_0) \) is well-defined for all \( t \geq 0 \). Then its \( \omega \)-limit set \( \Omega \) is defined as
\[
  \Omega := \{ x \in \mathbb{R}^n : \varphi_{t_k}(u_0) \to x \text{ for some sequence } t_k \to \infty \}.
\]

We can prove some easy properties.

**Proposition 0.4.** Let \( \Omega \) be an \( \omega \)-limit set associated to the flow \( \varphi_t(u_0) \). The following properties hold:

- \( \Omega \) is closed.
- If \( x \in \Omega \), then \( \varphi_t(x) \in \Omega \) (as long as \( \varphi_t(x) \) is well-defined).
- If \( \Omega \) is bounded, then it is connected.

**Proof.**
(1) By definition, we can write
\[
  \Omega = \cap_{t \geq 0} \{ u(t) : t \geq \tau \}.
\]

Since \( \Omega \) is an intersection of closed set, it is closed.

(2) If \( x \in \Omega \), then there exists \( s_k \to \infty \) such that \( x = \lim_{k \to \infty} \varphi_{s_k}(u_0) \). Then, as long as \( \varphi_t(x) \) is well-defined\(^1\),
\[
  \varphi_t(x) = \varphi_t(\lim_{k \to \infty} \varphi_{s_k}(u_0)) = \lim_{k \to \infty} \varphi_{t+s_k}(u_0).
\]

Hence, \( \varphi_t(x) \in \Omega \).

(3) This is left as an Exercise (cf. HW8).

\[\square\]

Here are a few examples of \( \omega \)-limit sets:

(1) Suppose \( u_0 \) is an equilibrium point. Then the \( \omega \)-limit set associated to \( \varphi_t(u_0) \) is \( u_0 \)

(2) Suppose \( u \) is an equilibrium point and \( u_0 \) is a point such that \( \varphi_t(u_0) \to u \) as \( t \to \infty \). Then the corresponding \( \omega \)-limit set is \( u \).

(3) Let \( \varphi_t(u_0) \) be a periodic solution. Then the corresponding \( \omega \)-limit set is exactly the image of \( \varphi_t(u_0) \).

\(^1\)Note that we have used that \( \varphi_t \) is continuous here, cf. Remark 0.2.
(4) A less obvious example is the following:

\[
x'(t) = \sin x(t) \left(-\frac{1}{10} \cos x(t) - \cos y(t)\right),
\]

\[
y'(t) = \sin y(t) \left(\cos x(t) - \frac{1}{10} \cos y(t)\right).
\]

It is easy to check that \((\frac{\pi}{2}, \frac{\pi}{2})\), \((0,0)\), \((0,\pi)\), \((\pi,0)\), \((\pi,\pi)\) are all equilibrium points. Moreover, consider any initial data point in \((0,\pi)\times(0,\pi)\}\setminus\{(\frac{\pi}{2}, \frac{\pi}{2})\}\}, the solution approaches the square bounded by \(x = 0, \pi, y = 0, \pi\). (This is not so easy to prove, but you will investigate some aspects of this in the homework.) In particular, the \(\omega\)-limit set, which is the square as above, consists of four equilibrium points (the vertices of the square) and four segments connecting them.

It is a remarkable fact that on \(\mathbb{R}^2\), bound \(\omega\)-limit sets in general do not get much more complicated than the examples that we have seen!

**Theorem 0.5** (Poincaré–Bendixson). Consider a planar ODE, i.e., assume \(n = 2\). Let \(\Omega\) be a non-empty bounded \(\omega\)-limit set. Then either \(\Omega\) contains at least one equilibrium point, or there exists a periodic solution \(u(t)\) such that the image of \(u(t)\) is exactly \(\Omega\).

**Remark 0.6.** Notice that this is false on \(\mathbb{R}^n\) for \(n > 2\)!

**Exercise:** Show that indeed in the setting of that problem, there exists a solution \(u(t)\) such that the \(\omega\)-limit set does not obey the conclusion of Theorem 0.5.

