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ON AN EFFICIENT TWO-STEP ESTIMATOR FOR DYNAMIC SIMULTANEOUS EQUATIONS MODELS WITH AUTOREGRESSIVE ERRORS*

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1. INTRODUCTION

WE CONSIDER THE MODEL:

$$y_{t\cdot} = y_{t\cdot}B + y_{t-1\cdot}C_0 + w_{t\cdot1} + u_{t\cdot}, \qquad t = 1, 2, \cdots T \tag{1}$$

where $y_{t\cdot}$ is an m-element vector of the dependent variables of the system and $w_{t\cdot}$ is an s-element vector of *exogenous* variables; the error term obeys

$$u_{t\cdot} = u_{t-1\cdot}R + \varepsilon_{t\cdot} \tag{2}$$

where R is a stable matrix and $\{\varepsilon'_{t\cdot}: t = 0, \pm 1, \pm 2, \cdots\}$ is a sequence of independent identically distributed (i.i.d.) random variables having mean zero and nonsingular covariance matrix Σ.

It is assumed that whatever the process generating $\{w'_{t\cdot}: t = 0, \pm 1, \pm 2, \cdots\}$ the latter sequence is independent of $\{\varepsilon'_{t\cdot}: t = 0, \pm 1, \pm 2, \cdots\}$.

In Dhrymes [**1**] under the additional assumption of normality for the ε-process the full information maximum likelihood (ML) estimator wasobtai ned as well as the three-stage-least-squares-like estimator, termed there the full information dynamic autoregressive (FIDA). The converging iterate of the latter (CIFIDA) was compared with the ML estimator and it was determined that the difference between the two lies in the way in which the (jointly) dependent variables of the system are purged of their stochastic component.

In Dhrymes and Erlat [**3**] the asymptotic distribution of the converging iterate of FIDA was obtained.

The purpose of this paper is twofold: First, to show that the asymptotic distributions of the converging iterate of FIDA and the ML estimator are identical and second, to provide a simple two step procedure which is fully as efficient as CIFIDA and ML estimators. This is a natural extension of the results in Dhrymes [**2**] and Hatanaka [**5**].

2. EQUIVALENCE OF ML AND CIFIDA ESTIMATORS

Write the Equation (1) compactly as

$$ZA = U$$

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where

$$Z = (Y, Y_{-1}, W),\ W = (w_{tj}) \quad t = 2, 3, \cdots, T,\ j = 1, 2, \cdots, s$$

$$Y = (y_{ti}),\ Y_{-1} = (y_{t-1,i}),\ U = (u_{ti}) \qquad t = 2, 3, \cdots, T,\ i = 1, 2, \cdots, m$$

$$A = (I - B', -C_0', -C_1')' .$$

Making use of the zero restrictions, we can write the i-th equation as

$$y_{\cdot i} = Z_i\delta_{\cdot i} + u_{\cdot i} \tag{3}$$

where $y_{\cdot i}$, $u_{\cdot i}$ are the i-th columns, respectively, of Y and U

$$Z_i = (Y_i, {}_iY_{-1}, W_i),\ \ \delta_{\cdot i} = (\beta'_{\cdot i}, \gamma^{*\prime}_{\cdot i}, \gamma'_{\cdot i})'$$

Y_i, ${}_iY_{-1}$, W_i being submatrices of Y, Y_{-1}, W, respectively, corresponding to the included right hand variables and $\beta_{\cdot i}$, $\gamma^*_{\cdot i}$, $\gamma_{\cdot i}$ being the i-th columns of B, C_0, C_1 respectively after elements known to be zero have been suppressed. The system may be written also as

$$y = Z^*\delta + u$$

where

$$y = \text{Vec}\,(Y),\ \ u = \text{Vec}\,(U),\ \ Z^* = \text{diag}\,(Z_1, Z_2, \cdots, Z_m)$$

$$\delta = (\delta'_{\cdot 1}, \delta'_{\cdot 2}, \cdots \delta'_{\cdot m})'$$

the notation Vec (A) denoting a column vector whose i-th subvector is the i-th column of A.

Using the representation above, it is shown in [1] that the converging iterate of FIDA is given as the solution (in δ and R) of the following equation system

$$\begin{aligned} \tilde{M}\tilde{\delta} &= \tilde{P}'(\tilde{\Sigma}^{-1} \otimes I)[y - (\tilde{R}' \otimes I)y_{-1}] \\ R &= (\tilde{A}'Z'_{-1}Z_{-1}\tilde{A})^{-1}\tilde{A}'Z'_{-1}Z\tilde{A} \end{aligned} \tag{4}$$

where

$$\begin{aligned} &\tilde{P} = \tilde{Z}^* - (\tilde{R}' \otimes I)Z^*_{-1},\ \ \tilde{M} = \tilde{P}'(\tilde{\Sigma}^{-1} \otimes I)\tilde{P} \\ &\tilde{Z} = (\tilde{Y}, Y_{-1}, W),\ \ \tilde{Y} = Q\tilde{F},\ \ Q = (Y_{-1}, Y_{-2}, W, W_{-1}) \\ &\tilde{F} = (Q'Q)^{-1}Q'Y \\ &\tilde{Z}_i = (\tilde{Y}_i, {}_iY_{-1}, W_i),\ \ \tilde{Z}^* = \text{diag}\,(\tilde{Z}_1, \tilde{Z}_2, \cdots, \tilde{Z}_m) \end{aligned} \tag{5}$$

and $\tilde{\Sigma}$ is a prior consistent estimator of $\tilde{\Sigma}$.

