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# ASYMPTOTIC PROPERTIES OF MULTIPERIOD CONTROL RULES IN THE LINEAR REGRESSION MODEL*

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## 1. INTRODUCTION

IN MULTIPERIOD CONTROL PROBLEMS with unknown parameters, current decisions affect not only current performance, but also the amount of information that is obtained about the unknown parameters. The purpose of this study is to investigate such aspects of multiperiod control in a simple linear regression model with one unknown parameter, where the independent variable is set at certain levels in order to bring the dependent variable to some desired level. The approach uses the methods and criteria of statistical estimation theory (such as strong consistency and efficiency) to investigate the properties of various control rules. This approach seems particularly useful in control problems of this type where estimation of unknown parameters plays an important role.

Previous investigations of this type of multiperiod control problem (Aoki [2]), Zellner [9], and Prescott [5]) have been from a Bayesian point of view. By specifying a loss function, prior distributions on the parameters, and a distribution for the random disturbance term, a Bayes control rule can be calculated, in principle, with the methods of dynamic programming. However, as these studies have shown, calculation or even characterization of Bayes control rules has proved quite difficult. The approach of this study is non-Bayesian. The methods and results should complement the usual Bayesian viewpoint in eventually leading to reasonable decisions in practical problems.

In Section 2 the model is introduced and two control rules are defined. In Section 3 we prove that these control rules converge with probability 1 to the value which would be used if the unknown parameter were known with certainty. In Section 4 we derive the asymptotic distribution of the control rules and parameter estimates, and in Section 5 we show that these control rules lead to parameter estimates which have as small an asymptotic variance as any other control rule in a fairly wide class. In particular this means that control rules which are designed for experimentation do not give parameter estimates which are any better asymptotically than the more simple control rules of this paper.

## 2. THE MODEL AND DEFINITION OF CONTROL RULES

We consider a linear regression model represented by

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$$x_t = \beta u_t + \varepsilon_t , \qquad t = 1, 2, \ldots, \tag{1}$$

where the control variable $u_t$ is used to control the endogenous variable $x_t$ about some desired level $a$, where $\beta$ is an unknown parameter, and where $\{\varepsilon_t\}$ is an independent sequence of (unobservable) random variables with zero mean and finite variance $\sigma^2$. Thus, we assume that the slope is unknown and the intercept is known with certainty. Since the intercept is known we can assume that it is zero without loss of generality.[3] This same model has been investigated from a Bayesian viewpoint by Prescott [5].

We define a *control rule* to be a sequence $\{u_t\}$, the elements of which are chosen sequentially on the basis of past observations. More specifically each element of $\{u_t\}$ is a function of all random variables observed prior to time $t$; that is

$$u_t = u_t(x_1, \ldots, x_{t-1}; u_1, \ldots, u_{t-1}) . \tag{2}$$

Thus a particular control rule can be thought of as a set of instructions which specifies the control action to take at each point in time for all possible developments of the process until that point in time. In this study we consider two control rules satisfying this definition.

One control rule that is particularly easy to calculate is the sequence defined by $u_1$ fixed and nonzero, but otherwise arbitrary, and

$$u_{t+1} = \frac{a}{\hat{\beta}_t} , \qquad t = 1, 2, \ldots, \tag{3}$$

where $\hat{\beta}_t$ is the least squares estimate of $\beta$ at time $t$ defined by

$$\hat{\beta}_t = \frac{\sum_{i=1}^{t} u_i x_i}{\sum_{i=1}^{t} u_i^2} , \qquad t = 1, 2, \ldots . \tag{4}$$

This is the value of the control rule which would be used if one treated $\beta$ as known with certainty and equal to the least squares estimate. We call this rule the *least squares certainty equivalence* control rule. It is of particular interest to investigate the properties of this rule since we expect that it is frequently used in practice.