The remainder of these notes will be devoted to a proof of Theorem 0.5. From now on, we will consider (0.1) and fix \(n = 2\).

1. Transversal line

Given the ODE (0.1) (with \(n = 2\)), we can define the notion of transversal lines.

**Definition 1.1** (Transversal lines). A line segment \(S = \{\lambda x_0 + (1-\lambda)x_1 : \lambda \in (0,1)\}\) is said to be transversal if for every \(x \in S\), \(F(x)\) is non-vanishing and is not parallel to \(S\).

The following is an easy fact about transversal lines:

**Lemma 1.2.** Let \(S\) be a transversal line and \(x_0 \in S\). Then there exists an open set \(U\) with \(x \in U\) and a \(C^1\) function \(\tau : U \to \mathbb{R}\) such that \(\varphi_{\tau(x)}(x) \in S\) for all \(x \in U\).

**Proof.** Let \((y,z)\) be the coordinates on \(\mathbb{R}^2\). Without loss of generality, assume that \(x_0 = (0,0)\) and \(S\) is a subset of \(y = 0\). For \(\epsilon\) sufficiently small to be determined, define a map \(\Psi : B_{\mathbb{R}}(0,\epsilon) \times B_{\mathbb{R}^2}(0,\epsilon) \to \mathbb{R}\) by

\[
\Psi(t,x) = \pi \varphi_t(x),
\]

where \(\pi : \mathbb{R}^2 \to \mathbb{R}\) is the map \((y,z) \mapsto y\).

**Fact:** \(\Psi\) is a \(C^1\) map. (This can be proven using the definitions and carefully studying (0.1), but we will omit the proof.)

Now, by definition \(\Psi(0,0) = 0\). Moreover, since \(S\) is transversal, \(\frac{\partial \Psi}{\partial x}(0,0) = \pi F(x_0) \neq 0\). The conclusion of the proposition hence follows from the implicit function theorem.

2. Main monotonicity property

The main technical proposition that builds towards the Poincaré–Bendixson theorem is the following monotonicity property.

**Proposition 2.1.** Let \(S\) be a transversal line and \(u(t)\) be a solution to the ODE. Suppose \(u(t_0), u(t_1)\) and \(u(t_2)\) are three points on \(S\) with \(t_0 < t_1 < t_2\) such that \(u(t_0) \neq u(t_1)\), then they are monotonic along \(S\).

**Proof.** Consider the curve \(\Gamma \subset \mathbb{R}^2\) given by

\[
\Gamma := \{u(t) : t \in [t_0, t_1]\} \cup \{x \in S : x \text{ is between } u(t_0) \text{ and } u(t_1)\}.
\]
By the Jordan curve theorem\(^2\), \(\Gamma\) divides \(\mathbb{R}^2\) into two components, \(D_1\) and \(D_2\). Since \(F(u(t_1))\) is transversal to \(S\), \(u\) must either enter \(D_1\) or \(D_2\) after \(t_1\). Suppose that \(u\) enters \(D_1\) after time \(t_1\), i.e., suppose there exists \(\epsilon > 0\) such that \(u(t) \in D_1\) for \(t \in (t_1, t_1 + \epsilon)\). We claim that \(u(t) \in D_1\) for all \(t > t_1\). Assume not, then there exists a time \(t_\epsilon > t_1\) such that \(u(t_\epsilon) \in \Gamma\), but this is impossible:

- we cannot have \(u(t_\epsilon) \in \{u(t) : t \in [t_0, t_1]\}\) since this contradicts the uniqueness of solutions; and
- we cannot have \(u(t_\epsilon) \in \{x \in S : x\) is between \(u(t_0)\) and \(u(t_1)\}\) since \(F\) points towards \(D_1\) on that set.