The ML estimator is shown to be a solution of

$$\begin{aligned} \tilde{M}_*\tilde{\delta} &= [\tilde{P}'_*(\tilde{\Sigma}^{-1} \otimes I) - \tilde{V}^{*\prime}(S^{-1} \otimes I)][y - (\tilde{R}' \otimes I)y_{-1}] \\ \tilde{R} &= (\tilde{A}'Z'_{-1}Z_{-1}\tilde{A})^{-1}\tilde{A}'Z'_{-1}Z\tilde{A} \end{aligned} \tag{6}$$

$$\tilde{\Sigma} = \frac{1}{T}\tilde{E}'\tilde{E}, \quad \tilde{E} = Z\tilde{A} - Z_{-1}\tilde{A}\tilde{R}$$

where

$$(7) \quad \begin{aligned} \tilde{P}_* &= [Z^* - (\tilde{R}' \otimes I)Z^*_{-1}], \\ \tilde{M}_* &= \tilde{P}'_*(\tilde{\Sigma}^{-1} \otimes I)\tilde{P}_* - \tilde{V}^{*\prime}(S^{-1} \otimes I)\tilde{V}^* \\ S &= \left(\frac{1}{T}\right)(I - \tilde{B}')\tilde{V}'\tilde{V}(I - \tilde{B}), \; \tilde{V} = Y - Q\tilde{F} \\ \tilde{V}^*_i &= (\tilde{V}_i, 0, 0), \quad \tilde{V}^* = \text{diag}\,(\tilde{V}^*_1, \tilde{V}^*_2, \cdots, \tilde{V}^*_m) \end{aligned}$$

the zeros in the definition of $\tilde{V}^*_i$ corresponding, in dimension, to ${}_iY_{-1}$, W_i.

It is also shown in [**1**] that the asymptotic distribution of the ML estimator of δ and R does not depend on any properties of the asymptotic distribution of the estimator of Σ, so long as the latter is estimated consistently.

Thus, for the purposes of comparison we shall consider only the first two sets of Equations in (6).

On the assumption that, in both Equations (4) and (6), a solution can be found by iteration *beginning with initial consistent estimators for δ and R, the solution is a consistent estimator of δ and R, and moreover, after slight rearrangement, we can write for the solution vector of CIFIDA*

$$(8) \quad \left(\frac{1}{T}\right)\begin{bmatrix} \tilde{M} & \tilde{P}'(\tilde{\Sigma}^{-1} \otimes U_{-1}) \\ (\tilde{\Sigma}^{-1} \otimes \tilde{U}'_{-1})\tilde{P} & \tilde{\Sigma}^{-1} \otimes \tilde{U}'_{-1}\tilde{U}_{-1} \end{bmatrix} \sqrt{T}\begin{pmatrix} \tilde{\delta} - \delta \\ \tilde{r} - r \end{pmatrix} = \left(\frac{1}{\sqrt{T}}\right)\begin{bmatrix} \tilde{P}'(\tilde{\Sigma}^{-1} \otimes I) \\ \tilde{\Sigma}^{-1} \otimes \tilde{U}'_{-1} \end{bmatrix}\varepsilon$$

where

$$\tilde{U}_{-1} = Z_{-1}\tilde{A}, \; \varepsilon = \text{Vec}\,(E), \; E = (\varepsilon_{ti})$$
$$t = 2, \cdots, T \text{ and } i = 1, 2, \cdots m\,.$$

The equations defining the ML estimator, on the other hand, are

$$(9) \quad \left(\frac{1}{T}\right)\begin{bmatrix} \tilde{M}_* & (\tilde{P}^{*\prime}(\tilde{\Sigma}^{-1} \otimes I) - \tilde{V}^{*\prime}(S^{-1} \otimes I))(I \otimes U_{-1}) \\ (\tilde{\Sigma}^{-1} \otimes \tilde{U}'_{-1})N & \tilde{\Sigma}^{-1} \otimes \tilde{U}'_{-1}\tilde{U}_{-1} \end{bmatrix} \sqrt{T}\begin{pmatrix} \tilde{\delta} - \delta \\ \tilde{r} - r \end{pmatrix} = \frac{1}{\sqrt{T}}\begin{bmatrix} \tilde{P}'_*(\tilde{\Sigma}^{-1} \otimes I) - \tilde{V}^{*\prime}(S^{-1} \otimes I) \\ \tilde{\Sigma}^{-1} \otimes U'_{-1} \end{bmatrix}\varepsilon$$

where

$$N = [Z^* - (R' \otimes I)Z^*_{-1}]\,.$$

To show the equivalence of the two estimators, we proceed somewhat formally giving in the form of Lemmata a number of preliminary required results. We begin by showing that the matrices in the left member of Equations (8) and (9) have identical probability limits. Thus, LEMMA 1. *Under the standard assumptions*

$$\operatorname*{plim}_{T\to\infty} S = \Sigma .$$

PROOF. By definition

$$\hat{V} = Y - Q\hat{F} = [I - Q(Q'Q)^{-1}Q']E(I - B)^{-1} .$$

Thus

$$\left(\frac{1}{T}\right)\hat{V}'\hat{V} = \left(\frac{1}{T}\right)(I - B')^{-1}E'[I - Q(Q'Q)^{-1}Q']E(I - B)^{-1} .$$

