1. (10 points) Let $A$ be an $(n \times n)$ real matrix. Recall that $\|A\|_{\text{op}} := \sup_{\|x\| \leq 1} \|Ax\|$. Denote by $a_{ij}$ the entry on the $i$-th row and the $j$-th column.

(a) Show that $\|A\|_{\text{op}} \leq \left( \sum_{i,j=1}^{n} a_{ij}^2 \right)^{\frac{1}{2}}$.

(b) Does equality hold in general? Justify your answer.

(c) Show that $\max_j \left( \sum_{i=1}^{n} a_{ij}^2 \right)^{\frac{1}{2}} \leq \|A\|_{\text{op}}$.

2. (5 points) Give an example of two $(2 \times 2)$ real matrices $A$ and $B$ such that $e^{A+B} \neq e^A e^B$. [Hint: You may find it easier to look for $A$ and $B$ with $A^2 = B^2 = 0$.]

3. (5 points) Prove that

\[
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
\]

is not diagonalizable.

4. (10 points) Let $A$ be an $(n \times n)$ complex matrix. Define $\text{tr} A = \sum_{i=1}^{n} a_{ii}$.

(a) Suppose $B = S^{-1} A S$ for some $S$. Show that $\text{tr} A = \text{tr} B$. [Hint: You may find it helpful to first show that $\text{tr}(CD) = \text{tr}(DC)$ for any $(n \times n)$ complex matrices $C$, $D$.]

(b) Prove that $\text{tr} A$ equals to the sum of the eigenvalues of $A$ (where the eigenvalues are counted with multiplicities).

(c) Show that $\det e^A = e^{\text{tr} A}$. [Hint: By (a), it suffices to consider the case where $A$ is an upper triangular matrix.]

5. (10 points) Problem 2.3 in Brendle. Fully simplify your answer — in particular, evaluate any matrix exponentials. [See also the example in the end of Section 2.5.]

6. (10 points) Problem 2.4 in Brendle.

7. (10 points) Problem 2.7 in Brendle. [You may use Theorem 2.11 in the textbook, even we have only stated it but have not proven it yet.]