A related control rule would be preferred to the least squares certainty equivalence control rule if there were some prior knowledge about the unknown parameter $\beta$. This prior knowledge might be due to some observations which have been made before the control problem starts. If the prior knowledge of $\beta$ could be represented by a normal prior distribution $N(b_0, \sigma_0^2)$ and if $\varepsilon_t$ were distributed according to $N(0, \sigma^2)$ with $\sigma^2$ known, then it can be shown[4] that the posterior distribution at any time $t$ will also be normal $N(b_t, \sigma_t^2)$ where

[3] If the known intercept were $\alpha \neq 0$ then, by redefining the endogenous variable $x_t^* = x_t - \alpha$ and the target $a^* = a - \alpha$, the model could be reduced to the zero intercept case of equation (1).

[4] See Raiffa and Schlaifer [6 (337)] for this calculation.

$$b_t = \frac{\dfrac{b_0}{\sigma_0^2} + \dfrac{1}{\sigma^2} \sum_{i=1}^{t} u_i x_i}{\dfrac{1}{\sigma_0^2} + \dfrac{1}{\sigma^2} \sum_{i=1}^{t} u_i^2}, \tag{5}$$

and

$$\frac{1}{\sigma_t^2} = \frac{1}{\sigma_0^2} + \frac{1}{\sigma^2} \sum_{i=1}^{t} u_i^2 . \tag{6}$$

If quadratic loss were the criterion of estimation, then the Bayes estimate of $\beta$ under these distribution assumptions would be the mean of the posterior distribution $b_t$. A control rule using this prior information is defined by the sequence

$$u_{t+1} = \frac{a}{b_t}, \qquad t = 0, 1, \ldots . \tag{7}$$

We call this control rule the *Bayesian certainty equivalence control rule*. In our study the Bayesian approach is *only* used to define the control rule (7); the estimate $b_t$ defined by (5) can alternatively be interpreted as a weighted average between the least squares estimate and a prior guess.

The theorems which follow are not based on the distribution assumptions of the Bayesian method. These assumptions are only used to suggest the Bayesian certainty equivalence control. The following analysis assumes only that the error terms are independently and identically distributed with zero mean and finite variance. Subject to these conditions the distribution may have any form.

## 3. CONVERGENCE OF CONTROL RULES

In this section we show that under suitable conditions the two control rules defined above converge to the value $a/\beta$ with probability 1. We choose to prove convergence with probability 1 rather than convergence in probability because we would like the control to converge to the true value and remain there with high probability. This is guaranteed by convergence with probability 1, but not by convergence in probability. The latter says only that, for any sufficiently large *fixed t*, the probability that the control is near its true value is arbitrarily close to one. Convergence in probability is enough in most econometric investigations because one is usually referring to some fixed sample. Further, the use of strong convergence allows one to use some non-probabilistic results for arbitrary sequences of numbers. Once a sequence of random variables is shown to have a certain property with probability 1, we then can ignore all sample points where the property does not occur and apply non-probabilistic results to the remaining points.

We first prove three preliminary lemmas, of which the first two are non-probabilistic.

LEMMA 1. *Let* $\{z_t\}$ *be an arbitrary sequence of numbers such that* $z_1 \neq 0$. *If*

(8)
$$s_t = \sum_{i=1}^{t} \left( \frac{z_i}{\sum_{j=1}^{i} z_j^2} \right)^2 ,$$

*then* $s_t < 2/z_1^2$ *for every* $t$.

PROOF. Define

$$r_k = \sum_{j=1}^{t-k} z_j^2 , \qquad k = 1, 2, \ldots, t-1 , \ r_t = 0 ,$$

and

$$w_i = \left[ \frac{z_i}{(r_{t-i+1} + z_i^2)} \right]^2 , \qquad i = 1, 2, \ldots, t ;$$

then $s_t = \sum_{i=1}^{t} w_i$. Since, for any $n > 0$, $z_{i+n}$ is not an argument of $w_i$, we can use backward induction to maximize $s_t$ with respect to $z_t, z_{t-1}, \ldots, z_2$ in turn and thus establish an upper bound. Let

(9)
$$v_{k+1} = \max_{z_{t-k}, \ldots, z_t} \left[ \sum_{i=t-k}^{t} w_i \right] ,$$

for $k = 0, 1, \ldots, t-2$. Then $s_t \leq 1/z_1^2 + v_{t-1}$ and to prove the lemma we must calculate $v_{t-1}$.

