A CRITERION FOR MULTIPERIOD CONTROLS IN ECONOMIC MODELS WITH UNKNOWN PARAMETERS

by

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1. Introduction

Economists are frequently faced with the practical problem of optimizing the behavior of an economic system about which little is known. Recently there has been much discussion and research about one such problem: the control of dependent variables over time in a model with unknown parameters but with a known structure. It is now well known that in these multiperiod problems, current control decisions affect not only current performance, but also the information available for estimating the unknown parameters. Sometimes called the problem of joint estimation and control, this problem has been investigated in the simple linear regression model with one unknown parameter by Prescott (1972) from a Bayesian point of view and by Taylor (1974) from a non-Bayesian point of view.

Because optimal Bayes control rules have been analytically as well as computationally difficult to find, except in the most simple cases, a large amount of research has been devoted to finding approximations to these rules (see Tse *et al.* (1973) and Chow (1973), for example). Most of these suggested approximations are accompanied by Monte Carlo experiments to evaluate their performance in a particular model with a given set of parameters. For the same reason that optimal Bayes rules are

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difficult to find analytically, there has been no analytic evaluation of these approximations.

The main purpose of this paper is to present and analyze a criterion for evaluating control rules which does provide some analytic results, at least for problems with an infinite time horizon and no discounting. The criterion is roughly defined as the limit, as the number of time periods approaches infinity, of the ratio of a sum of deviations from bliss to an appropriate function of time. The criterion is closely related to the concept of asymptotic efficiency of statistical estimation theory. It is because of such a relationship that some analytic results can be obtained. The criterion does not depend on placing prior distributions on the unknown parameters although an investigator may do so when the value of the criterion depends on the unknown parameters.

In order to analyze the infinite horizon no discount case when parameters are unknown, we must first consider the similar situation when the parameters are known. Therefore in Section 2 we consider a linear stochastic difference equation model with known parameters, and show that a stochastic version of the Ramsey deviation from bliss approach leads to a sum of undiscounted losses which converges.

In Section 3 we consider the unknown parameter case. The Ramsey deviation from bliss approach cannot be used in this case since the undiscounted sum diverges. We therefore modify the deviation from bliss approach by dividing the sum by an increasing function of time so that the limit of the ratio exists. In Section 4 we illustrate the use of the criterion with some previous results and in Section 5 we discuss how the criterion relates to the notion of good rules and the overtaking criterion of optimal growth theory.

2. No Discounting in the Known Parameters Case.

Consider a control problem of the following type. Let

(1) $$x_t = Ax_{t-1} + Cu_t + \varepsilon_t , \quad t = 1,2,\ldots,$$

where x_t is a vector of p elements consisting of current endogenous variables, lagged endogenous variables, current control variables and lagged control variables,[2] where u_t is a vector of q elements consisting of current control variables only, where $\{\varepsilon_t\}$ is a vector sequence of unobservable random variables which is independent and has zero mean and unknown finite variance Ω, and where A and C are unknown constant matrices of coefficients. We are concerned with bringing the elements of x_t close to certain target levels over an infinite time horizon with no discount rate, so that the loss function associated with deviations of x_t from these target levels is assumed to be independent of time.

This loss function is assumed to be the quadratic form

(2) $$L(x_t) = (x_t - a)'K(x_t - a),$$

where K is a known symmetric and positive semi-definite matrix and a is a known vector of targets.

A control rule is defined to be a sequence $\{u_t\}$ such that[3] $u_t = u_t\ (x_{t-1}, x_{t-2},\ldots)$ for $t = 1,2,\ldots$. Conceptually we would like to find a control rule which minimizes

(3) $$\lim_{T\to\infty} E \sum_{t=1}^{T} L(x_t).$$

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[2] For a discussion of the rationale behind this partitioning of the model see Chow (1970).

[3] By definition of the vector x_t this means that u_t is chosen sequentially on the basis of all past observations including observations on the control variable.

Using dynamic programming one may find that the limit as $T\to\infty$ of the solution to the functional equations for this problem exists. However, the criterion is meaningless because, with the additive disturbances in equation (1), the infinite sum diverges. In order to formulate the stochastic control problem more formally (which is necessary for the problem with unknown parameters introduced in the next section), we use a stochastic version of Ramsey's deviation from bliss approach which does lead to a well defined stochastic optimization problem.