Since

$$\operatorname*{plim}_{T\to\infty}\left(\frac{1}{T}\right)E'Q(Q'Q)^{-1}Q'E = 0, \quad \operatorname*{plim}_{T\to\infty}\left(\frac{1}{T}\right)E'E = \Sigma$$

we have, by the consistency of $\tilde{B}$

$$\operatorname*{plim}_{T\to\infty} S = \Sigma . \qquad \text{Q.E.D.}$$

LEMMA 2.

$$\operatorname*{plim}_{T\to\infty}\left(\frac{1}{T}\right)\hat{P}'(\tilde{\Sigma}^{-1} \otimes U_{-1}) = P_1'(\Sigma^{-1} \otimes I)$$

where

$$P_1 = \operatorname*{plim}_{T\to\infty}\left(\frac{1}{T}\right)\bar{P}'(I \otimes U_{-1}), \quad \bar{P} = \bar{Z}^* - (R' \otimes I)Z^*_{-1}$$

$$\bar{Z}_i = (QF_{i}, {}_iY_{-1}, W_i), \quad \bar{Z}^* = \operatorname{diag}(\bar{Z}_1, \cdots, \bar{Z}_m) ,$$

$$F = \operatorname*{plim}_{T\to\infty} \hat{F} .$$

PROOF. Obvious.

LEMMA 3.

$$\operatorname*{plim}_{T\to\infty}\left(\frac{1}{T}\right)[\hat{P}_*'(\tilde{\Sigma}^{-1} \otimes I) - \hat{V}^{*\prime}(S^{-1} \otimes I)][I \otimes U_{-1}] = P_1'(\Sigma^{-1} \otimes I) .$$

PROOF. By definition

$$\hat{P}_* = Z^* - (\tilde{R}' \otimes I)Z_{-1} = \bar{Z}^* - (\tilde{R}' \otimes I)Z_{-1} + V^* .$$

We also note that since

$$V = E(I - B)^{-1}, \quad \hat{V} = [I - Q(Q'Q)^{-1}Q']V$$

and

$$\operatorname*{plim}_{T\to\infty}\left(\frac{1}{T}\right)V'U_{-1} = 0$$

we have

$$\operatorname*{plim}_{T\to\infty}\left(\frac{1}{T}\right)\tilde{V}^{*\prime}(S^{-1}\otimes I)(I\otimes U_{-1}) = 0$$

and consequently

$$\operatorname*{plim}_{T\to\infty}\left(\frac{1}{T}\right)[\tilde{P}'_{*}(\tilde{\Sigma}^{-1}\otimes I) - \tilde{V}^{*\prime}(S^{-1}\otimes I)](I\otimes U_{-1})$$
$$= \operatorname*{plim}_{T\to\infty}\left(\frac{1}{T}\right)\tilde{P}'(\Sigma^{-1}\otimes I)(I\otimes U_{-1}) = P_1'(\Sigma^{-1}\otimes I)\,. \quad \text{Q.E.D.}$$

LEMMA 4.

$$\operatorname*{plim}_{T\to\infty}\left(\frac{1}{T}\right)(\tilde{\Sigma}^{-1}\otimes \tilde{U}'_{-1})\tilde{P} = \operatorname*{plim}_{T\to\infty}\left(\frac{1}{T}\right)(\tilde{\Sigma}^{-1}\otimes \tilde{U}'_{-1})\tilde{P}_{*} = (\Sigma^{-1}\otimes I)P_1\,.$$

PROOF. Obvious from Lemma 3.

Finally, letting

$$\Omega = \operatorname*{plim}_{T\to\infty}\frac{1}{T}\tilde{U}'_{-1}\tilde{U}_{-1}$$

and

$$M = \operatorname*{plim}_{T\to\infty}\frac{\tilde{M}}{T}$$

we have

LEMMA 5. *The matrices in the left members of Equations (8) and (9) have the same probability limit, which is given by*

$$C = \begin{bmatrix} M & P_1'(\Sigma^{-1}\otimes I) \\ (\Sigma^{-1}\otimes I)P_1 & \Sigma^{-1}\otimes\Omega \end{bmatrix}. \tag{11}$$

PROOF. Obvious from Lemmata 2, 3 and 4.