Considering the last term in $s_t$, we have by differentiation,

(10)
$$v_1 = \max_{z_t} w_t = \max_{z_t} \frac{z_t^2}{(r_1 + z_2^t)^2} = \frac{1}{a_1 r_1} ,$$

where $a_1 = 4$. Therefore, since $r_1 = r_2 + z_{t-1}^2$, by differentiation,

(11)
$$v_2 = \max_{z_{t-1}} (w_{t-1} + v_1) = \max_{z_{t-1}} \left[ \frac{z_{t-1}^2}{(r_2 + z_{t-1}^2)^2} + \frac{1}{a_1(r_2 + z_{t-1}^2)} \right] = \frac{1}{a_2 r_2} ,$$

where $a_2 = [1 + (a_1 - 1)/(a_1 + 1)]^2$. Now suppose that $v_k = 1/a_k r_k$ for some $k, 3 \leq k \leq t-1$. Then

(12)
$$v_{k+1} = \max_{z_{t-k}} (w_{t-k} + v_k) = \max_{z_{t-k}} \left[ \frac{z_{t-k}^2}{(r_{k+1} + z_{t-k}^2)^2} + \frac{1}{a_k(r_{k+1} + z_{t-k}^2)} \right] = \frac{1}{a_{k+1} r_{k+1}} ,$$

where $a_{k+1} = [1 + (a_k - 1)/(a_k + 1)]^2$, using the fact that the maximand in equation (12) is equivalent to that of equation (11) with $a_k$ replacing $a_1$ and $r_{k+1}$ replacing $r_2$. Setting $k = t-2$ in the recursive equation (12), we have that $v_{t-1} = 1/(a_{t-1} z_1^2)$, since $r_{t-1} = z_1^2$. Therefore

$$s_t \leq \frac{1}{z_1^2} \left( 1 + \frac{1}{a_{t-1}} \right) ,$$

and since $a_{t-1} > 1$ we have $s_t < 2/z_1^2$.

LEMMA 2. *Let $\{z_t\}$ be a sequence of numbers and let $\{a_t\}$ be an increasing sequence of positive numbers such that $\sum_{i=1}^{t} z_i/a_i$ converges,*
(i) *If $a_t \to \infty$, then $\lim_{t\to\infty} 1/a_t \sum_{i=1}^{t} z_i = 0$,*
(ii) *If $a_t \to M < \infty$, then $\lim_{t\to\infty} 1/a_t \sum_{i=1}^{t} z_i$ exists.*

PROOF. Part (i) is Kronecker's lemma. (See Feller [**3**(239)]). We need only consider part (ii). Define $s_0 = 0$ and let

$$s_t = \sum_{i=1}^{t} \frac{z_i}{a_i}, \qquad t = 1, 2, \ldots, \tag{13}$$

then

$$z_t = a_t(s_t - s_{t-1}), \tag{14}$$

and therefore

$$\frac{1}{a_t}\sum_{i=1}^{t} z_i = \frac{1}{a_t}\sum_{i=1}^{t} a_i(s_i - s_{i-1}) = s_t - \frac{1}{a_t}\sum_{i=1}^{t-1}(a_{i+1} - a_i)s_i . \tag{15}$$

Now, by assumption $s_t$ converges to $s$, say, so that to complete the proof of the lemma we must show that the second term on the right hand side of (15) converges.

For an arbitrary $\varepsilon > 0$ choose $t_0$ so that, for all $t > t_0$, $|s_t - s| < \varepsilon$. Such a $t_0$ exists by the convergence assumption. We then have

$$\frac{1}{a_t}\sum_{i=1}^{t-1}(a_{i+1} - a_i)s_i = \frac{1}{a_t}(a_t - a_1)s + \frac{1}{a_t}\sum_{i=1}^{t_0-1}(a_{i+1} - a_i)(s_i - s) + \frac{1}{a_t}\sum_{i=t_0}^{t-1}(a_{i+1} - a_i)(s_i - s) . \tag{16}$$

Now, because $a_t$ converges and $t_0$ is fixed, the first two terms on the right hand side converge. Further

$$\left|\frac{1}{a_t}\sum_{i=t_0}^{t-1}(a_{i+1} - a_i)(s_i - s)\right| < \frac{1}{a_t}(a_t - a_{t_0})\varepsilon < \varepsilon, \tag{17}$$

and, since $\varepsilon$ is arbitrary, the third term in (16) is arbitrarily small. Thus $1/a_t\sum_{i=1}^{t} z_i$ converges.