The control problem is thus formulated as: find a control rule $\{u_t\}$ to minimize

(3a) $$\lim_{T\to\infty} E \sum_{t=1}^{T} (L(x_t) - b)$$

where b is the minimized value of $EL(x_t)$ in the steady state.
That is,

(4) $$b = \min_{\{u_t\}} \lim_{t\to\infty} EL(x_t).$$

We refer to b as the bliss level of the expected loss function. A sufficient condition for this steady state value to exist is that all the roots of A be less than one in absolute value, but necessary conditions depend on K and C as well as A.

One method of obtaining the minimized value of $EL(x_t)$ in the steady state is the constrained minimization approach described in Chow (1970). Considering the case where a=0 for simplicity, this leads to the control rule

(5) $$u^*_t = Gx_{t-1}, \; t = 1,2,\ldots,$$

where $G = -(C'HC)^{-1}C'HA$ and H satisfies the equations

(6) $$H = K + A'P'HPA,$$

where $P = I-C(C'HC)^{-1}C'H$. Using this control rule, equation (1) becomes

$x_t = Rx_{t-1} + \varepsilon_t$ where $R = A + CG$. Therefore, if all the roots of R are less than one in absolute value, the steady state value of the loss function is

(7) $$b = \mathrm{tr}KM,$$

where $M = \lim_{t\to\infty} Ex_t x_t' = \sum_{t=0}^{\infty} R^t \Omega R'^t$ and can be found from the set of equations $M = \Omega + RMR'$.

We now show that the infinite sum in (3a) exists for the control rule defined in equation (5). This results from a simple generalization of the fact that a series whose terms consist of the remainders of a convergent geometric series is also a convergent geometric series.

Let $Ex_0 x_0' = M_0$, then for $t>0$,

(8) $$x_t = R^t x_0 + \sum_{i=0}^{t-1} R^i \varepsilon_{t-i},$$

and

(9) $$Ex_t' K x_t = \mathrm{tr}K \{R^t M_0 R'^t + \sum_{i=0}^{t-1} R^i \Omega R'^i\}.$$

Therefore the expected T period sum of deviations from bliss is

(10) $$E \sum_{t=0}^{T} (L(x_t)-b)$$
$$= \mathrm{tr}K \sum_{t=0}^{T} R^t M_0 R'^t + \mathrm{tr}K \sum_{t=0}^{T} \sum_{i=0}^{t-1} (R^i \Omega R'^i - M).$$

The first series in equation (10) converges if all the roots of R are less than one in absolute value.[4] Let $R = Q \Lambda Q^{-1}$, where Λ is a diagonal matrix of the characteristics roots of R. Then, the t^{th} term in the second series in equation (10) can be written:

(11) $$\mathrm{tr}K \sum_{i=0}^{t-1} (R^i \Omega R'^i - M)$$
$$= \mathrm{tr}\, K \sum_{i=0}^{t-1} (Q\Lambda^i Q^{-1}\Omega Q\Lambda^i Q^{-1} - \sum_{i=0}^{\infty} Q\Lambda^i Q^{-1}\Omega Q\Lambda^i Q^{-1})$$
$$= \mathrm{tr}K \Omega \sum_{i=0}^{t-1} (\Lambda^{2i} - \sum_{i=0}^{\infty} \Lambda^{2i}).$$

The jth diagonal element of this last sum can be written as $-\lambda_j^{2t}/(1-\lambda_j^2)$.

4 The argument can be modified in the case of multiple roots.

Thus, since $\sum_{t=1}^{T} \lambda_j^{2t}$ converges for $|\lambda_j| < 1$, the second series in equation (10) will converge if $|\lambda_j| < 1$ for all j.

Having shown that the criterion of (3a) exists for the control rule (5) which minimizes the expected loss in the steady state, we can now legitamately consider the minimization of (3a) using dynamic programming. The limit as $T \to \infty$ of the solution of the functional equations for (3) is equivalent to (3a), since we have only changed the scale of the loss function in order to obtain the finite sum. Chow (1972) has noted [5] that the limit of this solution gives the same control rule that is found by minimizing the expected loss in the steady state; namely that given by equation (5). One may interpret the result of this section as a justification for minimizing the expected loss in the steady state when one really is interested in minimizing an undiscounted sum of expected losses in each period.

3. The Unknown Parameters Case

When the coefficient matrices A and B are unknown the optimal control rule $\{u_t^*\}$ defined by (5) cannot be used since it is defined in terms of these matrices. Therefore, some alternative which we represent as $\hat{u}_t$ must be used. For example, $\hat{u}_t$ might have the same form as (5) with the least squares estimates of A and C replacing the unknown values. Alternatively, $\hat{u}_t$ might have an entirely different form than (5), perhaps reflecting risk and experimentation to obtain more information about the unknown parameters. Such an experimental rule might be a Bayes control rule calculated for a suitable prior distribution on the unknown parameters or it may be an approximation to such a rule as suggested in the papers mentioned in the introduction.