Thus, to establish the identity of the asymptotic distribution of the two estimators it will be sufficient to establish that the right members converge in distribution to the same limit. To this effect note that since

$$\tilde{\Sigma}^{-1}\otimes\tilde{U}'_{-1} = \tilde{\Sigma}^{-1}\otimes\tilde{A}'Z'_{-1} = (\tilde{\Sigma}^{-1}\otimes\tilde{A}')(I\otimes Z'_{-1})$$

we have that

$$\frac{1}{\sqrt{T}}[(\tilde{\Sigma}^{-1}-\Sigma)\otimes(\tilde{A}'-A')](I\otimes Z_1')\varepsilon$$
$$= \sqrt{T}(\tilde{\Sigma}-\Sigma)\otimes\sqrt{T}(\tilde{A}'-A')\left(\frac{1}{T}\right)^{3/2}(I\otimes Z'_{-1})\varepsilon$$

which converges in distribution to the degenerate random variable zero. Consequently in both (8) and (9) we can replace

$$(\tilde{\Sigma}^{-1} \otimes \tilde{U}'_{-1})$$

by

$$(\Sigma^{-1} \otimes A')(I \otimes Z_{-1}) .$$

Thus, the identity of the two asymptotic distributions depends only on the comparison between

$$\left(\frac{1}{\sqrt{T}}\right)\tilde{P}'(\tilde{\Sigma}^{-1} \otimes I)\varepsilon$$

and

$$\frac{1}{\sqrt{T}}[\tilde{P}'_*(\tilde{\Sigma}^{-1} \otimes I) - \tilde{V}^{*\prime}(S^{-1} \otimes I)]\varepsilon .$$

Consider first the quantity

$$\left(\frac{1}{\sqrt{T}}\right)[\tilde{P}'_*(\tilde{\Sigma}^{-1} \otimes I) - \tilde{V}^{*\prime}(\tilde{\Sigma}^{-1} \otimes I)]\varepsilon$$

and define

$$\left(\frac{1}{\sqrt{T}}\right)[\tilde{P}'_*(\tilde{\Sigma}^{-1}\otimes I) - \tilde{V}^{*\prime}(S^{-1}\otimes I) - \tilde{P}'_*(\tilde{\Sigma}^{-1}\otimes I) + \tilde{V}^{*\prime}(\tilde{\Sigma}^{-1}\otimes I)]\varepsilon$$
$$= \left(\frac{1}{\sqrt{T}}\right)\tilde{V}^{*\prime}[(\tilde{\Sigma}^{-1} - S^{-1}) \otimes I]\varepsilon .$$

If we show that

$$\sqrt{T}\,(\tilde{\Sigma}^{-1} - S^{-1})$$

converges in distribution to the degenerate random variable zero, we immediately conclude that

$$\left(\frac{1}{\sqrt{T}}\right)\tilde{V}^{*\prime}[(\tilde{\Sigma}^{-1} - S^{-1}) \otimes I]\varepsilon$$

similarly converges to the degenerate random variable zero. This is so since

$$\operatorname*{plim}_{T\to\infty}\left(\frac{1}{T}\right)\tilde{V}'E = (I - B')^{-1}\Sigma .$$

We have

LEMMA 6.

$$\sqrt{T}\,(\tilde{\Sigma}^{-1} - S^{-1})$$

is asymptotically a degenerate random variable.

PROOF.

$$\tilde{\Sigma}^{-1} - S^{-1} = \tilde{\Sigma}^{-1}(S - \tilde{\Sigma})S^{-1}$$

and thus

$$\sqrt{T}(\tilde{\Sigma}^{-1} - S^{-1}) \sim \Sigma^{-1}[\sqrt{T}(S - \tilde{\Sigma})]\Sigma^{-1}.$$

Now

$$\tilde{\Sigma} = \frac{1}{T}\tilde{E}'\tilde{E} = (I - \tilde{B}')(I - \tilde{B}')^{-1}\frac{\tilde{E}'\tilde{E}}{T}(I - \tilde{B})^{-1}(I - \tilde{B})$$

so

$$\sqrt{T}(S - \tilde{\Sigma}) = (I - \tilde{B}')\sqrt{T}\left[\frac{\tilde{V}'\tilde{V}}{T} - (I - \tilde{B}')^{-1}\frac{\tilde{E}'\tilde{E}}{T}(I - \tilde{B})^{-1}\right](I - \tilde{B}).$$

We observe that

$$\tilde{V} = Y - Q\tilde{F}$$

where $\tilde{F}$ is the unrestricted estimator of F. Similarly, we observe that

$$\tilde{E}(I - \tilde{B})^{-1} = Y - Q\hat{F}$$

where $\hat{F}$ is the restricted estimator of F derived from the ML estimator of the elements of A. Thus

$$\sqrt{T}(S - \tilde{\Sigma}) = (I - \tilde{B}')\Big[\sqrt{T}(\hat{F} - \tilde{F})'\frac{Q'Y}{T} + \frac{Y'Q}{T}\sqrt{T}(\hat{F} - \tilde{F})$$
$$- \sqrt{T}(\hat{F} - \tilde{F})\frac{Q'Q}{T}\hat{F} - \tilde{F}'\frac{Q'Q}{T}\sqrt{T}(\hat{F} - \tilde{F})\Big](I - \tilde{B}).$$

Provided $\sqrt{T}(\hat{F} - \tilde{F})$ has a well defined limiting distribution

$$\sqrt{T}(S - \tilde{\Sigma}) \sim (I - B)'[\sqrt{T}(\hat{F} - \tilde{F})'M_{QQ}F + F'M_{QQ}\sqrt{T}(\hat{F} - \tilde{F})$$
$$- \sqrt{T}(\hat{F} - \tilde{F})M_{QQ}F - FM_{QQ}\sqrt{T}(\hat{F} - \tilde{F})](I - B) = 0$$

where

$$M_{QQ} = \operatorname*{plim}_{T\to\infty}\frac{Q'Q}{T}.$$

Thus $\sqrt{T}(S - \tilde{\Sigma})$ converges in distribution to the zero random variable.