In particular, this implies \(u(t_2) \in \text{int}(D_1)\), from which it follows that the points \(u(t_0), u(t_1)\) and \(u(t_2)\) are monotonic along \(S\).

\[\square\]

**Proposition 2.2.** Let \(u(t)\) be a solution to (0.1) and \(\Omega\) be the corresponding \(\omega\)-limit set. Let \(S\) be a transversal line. Then \(S \cap \Omega\) has at most one point.

**Proof.** Suppose not, i.e., assume that \(x_1, x_2 \in S \cap \Omega\) are distinct. By Lemma 1.2, there exists \(U_1, U_2\) disjoint open neighborhoods of \(x_1, x_2\) respectively such that the conclusion of Lemma 1.2 holds. Since \(x_1, x_2 \in \Omega\), there exist sequences of times \(\{t_{1,n}\}_{n=1}^\infty\) and \(\{t_{2,n}\}_{n=1}^\infty\) with \(t_{1,n}, t_{2,n} \to \infty\) such that \(u(t_{1,n}) \to x_1\) and \(u(t_{2,n}) \to x_2\). Without loss of generality, we can assume that \(u(t_{1,n}) \in U_1\), \(u(t_{2,n}) \in U_2\) for all \(n \in \mathbb{N}\). Using Lemma 1.2, there exist sequences of times \(\{\tilde{t}_{1,n}\}_{n=1}^\infty\) and \(\{\tilde{t}_{2,n}\}_{n=1}^\infty\) with \(\tilde{t}_{1,n}, \tilde{t}_{2,n} \to \infty\) such that \(u(\tilde{t}_{1,n}) \to x_1\) and \(u(\tilde{t}_{2,n}) \to x_2\) and \(u(\tilde{t}_{1,n}), u(\tilde{t}_{2,n}) \in S\) for all \(n \in \mathbb{N}\). From this one easily concludes that there are three times, say, \(t_{1,1} < \tilde{t}_{2,n} < \tilde{t}_{1,n}\) (for some \(n, n_\ast \in \mathbb{N}\)) such that \(u(t_{1,1}) \neq u(\tilde{t}_{2,n})\), but the three points \(u(\tilde{t}_{1,1}), u(\tilde{t}_{2,n})\) and \(u(\tilde{t}_{1,n})\) are not monotonic along \(S\). This contradicts Proposition 2.1. \(\square\)

3. **Proof of Poincaré–Bendixson theorem**

We now prove the Poincaré–Bendixson theorem. We will consider a non-empty, bounded \(\omega\)-limit set \(\Omega\) with no equilibrium points. Our goal will be to show that \(\Omega\) coincides with the image of a periodic solution. This will be done in two steps. In the first step, we show that the flow starting from any point in \(\Omega\) is periodic. In the second step, we prove that the image of the periodic solution is in fact all of \(\Omega\).

**Proposition 3.1.** Let \(\Omega\) be the \(\omega\)-limit set associated to the solution \(\varphi(t)u_0\). Suppose \(\Omega\) is non-empty, bounded and has no equilibrium points. If \(y \in \Omega\), then \(\varphi(t)y\) is a periodic solution.

**Proof.** By Proposition 0.4, \(\varphi(t)y \in \Omega\) as long as it is defined. In particular, by the extension theorem, this implies \(\varphi(t)y\) is defined for all \(t \geq 0\). Since \(\{\varphi(t)y\}_{t \geq 0}\) is bounded, by the Bolzano–Weierstrass theorem, there exists a sequence \(\{t_k\}_{k=1}^\infty\) with \(t_k \to \infty\) and a point \(z\) such that \(\varphi(t_k)y \to z\). Since \(\varphi(t_k)y \in \Omega\) for all \(k \in \mathbb{N}\) and \(\Omega\) is closed (by Proposition 0.4), this implies \(z \in \Omega\). In particular, \(z\) is not an equilibrium point.