In order to evaluate rules in this situation, a first inclination is to use the same criterion (3a) that was introduced in Section 2.

However, although the sum in (3a) is

[5] Merton (1973) obtains the same result in a continous time stochastic growth model.

finite when the parameters are known, it will not be finite when the parameters are unknown because $E(L(x_t)-b)$ does not converge to zero quickly enough. This is demonstrated in the next section of this paper in a model with one unknown parameter where $E(L(x_t)-b) = 0(1/t)$ so that $\sum_{t=1}^{T} E(L(x_t)-b)$ does not converge. Since in a model with more than one unknown parameter there will be even less information, the convergence cannot be any faster than 1/t so we will have divergence of this sum in more general situations also.

The suggested criterion of control is therefore

$$(13) \qquad \lim_{T\to\infty} \frac{E \sum_{t=1}^{T} (L(x_t)-b)}{f(T)} ,$$

where f(T) is a positive increasing function of T defined so that the limit exists and is nonzero, and where b is given in equation (7). Of two control rules, the one which results in a smaller value of (13) is preferred. It should be emphasized that the expectation is with respect to the sequence of random variables $\{\varepsilon_t\}$. A and C and therefore b are fixed constants although unknown. Note that f(T) should be chosen to increase more slowly than T whenever there exists a control rule for which $E(L(x_t)-b)\to 0$; therefore, the criterion is generally different from an average deviation from bliss.

It would seem that a minimum prerequisite for a control rule would be that $EL(x_t)\to b$, so that one might limit the admissible class of control rules to rules which have this convergence property. In situations where such mean square convergence is difficult to establish, one might use some other convergence concept. If $\{\hat{x}_t\}$ and $\{x^*_t\}$ represent the path of x_t when $\{\hat{u}_t\}$ and $\{u^*_t\}$ are used respectively, then this might entail showing that $\hat{x}_t - x^*_t \to 0$ in probability or with probability one.

It may be argued that this convergence prerequisite is not worth considering since it is already well known that many classical or Bayesian estimates are consistent, and therefore $\hat{x}_t$ which is a function of these estimates will converge to x_t^* in some sense. Such a sanguine view is not warranted, however, because the usual consistency theorems cannot be employed. The data used for estimation are the control variables of previous periods and are therefore random variables with a possibly complicated structure. In other words, the data cannot be assumed to be fixed and given since it is manipulated in a prescribed way by the control rule. One possible result is that as the control variable converges to a deterministic function of the lagged endogenous variables (note that u_t^* is a linear function of x_{t-1}), so that a multicollinearity problem becomes so serious as to prevent consistent estimates of the parameters. If the estimates of A and B upon which $\hat{u}_t$ is based are not consistent, then $\hat{x}_t$ may not converge to x_t^*. The simulation results of Tse, et. al. (1973) are interesting in this regard, because the parameter estimates do not seem consistent.

The criterion in (13) is also related to the notion of asymptotic efficiency in statistical estimation theory. Suppose that for some increasing function d_t a class of control rules has the property that $\lim_{t\to\infty} d_t E(L(x_t)-b)$ exists and is nonzero. Then, analogous to estimation theory, a control rule $\{\hat{u}_t\}$ is said to be asymptotically efficient out of this class, if $\lim_{t\to\infty} d_t E(L(\hat{x}_t)-b)$ is a minimum out of all other rules in the class. Let m be this minimum value. Then, assuming that $\sum_{t=1}^{T} d_t^{-1}$ diverges,[6]

$$(14) \qquad \lim_{T\to\infty} \frac{E\sum_{t=1}^{T}(L(\hat{x}_t)-b)}{f(T)} = m,$$

where $f(T) = \sum_{t=1}^{T} d_t^{-1}$. Thus an asymptotically efficient control rule minimizes

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[6] This assumption holds if $d_t=t$. The following lemma is used: Let $\{p_t\}$ be a sequence of positive numbers with $\sum_{t=1}^{T} p_t$ diverging to infinity and suppose that a sequence $\{z_t\}$ converges to z, then

$$\sum_{t=1}^{T} p_t z_t / \sum_{t=1}^{T} p_t \to z.$$ (See Knopp (1956), p. 32.)

the criterion of this paper.