Q.E.D.

COROLLARY.

$$\frac{1}{\sqrt{T}}[\tilde{P}'_*(\tilde{\Sigma}^{-1} \otimes I) - \tilde{V}^{*\prime}(S^{-1} \otimes I)]\varepsilon$$

and

$$\frac{1}{\sqrt{T}}[\tilde{P}'_*(\tilde{\Sigma}^{-1}\otimes I) - \tilde{V}^{*\prime}(\tilde{\Sigma}^{-1}\otimes I)]\varepsilon$$

have the same limiting distribution.

PROOF. Obvious.

We now observe that

$$\tilde{P}'_*(\tilde{\Sigma}^{-1}\otimes I) - \tilde{V}^{*\prime}(\tilde{\Sigma}^{-1}\otimes I) = \tilde{P}(\tilde{\Sigma}^{-1}\otimes I)$$

where, *mutatis mutandis*, $\tilde{P}$ is the same matrix as that appearing in the CIFIDA estimator. We now have the following obvious results applying to both the CIFIDA and ML estimators.

LEMMA 7. Asymptotically,

$$\frac{1}{\sqrt{T}}\tilde{P}'(\tilde{\Sigma}^{-1}\otimes I)\varepsilon \sim \frac{1}{\sqrt{T}}\tilde{P}'(\Sigma^{-1}\otimes I)\varepsilon .$$

PROOF.

$$\frac{1}{\sqrt{T}}\tilde{P}[(\tilde{\Sigma}^{-1} - \Sigma^{-1})\otimes I]\varepsilon = \frac{1}{T}\tilde{P}'[\sqrt{T}(\tilde{\Sigma}^{-1} - \Sigma^{-1})\otimes I]\varepsilon$$

which obviously converges in distribution to the degenerate random variable zero.

LEMMA 8. *Asymptotically,*

$$\frac{1}{\sqrt{T}}\tilde{P}'(\Sigma^{-1}\otimes I)\varepsilon \sim \frac{1}{\sqrt{T}}\bar{P}'(\Sigma^{-1}\otimes I)\varepsilon .$$

PROOF.

$$\tilde{P} - \bar{P} = \tilde{Z}^* - (\tilde{R}'\otimes I)Z_{-1} - \bar{Z}^* + (R'\otimes I)Z_{-1} .$$

But we note

$$\tilde{Z} - \bar{Z} = (\tilde{Y} - \bar{Y}, 0, 0) = (-Q(\tilde{F} - F), 0, 0) .$$

The desired result is then obtained if we note that $\sqrt{T}(\tilde{F} - F)$ has a well behaved limiting distribution and moreover that

$$\operatorname*{plim}_{T\to\infty}\frac{Q'E}{T} = 0, \quad \operatorname*{plim}_{T\to\infty}\frac{Z'_{-1}E}{T} = 0 .$$

Using the results in Dhrymes and Erlat [3] we observe

LEMMA 9. *Asymptotically*

$$\frac{1}{\sqrt{T}}\begin{bmatrix}\bar{P}'(\Sigma^{-1}\otimes I)\\ \Sigma^{-1}\otimes A'Z'_{-1}\end{bmatrix}\varepsilon \sim N(0, C) .$$

PROOF. See [3].

We have therefore proved

THEOREM 1. *Consider the model in (1) and (2) subject to the following conditions:*

(i) *$I - B$ is nonsingular.*
(ii) *R, $C_0(I - B)^{-1}$ are both stable.*
(iii) $\operatorname{plim}_{T\to\infty} (Q'Q/T)$ *exists and is nonsingular.*
(iv) *The sequence $\{w'_{t\cdot}: t = 0, \pm 1, \pm 2, \cdots\}$ is independent of the error process $\{\varepsilon'_{t\cdot}: t = 0, \pm 1, \pm 2, \cdots\}$.*
(v) *The process $\{\varepsilon'_{t\cdot}: t = 0, \pm 1, \pm 2, \cdots\}$ is one of independent i.i.d. (normal) random vectors with zero mean and nonsingular covariance matrix Σ.*
(vi) *The ML estimators $\sqrt{T}(\tilde{\Sigma} - \Sigma)$, $\sqrt{T}(\tilde{A} - A)$ have well behaved limiting distinctions.*

Then, the converging iterate of the full information dynamic autoregressive (CIFIDA) and ML estimators of δ have the same asymptotic distribution which is

$$N(0, C^{-1})$$

where

$$C = \begin{bmatrix} M & P_1'(\Sigma^{-1} \otimes I) \\ (\Sigma^{-1} \otimes I)P_1 & \Sigma^{-1} \otimes \Omega \end{bmatrix}.$$

PROOF. Lemmata 5, 6, 7, 8, 9.

