Spring 2018 Math 63CM: Midterm 2

Answer all questions. This is a one hour exam. This is a closed-book, closed-notes and closed-internet exam. Please write all your answers in (one or multiple) blue books.

Unless explicitly stated otherwise, you may quote standard theorems from lectures or the textbook, as long as you state them clearly.

1. (3 points) Let \( A \) be an \((n \times n)\) complex matrix (for some \( n \in \mathbb{N} \)). Prove that every eigenvalue \( \lambda \) of \( A \) satisfies \( |\lambda| \leq \|A\|_{\text{op}} \).

2. (6 points) Let \( A \) be an \((n \times n)\) complex matrix (for some \( n \in \mathbb{N} \)). Suppose \( q(\lambda) \) is a polynomial with complex coefficients. Prove that \( \mu \) is an eigenvalue of \( q(A) \) if and only if \( \mu = q(\sigma) \) where \( \sigma \) is an eigenvalue of \( A \). [Hint: First consider the case where \( A \) is an upper triangular matrix.]

3. (6 points) Suppose \( A \) is a \((2 \times 2)\) complex-valued matrix such that all its eigenvalues have non-positive real parts (i.e. every eigenvalue \( \lambda \) satisfies \( \text{Re}(\lambda) \leq 0 \)). Consider the ODE \( x'(t) = Ax \) in \( \mathbb{C}^2 \). Prove that there exists a constant \( C > 0 \) such that for all initial values \( x(0) = x_0 \) with \( \|x_0\| < 1 \), one has \( \|x(t)\| \leq Ct \) for all \( t \geq 1 \).

4. Sketch a phase portrait for \( x'(t) = Ax \) in each of the following cases: (Note that the conditions in (a) and (b) below do not uniquely determine the phase portrait. You only need to sketch a phase portrait which is consistent with the given conditions.)

   (a) (2 points) \( A \) is a \((2 \times 2)\) real-valued matrix with eigenvalues 1 and \(-1\).

   (b) (3 points) \( A \) is a \((2 \times 2)\) real-valued matrix with eigenvalues \(-1 \pm i\).

   [Draw your sketch as legibly as possible, but you do not need to give any justifications.]