If the limit of the normalized variance is difficult to obtain one might be satisfied in determining the limiting distribution of $d_t(\hat{x}_t - x_t^*)$ and compare control rules on the basis of the variance of this limiting distribution. It should be mentioned that a byproduct of this derivation is the distribution of the parameter estimates. These distributions will be different than in the non-control case where the data is not manipulated. The parameter estimates may converge at a rate considerably[7] less than d_t.

The similarity between our criterion and asymptotic efficiency indicates that the approach is useful in evaluating the advantages of control rules designed specifically for experimentation. The idea behind such control rules is that, by experimenting, more information can be obtained about the parameters to improve control performance in later periods. It might be argued intuitively that experimental control rules are designed for short run problems, and that in an infinite horizon model they are not important. However, in an infinite horizon problem experimentation in early periods has a much greater reward in the future than in short horizon problems, simply because there are more future periods. Thus, the intuitive argument can go either way, and a mathematical analysis using the criterion suggested here would be of interest.

Finally, the similarity between our criterion and asymptotic efficiency indicates that the approach will fail to discriminate between some control rules (just as more than one estimator may be asymptotically efficient). This is a difficulty which may be overcome by using asymptotic expansions to further discriminate between control rules as is currently being done in simultaneous equations problems. On the other hand,

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[7] Preliminary analysis of a model with two unknown parameters indicates (though does not rigorously establish) that if $d_t = t$, then the variance of the parameter estimates converges as $1/\log t$. The econometrician should be aware of such poor estimates if his model is ever to be used for purposes other than control.

the criterion is of some value even when it fails to discriminate, if the control rules were previously ranked differently on intuitive grounds. For example, if by this criterion a certainty equivalence control rule performs just as well as a computationally cumbersome control rule then in many problems a policy maker may be satisfied with using the certainty equivalence rule. In the following section we consider a particular case where this occurs.

4. The Case of One Unknown Parameter

As a particular case of equation (1) consider the scalar model

(15) $$x_t = \beta u_t + \varepsilon_t, \quad t = 1,2,\ldots,$$

where u_t is a scalar control variable, x_t is the variable to be controlled, β is an unknown scalar, and $\{\varepsilon_t\}$ is an independent sequence of random variables with zero mean and finite variance σ^2. Further assume that the loss function is

(16) $$L(x_t) = (x_t - a)^2,$$

so that there is no cost of control.[8] Thus, $u_t^* = a/\beta$ and $b = \sigma^2$ and the deviation from bliss due to β being unknown is

(17) $$E(L(\hat{x}_t) - b) = E(\beta\hat{u}_t - a)^2$$

when $\hat{u}_t$ is used instead of u_t^*. In a previous paper {Taylor (1974)} two such control rules $\{\hat{u}_t\}$ were considered; a least squares certainty equivalence control rule defined by $\hat{u}_1$ non-zero but otherwise arbitrary, and

(18) $$\hat{u}_t = \frac{a}{\hat{\beta}_{t-1}}, \quad t = 2,3,\ldots,$$

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[8] The following discussion could be easily modified if u_t also appeared in the loss function.

where $\hat{\beta}_{t-1}$ is the least squares estimate of β based on $(x_{t-1}, u_{t-1},\ldots,x_1,u_1)$; and a Bayesian certainty equivalence control rule

$$(19) \qquad \hat{u}_t = \frac{a}{b_{t-1}} \quad t = 1,2,\ldots,$$

where b_{t-1} is the Bayes estimate of β when the assumed prior distribution of β is normal with mean b_o, ε_t is normally distributed and the loss function for estimation is quadratic. For both rules it is shown that

$$(20) \qquad \hat{x}_t - x^*_t \to 0$$

with probability one, and that

$$(21) \qquad \sqrt{t}(\hat{x}_t - x^*_t) \overset{d}{\to} N(0,\sigma^2).$$

If we assume that $\hat{u}_t$ is bounded, then

$$(22) \qquad tE(\beta\hat{u}_t - a)^2 \to \sigma^2,$$

and using equation (14)

$$(23) \qquad \lim_{T\to\infty} \frac{\sum_{t=1}^{T} E(\beta\hat{u}_t - a)^2}{\log T} = \sigma^2,$$

when we use the fact that $\Sigma_{t=1}^{T}\ 1/t = O(\log T)$. Thus by the criterion of this paper both control rules perform equally well.

But a stronger result also holds. The earlier paper shows that <u>any</u> control rule $\{\hat{u}_t\}$ for which $\beta\hat{u}_t - a \to 0$ with probability one, cannot lead to estimates of β which have a smaller asymptotic variance. Since the control value of $\hat{u}_t$ must be based on parameter estimates of β we conclude that any such control rule cannot have a smaller asymptotic variance than that given in (21). Thus, any control rule which converges to the value a/β with probability one cannot give a smaller value of the criterion than that given by (23).