3. A SIMPLIFIED TWO STEP ESTIMATOR

In demonstrating the asymptotic equivalence of a simplified estimator to the CIFIDA and ML estimators the following lemma is quite useful:

LEMMA 10. *Let θ^* be a consistent estimator of a parameter vector θ_0 obtained by minimizing some function, say $S(\theta)$, and suppose that $\sqrt{T}(\theta^* - \theta_0)$ has a limiting distribution. Let $\tilde{\theta}$ be a consistent estimator of θ_0 such that $\sqrt{T}(\tilde{\theta} - \theta_0)$ has a limiting distribution and let $\Gamma(\tilde{\theta})$ be a matrix such that*

$$\Gamma(\tilde{\theta}) = \frac{\partial^2 S}{\partial\theta\partial\theta}(\tilde{\theta}) + o_p(T)$$

where T is the sample size. Define the estimator $\hat{\theta}$ by

$$\hat{\theta} = \tilde{\theta} - \Gamma^{-1}(\tilde{\theta})\frac{\partial S}{\partial\theta}(\tilde{\theta}). \tag{12}$$

Then $\hat{\theta}$ has the same asymptotic distribution as θ^*.

PROOF. By the mean value theorem

$$\frac{\partial S}{\partial \theta}(\tilde{\theta}) = \frac{\partial S}{\partial \theta}(\theta_0) + \frac{\partial^2 S}{\partial \theta \partial \theta}(\bar{\theta})(\tilde{\theta} - \theta_0) \tag{13}$$

where θ_0 is the true parameter vector and $\bar{\theta}$ obeys

$$|\bar{\theta} - \theta_0| < |\tilde{\theta} - \theta_0| .$$

Substituting (13) into (12), we have

$$\begin{aligned} \sqrt{T}(\hat{\theta} - \theta_0) = \Big[I - \Big(\frac{\Gamma(\tilde{\theta})}{T}\Big)^{-1}\Big(\frac{1}{T}\frac{\partial^2 S}{\partial \theta \partial \theta}(\bar{\theta})\Big)\Big]\sqrt{T}(\tilde{\theta} - \theta_0) \\ - \Big(\frac{\Gamma(\tilde{\theta})}{T}\Big)^{-1}\frac{1}{\sqrt{T}}\frac{\partial S}{\partial \theta}(\theta_0) . \end{aligned} \tag{14}$$

Under the assumptions of the lemma the first term above converges (in distribution) to the degenerate random variable zero. Since it can be shown that, asymptotically,

$$\sqrt{T}(\theta^* - \theta_0) \sim -\Big(\frac{\Gamma(\theta_0)}{T}\Big)^{-1}\frac{\partial S}{\partial \theta}(\theta_0)$$

we immediately conclude

$$\sqrt{T}(\theta^* - \theta_0) \sim \sqrt{T}(\hat{\theta} - \theta_0) . \qquad \text{Q.E.D.}$$

To obtain the simplified estimator using Lemma 10, recall that the CIFIDA estimator is obtained by minimizing

$$S(\theta) = tr\tilde{\Sigma}^{-1}(\tilde{Z}A - Z_{-1}AR)'(\tilde{Z}A - Z_{-1}AR) \tag{15}$$

with

$$\theta = (\delta', r')' .$$

A matrix satisfying the properties of Γ in Lemma 10, has been determined in the Appendix—Equation (A. 9). Moreover, we observe that

$$\begin{aligned} \frac{\partial S}{\partial \delta}(\tilde{\theta}) &= -2\tilde{P}'(\tilde{\Sigma}^{-1} \otimes I)[y - (\tilde{R}' \otimes I)y_{-1} - \tilde{P}\tilde{\delta}] \\ \frac{\partial S}{\partial r}(\tilde{\theta}) &= -2(I \otimes \tilde{U}'_{-1})(\tilde{\Sigma}^{-1} \otimes I)[y - (\tilde{R}' \otimes I)y_{-1} - \tilde{P}\tilde{\delta}] . \end{aligned} \tag{16}$$

In view of (16) we can write (12)—after minor rearrangement, as

$$\begin{aligned} &\tilde{P}'(\tilde{\Sigma}^{-1} \otimes I)\tilde{P}(\hat{\delta} - \tilde{\delta}) + \tilde{P}'(\tilde{\Sigma}^{-1} \otimes I)(I \otimes \tilde{U}_{-1})(\hat{r} - \tilde{r}) \\ &\qquad = \tilde{P}'(\tilde{\Sigma}^{-1} \otimes I)[y - (\tilde{R}' \otimes I)y_{-1} - \tilde{P}\tilde{\delta}] \\ &(I \otimes \tilde{U}_{-1})(\tilde{\Sigma}^{-1} \otimes I)\tilde{P}(\hat{\delta} - \tilde{\delta}) + (I \otimes \tilde{U}'_{-1})(\tilde{\Sigma}^{-1} \otimes I)(I \otimes \tilde{U}_{-1})(\hat{r} - \tilde{r}) \\ &\qquad = (I \otimes \tilde{U}'_{-1})(\tilde{\Sigma}^{-1} \otimes I)[y - (\tilde{R}' \otimes I)y_{-1} - \tilde{P}\tilde{\delta}] \end{aligned}$$

or, upon cancellation of terms involving $\tilde{P}\tilde{\delta}$ on both sides, as

$$
(17)\quad \begin{bmatrix} \tilde{P}'(\tilde{\Sigma}^{-1}\otimes I)\tilde{P} & \tilde{P}'(\tilde{\Sigma}^{-1}\otimes I)(I\otimes \tilde{U}_{-1}) \\ (I\otimes \tilde{U}'_{-1})(\tilde{\Sigma}^{-1}\otimes I)\tilde{P} & (I\otimes \tilde{U}'_{-1})(\tilde{\Sigma}^{-1}\otimes I)(I\otimes \tilde{U}_{-1}) \end{bmatrix} \begin{bmatrix} \hat{\delta} \\ \hat{r} - \tilde{r} \end{bmatrix} = \begin{pmatrix} \tilde{P}' \\ I\otimes \tilde{U}'_{-1} \end{pmatrix} (\tilde{\Sigma}^{-1}\otimes I)\cdot[y - (\tilde{R}\otimes I)y_{-1}].
$$