The following lemma is probabilistic and uses the martingale convergence theorem. (See Feller [**3** (236)]).

LEMMA 3. *Let $\{\varepsilon_i\}$ be an independent sequence of random variables with $E\varepsilon_i = 0$ and $E\varepsilon_i^2 = \sigma^2 < \infty$ and let $\{u_i\}$ be a sequence of random variables with $u_1$ fixed and nonzero and $\varepsilon_i$ independent of $\{u_i, u_{i-1}, \ldots, u_1, \varepsilon_{i-1}, \ldots, \varepsilon_1\}, i = 2, 3, \ldots$. Then*

$$s_t = \sum_{i=1}^{t} \frac{u_i\varepsilon_i}{\sum_{j=1}^{i} u_j^2} \tag{18}$$

*converges with probability* 1.

PROOF. We have that

$$(19)\quad E\left(\frac{u_i\varepsilon_i}{\sum_{j=1}^{i} u_j^2}\middle| s_{i-1},\ldots,s_1\right) = E\left(\frac{u_i}{\sum_{j=1}^{i} u_j^2}\middle| s_{i-1},\ldots,s_1\right)E(\varepsilon_i \mid s_{i-1},\ldots,s_1)$$

$$= E\left(\frac{u_i}{\sum_{j=1}^{i} u_j^2}\middle| s_{i-1},\ldots,s_1\right)\cdot 0$$

$$= 0$$

since $|u_i|/\sum_{j=1}^{i} u_j^2 \leq 1/|u_1|$. Thus $\{s_t\}$ is a martingale and to use the martingale convergence theorem it must be shown that $Es_t^2$ remains bounded for all $t$. From the independence assumptions we have

$$(20)\quad E\left(\sum_{i=1}^{t}\frac{u_i\varepsilon_i}{\sum_{j=1}^{i} u_j^2}\right)^2 = \sigma^2 E\left(\frac{u_i}{\sum_{j=1}^{i} u_j^2}\right)^2 \leq \sigma^2\frac{2}{u_1^2},$$

where the last inequality follows from Lemma 1 with $z_1 = u_1$. Thus the variance remains bounded for all $t$ and by the martingale convergence theorem $s_t$ converges with probability 1.

The following theorem contains the main convergence results about the multiperiod control rules. The proof involves showing that, with probability 1, each control rule does not stop obtaining information about the unknown parameter.

THEOREM 1. *In the model $x_t = \beta u_t + \varepsilon_t$, if $\{\varepsilon_t\}$ is an independent sequence of random variables with $E\varepsilon_t = 0$ and $E\varepsilon_t^2 = \sigma^2 < \infty$ and $\beta \neq 0$, then (i) the least squares certainty equivalence control rule converges to $a/\beta$ with probability 1, and (ii) if $b_0 \neq 0$ and $\sigma_0^2 \neq 0$ then the Bayesian certainty equivalence control rule converges to $a/\beta$ with probability 1.*

PROOF. (i) The least squares certainty equivalence control rule can be written

$$(21)\quad u_{t+1} = \frac{a}{\beta + \dfrac{\sum_{i=1}^{t} u_i\varepsilon_i}{\sum_{i=1}^{t} u_i^2}}.$$

We first must establish that $\sum_{i=1}^{t} u_i^2 \to \infty$ with probability 1. Let $\omega$ be any sample point in the sample space $\Omega$. Then we have from Lemma 3 that

$$(22)\quad P\left[\omega \middle| \sum_{i=1}^{t}\frac{u_i(\omega)\varepsilon_i(\omega)}{\sum_{j=1}^{i} u_j^2(\omega)} \text{ converges}\right] = 1.$$

