1. (10 points) Let \( k \in \mathbb{N} \cup \{0\} \) and \( A \) be an \((n \times n)\) complex matrix.

(a) Show that \( A^k, A^{k+1}, \ldots, A^{k+n} \) are linearly dependent.

(b) For every \( n \geq 2 \), give an example of an \((n \times n)\) matrix \( A \) such that \( I, A, \ldots, A^{n-1} \) are linearly independent.

2. (5 points) Let \( A \) be an \((n \times n)\) complex matrix. Suppose \( \lambda \) is an eigenvalue of multiplicity \( k \) (i.e., \( \lambda \) is a root of \( \det(\lambda I - A) \) of multiplicity \( k \)). Show that for any \( \ell \geq k \),

\[
\ker(\lambda I - A)^k = \ker(\lambda I - A)^\ell.
\]

3. (10 points) Let \( A \) be an \((n \times n)\) complex matrix. Suppose \( \lambda_1, \ldots, \lambda_m \) are (all) the distinct (complex) eigenvalues of \( A \). Show that \( A \) is diagonalizable if and only if \( p(A) = 0 \) where \( p \) is the polynomial \( p(\lambda) = \prod_{i=1}^{m} (\lambda - \lambda_i) \).

4. (10 points) Let \( A \) be an \((n \times n)\) complex matrix such that \( \det(\lambda I - A) = \prod_{i=1}^{m} (\lambda - \lambda_i)^{n_i} \), where \( \lambda_i \) are distinct. Find \( \det(\lambda I - e^A) \). Justify your answer.

5. (15 points)

(a) Problem 2.2 in Brendle.

(b) Consider the matrix \( A \) as in Problem 2.2 in Brendle. Find a matrix \( S \) such that \( S^{-1} A S \) is in the Jordan canonical form.