We have therefore proved

THEOREM 2. *Consider the model stated in Theorem 1; an estimator which is asymptotically equivalent to the ML and the converging iterate of FIDA can be obtained as follows:*

(i) *form a consistent estimator, say by instrumental variables, of the unknown elements of A. Denote this by $\tilde{A}$.*

(ii) *Compute the residual matrix*

$$\tilde{U} = Z\tilde{A}.$$

(iii) *Obtain the estimator*

$$\tilde{R} = (\tilde{A}'Z'_{-1}Z_{-1}\tilde{A})^{-1}\tilde{A}Z'_{-1}Z\tilde{A}.$$

(iv) *Compute*

$$\tilde{E} = \tilde{U} - \tilde{U}_{-1}\tilde{R}.$$

(v) *Estimate*

$$\tilde{\Sigma} = \left(\frac{1}{T}\right)\tilde{E}'\tilde{E}.$$

(vi) *Form the quantities*

$$\tilde{Z}^* - (\tilde{R}'\otimes I)Z^*_{-1},\quad I\otimes \tilde{U}_{-1},\quad y - (\tilde{R}\otimes I)y_{-1}.$$

(vii) *Obtain the feasible Aitken type estimator by regressing $y - (\tilde{R}'\otimes I)y_{-1}$ on $\tilde{Z}^* - (\tilde{R}'\otimes I)Z^*_{-1}$, $(I\otimes \tilde{U}_{-1})$ using $(\tilde{\Sigma}^{-1}\otimes I)$ as the estimated covariance matrix.*

(viii) *Add to the solution vector* $(0, \tilde{r}')'$.

Remark 1. The theorem above outlines an estimating procedure which involves estimation of the structural parameters by instrumental variables, computation of the residuals and the elements of the autocorrelation matrix R, followed by transformation of the data and another regression. No iteration is involved, although as a practical matter one might wish to iterate at least once in order to reduce the dependence of the procedure on the initial choice of instruments—which is rather arbitrary.

Remark 2. Even though $\tilde{Z}$ as it appears in (iv) of Theorem 2 was earlier defined with the $\tilde{Y}$ component given by $\tilde{Y} = Q\tilde{F}$, $\tilde{F} = (Q'Q)^{-1}Q'Y$, it is clear that we can define $\tilde{Y}$ by $Q\hat{F}$ where $\hat{F}$ is the *restricted reduced form* obtained from $\tilde{A}$ and $\tilde{R}$ as given in (i) and (iii) respectively. *Thus, we only need an initial instrumental variables estimator and a feasible Aitken procedure to obtain this two-step* *estimator.*

Remark 3. The limited information analogue may be obtained by setting $\tilde{\Sigma} = I$ and then minimizing

$$tr(\tilde{Z}A - Z_{-1}AR)'(\tilde{Z}A - Z_{-1}AR)$$

with respect to the unknown elements of A and R. The analogue in the two-step procedure will occur if in (17) we set $\tilde{\Sigma} = I$. Needless to say unless R is a *diagonal* matrix this is *not* a single equation procedure.

Remark 4. Outside the almost tautological condition that the matrix M be invertible, it is not clear what corresponds precisely to the rank and order condition (in the standard model) in setting forth the identifiability characteristics of the model.

Remark 5. The instrumental variables procedure given in Fair [**4**] is not an efficient one unless it is iterated to convergence. What Fair proposes, using the notation of this paper, is to write the model as

$$y - (R' \otimes I)y_{-1} = [Z^* - (R' \otimes I)Z^*_{-1}]\delta + \varepsilon$$

and obtain instrumental variables estimators using the instrumental matrix

$$\tilde{P}'(\tilde{\Sigma}^{-1} \otimes I)$$

substituting for R a prior consistent estimate thereof. It can be shown that the resulting estimator will depend on the asymptotic distribution properties of the particular estimate of R and will, in general, be inefficient unless the procedure is iterated with a new estimate of R until convergence.

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APPENDIX

In the discussion of the paper we have used a number of results involving matrix differentiation. For completeness we give a summary derivation here.