Thus we can apply Lemma 2, parts (i) and (ii), at each sample point with

$z_i = u_i(\omega)\varepsilon_i(\omega)$ and $a_i = \sum_{j=1}^{i} u_j^2(\omega)$ to obtain

$$(23) \qquad P\left[\omega \left| \frac{\sum_{i=1}^{t} u_i(\omega)\varepsilon_i(\omega)}{\sum_{i=1}^{t} u_i^2(\omega)} \text{ converges} \right.\right] = 1 .$$

But this implies that

$$(24) \qquad P\left[\omega \left| \lim_{t\to\infty} a\left(\beta + \frac{\sum_{i=1}^{t} u_i(\omega)\varepsilon_i(\omega)}{\sum_{i=1}^{t} u_i^2(\omega)}\right)^{-1} \neq 0 \right.\right] = 1$$

and, from (21),

$$(25) \qquad P[\omega \,|\, \lim_{t\to\infty} u_{t+1}(\omega) \neq 0] = 1 ,$$

and therefore, we have that

$$(26) \qquad P[\omega \,|\, \sum_{i=1}^{t} u_i^2(\omega) \text{ diverges}] = 1 .$$

Having proved that $\sum_{i=1}^{t} u_i^2 \to \infty$ with probability 1, we can now apply Lemma 2(i) at every sample point to obtain

$$(27) \qquad P\left[\omega \left| \frac{\sum_{i=1}^{t} u_i(\omega)\varepsilon_i(\omega)}{\sum_{i=1}^{t} u_i^2(\omega)} \to 0 \right.\right] = 1 ,$$

and from (21) this implies that

$$(28) \qquad u_{t+1} \to \frac{a}{\beta}$$

with probability 1.

(ii) The argument for the Bayesian certainty equivalence control rule is similar, except that we must insure that the weights on the prior parameters converge to zero with probability one. With Bayesian estimates we have

$$(29) \qquad u_{t+1} = a\left[\frac{\dfrac{1}{\sigma_0^2} + \dfrac{1}{\sigma^2}\sum_{i=1}^{t} u_i^2}{\dfrac{b_0}{\sigma_0^2} + \dfrac{1}{\sigma^2}\left(\beta \sum_{i=1}^{t} u_i^2 + \sum_{i=1}^{t} u_i\varepsilon_i\right)}\right]$$

$$= \frac{a\left(\dfrac{\sigma^2}{\sigma_0^2}\dfrac{1}{\sum_{i=1}^{t} u_i^2} + 1\right)}{\dfrac{b_0}{\sigma_0^2}\dfrac{\sigma^2}{\sum_{i=1}^{t} u_i^2} + \beta + \dfrac{\sum_{i=1}^{t} u_i\varepsilon_i}{\sum_{i=1}^{t} u_i^2}} .$$

Now, since

$$\frac{\sigma^2}{\sigma_0^2} \frac{1}{\sum_{i=1}^{t} u_i^2} + 1 \tag{30}$$

is nonzero, we can use the argument of equations (22) and (23) to show that

$$P[\omega \mid \lim_{t\to\infty} u_{t+1}(\omega) \neq 0] = 1 , \tag{31}$$

and therefore

$$P[\omega \mid \sum_{i=1}^{t} u_i^2(\omega) \text{ diverges}] = 1 . \tag{32}$$

Thus, from Lemma 2(i) applied at every sample point,

$$P\left[\omega \left| \frac{\sum_{i=1}^{t} u_i(\omega)\varepsilon_i(\omega)}{\sum_{i=1}^{t} u_i^2(\omega)} \to 0 \right.\right] = 1 , \tag{33}$$

and also,

$$P\left[\omega \left| \frac{1}{\sum_{i=1}^{t} u_i^2(\omega)} \to 0 \right.\right] = 1 . \tag{34}$$

From equation (29) this implies that

$$u_t \to \frac{a}{\beta} \tag{35}$$

with probability 1.

This completes the proof of Theorem 1 from which we have the following corollary which states that both the least squares estimate and the Bayesian estimate of $\beta$ are strongly consistent.

COROLLARY 1. *Under the assumptions of Theorem* 1
(i) $\hat{\beta}_t \to \beta$ *with probability* 1, *and*
(ii) $b_t \to \beta$ *and* $\sigma_t^2 \to 0$ *with probability* 1.

