1. (10 points) Let
\[ A = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \]
and consider the initial value problem
\[ x'(t) = Ax(t), \quad x(0) = x_0. \]
Find all values of \( a, b \in \mathbb{R} \) such that
\begin{enumerate}
\item (a) 0 is a stable equilibrium.
\item (b) 0 is an asymptotically stable equilibrium.
\item (c) 0 is an unstable equilibrium.
\end{enumerate}
Justify your answers.
Finally, choose some values of \( a \) and \( b \) such that (c) above holds and sketch the corresponding phase portrait.

2. (10 points) Let \( A \) be a \((4 \times 4)\) complex matrix with eigenvalues \(-1 \pm i, \pm i\).
\begin{enumerate}
\item (a) Show that there exists a non-zero periodic solution \( x(t) \) to \( x'(t) = Ax(t) \).
\item (b) Show that there exists a non-zero solution \( x(t) \) to \( x'(t) = Ax(t) \) such that \( \|x(t)\| \to 0 \) as \( t \to \infty \).
\item (c) Show that there is an open and dense subset \( U \subset \mathbb{C}^4 \) such that if \( x_0 \in U \), then the unique solution to
\[ x'(t) = Ax(t), \quad x(0) = x_0 \]
is not periodic and does not satisfy \( \|u(t)\| \to 0 \) as \( t \to \infty \).
\end{enumerate}

3. (20 points) Consider the system
\[ \begin{cases} 
y''_1(t) = -\omega_1^2 y_1(t), \\
y''_2(t) = -\omega_2^2 y_2(t),
\end{cases} \]
where \( \omega_1, \omega_2 \in \mathbb{R} \setminus \{0\} \) satisfies \( \frac{\omega_1}{\omega_2} \notin \mathbb{Q} \) and \( y_1, y_2 : \mathbb{R} \to \mathbb{R} \) are the unknowns. This system describes two particles attached to two different springs. The purpose of the problem is to show that for \( \omega_1, \omega_2 \) as above, the solution is in general not periodic, but is always "almost" periodic (in a sense to be made precise below).
(a) Show that $(y_1)^2 + \omega_1^{-2}(y'_1)^2$ and $(y_2)^2 + \omega_2^{-2}(y'_2)^2$ are independent of $t$.

(b) From now on, we fix the initial conditions $y_1(0), y'_1(0), y_2(0), y'_2(0)$ so that

$$(y_1(0))^2 + \omega_1^{-2}(y'_1(0))^2 \neq 0, \quad (y_2(0))^2 + \omega_2^{-2}(y'_2(0))^2 \neq 0.$$ 

Define an equivalence relation $x \sim y$ if $x - y \in 2\pi\mathbb{Z} = \{2\pi n : n \in \mathbb{Z}\}$. Define also the distance on $\mathbb{R}/\sim$ by $d(\theta, \phi) = \min_{\phi' \sim \theta} |\theta' - \phi'|$. (Note that this depends only on the equivalence classes of $\theta$ and $\phi$.) Let $(\theta_1, \theta_2) : \mathbb{R} \to \mathbb{R}/\sim$ be defined by

$$\cos \theta_i(t) = \frac{y_i(t)}{\sqrt{(y_i(t))^2 + \omega_i^{-2}(y'_i(t))^2}},$$

$$\sin \theta_i(t) = \frac{\omega_i^{-1}y'_i(t)}{\sqrt{(y_i(t))^2 + \omega_i^{-2}(y'_i(t))^2}},$$

for $i = 1, 2$.

By deriving equations for $\theta_1$ and $\theta_2$, or otherwise, show that for $i = 1, 2$, $(y_i, y'_i) : \mathbb{R} \to \mathbb{R}^2$ is periodic of period $\frac{2\pi}{\omega_i}$, but the solution $(y_1, y'_1, y_2, y'_2) : \mathbb{R} \to \mathbb{R}^4$ is not periodic.

(c) Show that for every $\delta > 0$, there exists $T \in \mathbb{R}$ such that $d(\theta_1(t + T), \theta_1(t)) < \delta$ and $d(\theta_2(t + T), \theta_2(t)) < \delta$ for every $t \in \mathbb{R}$. [Hint: First show by a pigeon hole principle argument that there exist $m_1, m_2 \in \mathbb{N}$ with $m_1 < m_2$ such that $d(\frac{2\pi m_1 \omega_1}{\omega_2}, \frac{2\pi m_2 \omega_1}{\omega_2}) < \delta$. Conclude that $d(0, \frac{2\pi (m_2 - m_1) \omega_1}{\omega_2}) < \delta$. Let $m = m_2 - m_1$. Now take $T = \frac{2\pi m}{\omega_2}$.]

(d) We say that a function $f : \mathbb{R} \to \mathbb{R}^n$ is almost periodic if for every $\epsilon > 0$, there exists $T > 0$ such that $\|f(t + T) - f(t)\| < \epsilon$ for all $t \in \mathbb{R}$. Conclude, using part (c) or otherwise, that the solution $(y_1, y'_1, y_2, y'_2) : \mathbb{R} \to \mathbb{R}^4$ is almost periodic.

4. (10 points) Problem 2.5 in Brendle.

5. (10 points) Problem 2.6 in Brendle.