Therefore, we can find a transversal line \(S\) passing through \(z\).

Since \(\varphi(t_k)y \to z\), by Lemma 1.2, there exists \(t_k \to \infty\) such that \(\varphi(t_k)y \to z\) and \(\varphi(t_k)y \in S\). Since \(\varphi(t_k)y \in S \cap \Omega\), by Proposition 2.2, they must in fact be the same point for all \(k \in \mathbb{N}\).

In particular, there exist \(t_{k_1} \neq t_{k_2}\) such that \(\varphi(t_{k_1}y) = \varphi(t_{k_2}y)\). This implies that \(\varphi(t)y\) is a periodic solution. \(\square\)

**Proposition 3.2.** Let \(\Omega\) be the \(\omega\)-limit set associated to the solution \(\varphi(t)u_0\). Suppose \(\Omega\) is non-empty, bounded and has no equilibrium points and \(y \in \Omega\). Then \(\Omega \setminus \bigcup_{t \geq 0}\{\varphi(t)y\} = \emptyset\).

**Proof.** It suffices to show that \(\Omega \setminus \bigcup_{t \geq 0}\{\varphi(t)y\}\) is closed. This is because by Proposition 0.4, \(\Omega\) is connected. Therefore, if we can write \(\Omega = (\bigcup_{t \geq 0}\{\varphi(t)y\} \cup (\Omega \setminus \bigcup_{t \geq 0}\{\varphi(t)y\}))\) as a union of two disjoint closed sets, one of them must be empty. Since \(\bigcup_{t \geq 0}\{\varphi(t)y\} \neq \emptyset\), it follows that \(\Omega \setminus \bigcup_{t \geq 0}\{\varphi(t)y\} = \emptyset\).

Now take a sequence of points \(\{z_k\}_{k=1}^\infty \subset (\Omega \setminus \bigcup_{t \geq 0}\{\varphi(t)y\})\) and suppose \(z_k \to z\). Our goal is to show that \(z \in \Omega \setminus \bigcup_{t \geq 0}\{\varphi(t)y\}\). This would imply that \(\Omega \setminus \bigcup_{t \geq 0}\{\varphi(t)y\}\) is closed.

First, we note that since \(\Omega\) is closed, \(z \in \Omega\). In particular, \(z\) is not an equilibrium point. We can therefore find a traversal line \(S\) such that \(z \in S\). For every \(k \in \mathbb{N}\), since \(z_k \in \Omega\), there exists a sequence of times \(\{t_{k,\ell}\}_{\ell=1}^\infty\) (with \(t_{k,\ell} \to \infty\) as \(\ell \to \infty\)) such that \(\varphi(t_{k,\ell}y) \to z_k\) as \(\ell \to \infty\). By Lemma 1.2, for sufficiently large \(k\), we can find \(\{\tilde{t}_{k,\ell}\}_{\ell=1}^\infty\) (with \(\tilde{t}_{k,\ell} \to \infty\)) such that \(\varphi(\tilde{t}_{k,\ell}y) \to z_k\) as \(\ell \to \infty\) and moreover

\(^2\)The Jordan curve theorem says that any continuous simple closed curve in the plane divides the plane into two disjoint components (the “inside” and the “outside”).
\( \varphi_{i_k,t}(u_0) \in S \) for all \( \ell \in \mathbb{N} \). By Proposition 2.2, all these points must coincide, and therefore \( \varphi_{i_k,t}(u_0) = z \) for all \( k \) sufficiently large and for all \( \ell \in \mathbb{N} \). In particular, this implies \( z_k = z \) for all \( k \) sufficiently large.

Since \( z_k \in \Omega \setminus \cup_{t \geq 0} \{ \varphi_t(y) \} \), it follows that \( z \in \Omega \setminus \cup_{t \geq 0} \{ \varphi_t(y) \} \). This concludes the proof.

Combining Propositions 3.1 and 3.2 gives Theorem 0.5.