PROOF. Once we have established that $\sum_{i=1}^{t} u_i^2 \to \infty$ with probability one, the corollary follows immediately from Lemma 2(i) as described in the proof of Theorem 1.

## 4. THE ASYMPTOTIC DISTRIBUTION OF CONTROL RULES

Additional information about the behavior of multiperiod control rules can be obtained by examining their asymptotic distributions. To obtain these distributions we first derive the asymptotic distributions of the estimates of the unknown parameter $\beta$. We begin by proving a preliminary lemma[5].

LEMMA 4. *Let* $\{\nu_i\}$ *be a sequence of random variables such that* $\nu_i \to 0$ *with probability* 1 *and let* $\{\varepsilon_i\}$ *be an independent sequence of random variables with* $E\varepsilon_i = 0$ *and* $E\varepsilon_i^2 = \sigma^2 < \infty$ *and* $\varepsilon_i$ *independent of* $\{\varepsilon_{i-1}, \ldots, \varepsilon_1, \nu_i, \nu_{i-1}, \ldots, \nu_1\}$, $i = 2, 3, \ldots$.

*Then*

$$\frac{\sum_{i=1}^{t} \nu_i \varepsilon_i}{\sqrt{t}} \xrightarrow{p} 0 .$$

PROOF. We use a method of truncation. Define $\nu_i'$ and $\nu_i''$ as

$$\begin{aligned} &\nu_i' = \nu_i , \quad \nu_i'' = 0 \quad \text{if} \quad |\nu_i| < c , \\ &\nu_i' = 0 , \quad \nu_i'' = \nu_i \quad \text{if} \quad |\nu_i| \geq c , \end{aligned} \tag{36}$$

then $\nu_i = \nu_i' + \nu_i''$ and

$$\frac{1}{\sqrt{t}} \sum_{i=1}^{t} \nu_i \varepsilon_i = \frac{1}{\sqrt{t}} \sum_{i=1}^{t} \nu_i' \varepsilon_i + \frac{1}{\sqrt{t}} \sum_{i=1}^{t} \nu_i'' \varepsilon_i . \tag{37}$$

Now, since $\nu_i'$ is bounded and since $\nu_i' \to 0$ with probability 1 we have that $E(\nu_i')^2 \to 0$. Therefore

$$E\left( \frac{\sum_{i=1}^{t} \nu_i' \varepsilon_i}{\sqrt{t}} \right)^2 = \frac{\sigma^2 \sum_{i=1}^{t} E(\nu_i')^2}{t} \to 0 , \tag{38}$$

and by Chebyshev's inequality

$$\frac{\sum_{i=1}^{t} \nu_i' \varepsilon_i}{\sqrt{t}} \xrightarrow{p} 0 .$$

It remains to consider

$$\frac{\sum_{i=1}^{t} \nu_i'' \varepsilon_i}{\sqrt{t}} .$$

From the definition of $\nu_i''$ we have

$$P[\nu_i'' \neq 0] = P[|\nu_i| \geq c] . \tag{39}$$

But since $\nu_i \to 0$ with probability 1

$$P[\omega \,|\, |\nu_i(\omega)| \geq c \text{ infinitely often}] = 0 . \tag{40}$$

[5] In the following lemma and theorems we use the notation "$\xrightarrow{p}$" for converges in probability and "$\xrightarrow{d}$" for converges in distribution. With the exception of Lemma 4 we use only the weak consistency property of the control rules and parameter estimates to derive the asymptotic distributions.

Therefore,

$$P[\omega \mid \nu_i''(\omega) \neq 0 \text{ infinitely often}] = 0\,, \tag{41}$$

so that

$$P[\omega \mid \sum_{i=1}^{\infty} \nu_i''(\omega)\varepsilon_i(\omega) \text{ has a finite number of nonzero terms}] = 1\,, \tag{42}$$

and therefore

$$P\left[\omega \left| \frac{\sum_{i=1}^{t} \nu_i''(\omega)\varepsilon_i(\omega)}{\sqrt{t}} \to 0 \right.\right] = 1\,. \tag{43}$$