Therefore, under the assumptions of this model and criterion, the certainty equivalence control rules perform as well as any control rule designed for experimentation. If this result also holds in more general models, then the practical use of certainty equivalence rules in models with a long time horizon might be acceptable afterall. But the more general results remain to be shown.

5. The Overtaking Criterion and Good Rules.

In the literature of optimal economic growth it has become common to use the overtaking criterion introduced by von Weiszacker (1965) in undiscounted infinite horizon problems. A stochastic version of the overtaking criterion is that a control rule $\{\hat{u}_t\}$ overtakes $\{u_t\}$ if there exists a T_o such that $E\sum_{t=1}^{T}\{L(\hat{x}_t)-L(x_t)\} \leqq 0$ for all $T > T_o$. A related criterion discussed by Gale (1967) is that of a good rule. A control rule $\{\hat{u}_t\}$ is good relative to $\{u_t\}$ if there exists a T_o and an M such that $E\sum_{t=1}^{T}\{L(\hat{x}_t)-L(x_t)\} \leqq M$ for all $T > T_o$. If $\{\hat{u}_t\}$ overtakes $\{u_t\}$, then $\{\hat{u}_t\}$ is good relative to $\{u_t\}$ but the converse is not true since M may be greater than zero.

Both the overtaking criterion and criterion (13) of this paper are generalizations of the usual methods of comparing sums. That is, when the expected sum of losses in expression (3) happens to converge then both criteria give the same partial ordering and one which coincides with the common sense method of comparing convergent sums. A priori it is therefore difficult to decide which generalization makes more economic sense.

To facilitate comparison of these criteria in nonconvergent cases, we note that, if the value of criterion (13) at $\{\hat{u}_t\}$ is less than or equal to the value at $\{u_t\}$, then

$$\lim_{T\to\infty} \frac{E\sum_{t=1}^{T}\{L(\hat{x}_t) - L(x_t)\}}{f(T)} \leqq 0 . \tag{24}$$

Therefore, whenever lim f(T) exists, the overtaking criterion is equivalent to criterion (13) since we can focus on the numerator of the above expression. In particular when the coefficient matrices A and C in equation (1) are known, f(T) may be chosen to be a constant (since the expected sum of deviations from bliss converges) and criterion (13) is equivalent to the overtaking criterion.

However, if $f(T)$ is divergent, as is the case when the coefficient matrices A and C are unknown, then the criterion (13) and the overtaking criterion are not equivalent. If there is a strict preference of one control rule over another by criterion (13); that is, if strict inequality holds in (24), then there will also be strict inequality in the overtaking criterion. However, as the following example shows, if equality holds in (24), then not only may the overtaking criterion be violated, but one rule may not even be good relative to the other. For example, suppose that

$$(25) \qquad \hat{r}_t = b + \frac{\delta}{t} + \frac{\hat{\theta}}{t\log t}, \quad t = 2,3,..,$$

and

$$(26) \qquad r_t = b + \frac{\delta}{t} + \frac{\theta}{t\log t}, \quad t = 2,3,...,$$

is the expected loss at any time t associated with the control rule $\{\hat{u}_t\}$ and $\{u_t\}$ respectively, where $\hat{\theta} > \theta$. Then $t(\hat{r}_t - b)$ and $t(r_t - b)$ both converge to δ so that the value of criterion (13) under both rules is the same, and we have equality in (24) with $f(T) = \log T$. However,

$$(27) \qquad \sum_{t=2}^{T} (\hat{r}_t - r_t) = (\hat{\theta} - \theta) \sum_{t=2}^{T} \frac{1}{t\log t},$$

which diverges to infinity. Thus $\{\hat{u}_t\}$ is not good relative to $\{u_t\}$. The conditions of a particular problem may exclude examples such as this, but nevertheless it illustrates that control rules which perform equally well under criterion (13) might be ranked quite differently by another criterion. Such difficulties seem inherent in any criterion proposed to compare infinite sums.

The criterion proposed in this paper is useful because it can provide analytic results about control rules as illustrated in Section 4 using the methods of statistical estimation theory. If a particular control rule is under consideration in a practical problem, either because it is intuitively pleasing or because it is an approximation to a Bayes control rule, it should be investigated under the basis of this criterion if possible. Any control rule which is strictly inferior by this criterion should be eliminated from consideration. For rules which perform equally well, other critera might also be taken into account.

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