From (37) we therefore have that

$$\frac{\sum_{i=1}^{t} \nu_i \varepsilon_i}{\sqrt{t}} \xrightarrow{p} 0\,.$$

THEOREM 2. *Under the assumption of Theorem* 1,

$$\text{(i)} \qquad \sqrt{t}\,(\hat{\beta}_t - \beta) \xrightarrow{d} N\left(0, \frac{\beta^2}{a^2}\sigma^2\right)$$

*and*

$$\text{(ii)} \qquad \sqrt{t}\,(b_t - \beta) \xrightarrow{d} N\left(0, \frac{\beta^2}{a^2}\sigma^2\right).$$

PROOF. (i) To find the limiting distribution of $\sqrt{t}\,(\hat{\beta}_t - \beta)$, we have from the definition of the least squares estimate,

$$\begin{aligned} \sqrt{t}\,(\hat{\beta}_t - \beta) &= \frac{t}{\sum_{i=1}^{t} u_i^2} \frac{\sum_{i=1}^{t} u_i \varepsilon_i}{\sqrt{t}} \\ &= \frac{t}{\sum_{i=1}^{t} u_i^2} \left[ \frac{\sum_{i=1}^{t}\left(u_i - \frac{a}{\beta}\right)\varepsilon_i}{\sqrt{t}} + \frac{a}{\beta} \frac{\sum_{i=1}^{t} \varepsilon_i}{\sqrt{t}} \right]. \end{aligned} \tag{44}$$

From Lemma 4 the first term in brackets converges to zero in probability with $\nu_i = u_i - a/\beta$. In addition, from Theorem 1 we have that $u_i^2 \to (a/\beta)^2$ with probability 1, so that[6]

$$\frac{\sum_{i=1}^{t} u_i^2}{t} \longrightarrow \left(\frac{a}{\beta}\right)^2 \tag{45}$$

[6] If a sequence converges then the arithmetic mean of the sequence also converges to the same point. See Knopp [**4** (35)]. We apply this result at every sample point to obtain the result of equation (45).

with probability 1. Therefore, the difference between the right hand side of equation (44) and

$$\frac{\beta}{a}\frac{\sum_{i=1}^{t}\varepsilon_i}{\sqrt{t}} \tag{46}$$

converges in probability to zero. By the central limit theorem (46) converges in distribution to $N(0, \beta^2\sigma^2/a^2)$.

(ii) To find the limiting distribution of $\sqrt{t}\,(b_t - \beta)$ we have, from the definition of $b_t$

$$\sqrt{t}\,(b_t - \beta) = \frac{\sqrt{t}\left(\dfrac{(b_0 - \beta)\sigma^2}{\sigma_0^2 \sum_{i=1}^{t} u_i^2} + \dfrac{\sum_{i=1}^{t} u_i\varepsilon_i}{\sum_{i=1}^{t} u_i^2}\right)}{\dfrac{\sigma^2}{\sigma_0^2\sum_{i=1}^{t} u_i^2} + 1}\,. \tag{47}$$

Now, since $u_i \to a/\beta$ with probability 1, we have

$$\frac{\sqrt{t}}{\sum_{i=1}^{t} u_i^2} \longrightarrow 0 \tag{48}$$

with probability 1, using the result of equation (45). Therefore equation (47) converges in probability to equation (44) and we can apply the same argument as in part (i) to show that $\sqrt{t}\,(b_t - \beta)$ has the same limiting distribution as $\sqrt{t}\,(\hat{\beta}_t - \beta)$.

The results of Theorem 2 can now be used to derive the asymptotic distribution of the control rules themselves in the following theorem.

THEOREM 3. *Under the assumptions of Theorem* 1, *if* $\{u_t\}$ *is defined as either* (i) *the least squares certainty equivalence control rule, or* (ii) *the Bayesian certainty equivalence control rule, then*

$$\sqrt{t}\left(u_t - \frac{a}{\beta}\right) \xrightarrow{d} N\left(0, \frac{\sigma^2}{\beta^2}\right). \tag{49}$$

PROOF. (i) The limiting distribution of $\sqrt{t}\,(u_t - a/\beta)$ in the least squares case follows from

$$\sqrt{t}\left(\frac{a}{\hat{\beta}_t} - \frac{a}{\beta}\right) = \frac{a}{\hat{\beta}_t\beta}\sqrt{t}\,(\beta - \hat{\beta}_t)\,. \tag{50}$$

From Corollary 1(i) $\hat{\beta}_t \xrightarrow{p} \beta$, so that the difference between the right hand side of equation (50) and $\sqrt{t}\,(\beta - \hat{\beta}_t)a/\beta^2$ converges to zero in probability. The first part of the Theorem then follows from

(51) $$\sqrt{t}\,(\beta - \hat{\beta}_t)a/\beta^2 \xrightarrow{d} N\left(0, \frac{\sigma^2}{\beta^2}\right),$$

which follows directly from Theorem 2(i).

(ii) Similarly in the case of Bayesian certainty equivalence control we have

(52) $$\sqrt{t}\left(\frac{a}{b_t} - \frac{a}{\beta}\right) = \frac{a}{b_t\beta}\sqrt{t}\,(b_t - \beta)\,,$$

and from Corollary 1(ii), $b_t \xrightarrow{p} \beta$, so that the difference between the right hand side of equation (52) and $\sqrt{t}\,(\beta - b_t)a/\beta^2$ converges to zero in probability. From Theorem 2(ii) we have

(53) $$\frac{\sqrt{t}\,(\beta - b_t)a}{\beta^2} \xrightarrow{d} N\left(0, \frac{\sigma^2}{\beta^2}\right)$$

which completes the proof of the Theorem.

## 5. ASYMPTOTIC EFFICIENCY OF CONTROL RULES

In this section we consider how the asymptotic normality results of Section 4 might be used as criteria for judging the effectiveness of control rules, as well as for suggesting whether there exist other control rules which might do better. Once a particular control rule has been decided upon, its behavior over time will depend on the data generated by the random disturbance term. The situation is similar to problems in the theory of estimation where the sampling distribution of an estimate is investigated. In that theory an estimate is considered good if its sampling distribution is concentrated in some sense about the true parameter being estimated. In problems where the exact sampling distribution is difficult or impossible to determine, one might be able to find the asymptotic distribution of the estimate and examine its asymptotic efficiency. Since such criteria have been useful in the theory of estimation, it seems likely that they would be useful in the theory of control with unknown parameters where estimation plays an important part.[7]

Because these results are asymptotic, they are more useful in control problems with a long time horizon and small discount rate. However, there are many control problems, such as stabilizing the rate of inflation, where there is no natural terminal data nor any reason to discount the future. In such problems these results would be especially useful, but in problems of short duration they should be used with caution.

The following theorem is a formal statement of how the control rules defined and studied in this paper are asymptotically efficient.

THEOREM 4. *Under the assumptions of Theorem* 1 *let* $\{u_t\}$ *be any control rule*

[7] For a more complete discussion on the usefulness of this criterion in the control problem see Taylor [7].

*which converges to* $a/\beta$ *with probability* 1. *Then the limiting distribution of* $\sqrt{t}(\hat{\beta}_t - \beta)$ *and of* $\sqrt{t}(b_t - \beta)$ *is* $N(0, (\beta^2/a^2)\sigma^2)$.

PROOF. In the proof of Theorem 2 the only property of the least squares certainty equivalence rule and the Bayesian certainty equivalence rule which we use is convergence to the true value $a/\beta$ with probability one. This is enough to show that the first term in brackets in equation (44) converges in probability to zero and that $\sum_{i=1}^{t} u_i^2/t \to a^2/\beta^2$. Since by assumption any control rule in the class defined in this theorem has this convergence property, we obtain the same results about the limiting distributions of $\sqrt{t}(\hat{\beta}_t - \beta)$ and $\sqrt{t}(b_t - \beta)$.

The importance of this theorem is that the least squares certainty equivalence control and the Bayesian certainty equivalence control lead to parameter estimates which have as small an asymptotic variance as any other control rule in the class of rules having the property of convergence to the true value with probability one. This class includes controls designed especially for experimentation as long as the control converges with probability one to $a/\beta$. The implication is that asymptotically there is nothing to gain by experimenting with controls to obtain more information about parameter estimates. In the long run as much information can be obtained by the more easily calculated control rules of this paper.

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