In particular we have to obtain the matrix of partial derivatives of

$$S = tr\tilde{\Sigma}^{-1}(\tilde{Z}A - Z_{-1}AR)'(\tilde{Z}A - Z_{-1}AR)$$

with respect to the unknown elements of A and R. No restrictions are assumed to be placed on R and certain zero restrictions are known to hold with respect to the elements of A. We first observe that, using the restrictions on A we can write, in the notation of the text,

$$\text{(A.1)} \qquad S = [y - (R' \otimes I)y_{-1} - (\tilde{Z}^* - (R' \otimes I)Z^*_{-1})\delta]'(\tilde{\Sigma}^{-1} \otimes I) \times [y - (R' \otimes I)y_{-1} - (\tilde{Z}^* - (R' \otimes I)Z^*_{-1})\delta].$$

From this we easily establish

(A.2) $$\frac{\partial S}{\partial \delta} = -2(\tilde{Z}^* - (R' \otimes I)Z^*_{-1})'(\tilde{\Sigma}^{-1} \otimes I) \times [y - (R' \otimes I)y_{-1} - (\tilde{Z}^* - (R' \otimes I)Z^*_{-1})\delta]$$

and thus

(A.3) $$\frac{\partial^2 S}{\partial \delta \partial \delta} = [2\tilde{Z}^* - (R' \otimes I)Z^*_{-1}]'(\tilde{\Sigma}^{-1} \otimes I)[\tilde{Z}^* - (R' \otimes I)Z^*_{-1}] .$$

To establish the other derivatives it is convenient to proceed as follows. We first observe that

(A.4) $$\frac{\partial S}{\partial R} = -2[A'Z_{-1}\tilde{Z}A - A'Z'_{-1}Z_{-1}AR]\tilde{\Sigma}^{-1}$$

and as a matter of notation we define

$$\frac{\partial S}{\partial r} = \mathrm{Vec}\left(\frac{\partial S}{\partial R}\right).$$

where for any matrix D, Vec (D) denotes the vector whose i-th subvector is the i-th column of D. Thus,

$$\frac{\partial^2 S}{\partial r \partial \delta} = \frac{\partial \,\mathrm{Vec}\,(\partial S/\partial R)}{\partial \,\mathrm{Vec}\,(A)} .$$

We note, for compatible matrices B_1, B_2, B_3, that

$$\mathrm{Vec}\,(B_1B_2B_3) = (I \otimes B_1)(I \otimes B_2)b_3$$

where

$$b_3 = \mathrm{Vec}\,(B_3) .$$

Moreover,

$$(I \otimes B_2)b_3 = (B_3' \otimes I)b_2 .$$

Using these results we easily see

(A.5) $$\mathrm{Vec}\,(A'Z'_{-1}\tilde{Z}A\tilde{\Sigma}^{-1}) = (\tilde{\Sigma}^{-1} \otimes I)(I \otimes A'Z'_{-1}\tilde{Z})a$$

(A.6) $$\mathrm{Vec}\,(A'Z'_{-1}Z_{-1}AR\tilde{\Sigma}^{-1}) = (\tilde{\Sigma}^{-1} \otimes I)(R' \otimes I)(I \otimes A'Z'_{-1}Z_{-1})a .$$

To differentiate (A.5) w.r.t. a, we note that one component can be obtained immediately as

$$(\tilde{\Sigma}^{-1} \otimes I)(I \otimes A'Z'_{-1}\tilde{Z}) .$$

The other component is obtained by noting that the i-th subvector is given by

$$A'Z'_{-1}\tilde{Z}\sum_{j=1}^{m} \sigma^{ji}a_{\cdot j} = A'Z'_{-1}\tilde{Z}A\tilde{\sigma}^{\cdot i}$$

which yields upon differentiation with respect to a *as it enters the component*

$A'Z'_{-1}\tilde{Z}$,

$$(I \otimes \tilde{\sigma}^{i\cdot})(I \otimes A'\tilde{Z}'Z_{-1}), \qquad i = 1, 2, \cdots m$$

where $\tilde{\sigma}^{i\cdot}$ is the i-th row of $\tilde{\Sigma}^{-1}$. Differentiating (A.6) similarly we obtain

$$(\tilde{\Sigma}^{-1} \otimes I)(R' \otimes I)(I \otimes A'Z'_{-1}Z_{-1})$$

and

$$(I \otimes \tilde{\sigma}^{i\cdot})(I \otimes R'A'Z'_{-1}Z_{-1}), \qquad i = 1, 2, \cdots m.$$

Writing

$$\tilde{\Sigma}_i^* = (I \otimes \tilde{\sigma}^{i\cdot}), \quad \tilde{\Sigma}^* = (\tilde{\Sigma}_1^{*\prime}, \tilde{\Sigma}_2^{*\prime}, \cdots \tilde{\Sigma}_m^{*\prime})'$$

we can express the derivative compactly as

$$\frac{\partial \operatorname{Vec}(\partial S/\partial R)}{\partial a} = -2(\tilde{\Sigma}^{-1} \otimes I)(I \otimes A'Z'_{-1})[(I \otimes \tilde{Z}) - (R' \otimes Z_{-1})] - 2\tilde{\Sigma}^*(I \otimes (\tilde{Z}A - Z_{-1}AR)'Z_{-1}).$$

Noting that upon division by T the second member above has a null probability limit and imposing the condition that some elements of A are null, we can write equivalently

$$\text{(A.7)} \qquad \frac{\partial^2 S}{\partial r \partial \delta} = -2(\tilde{\Sigma}^{-1} \otimes I)(I \otimes U'_{-1})(\tilde{Z}^* - (R' \otimes I)Z^*_{-1}) + o_p^{(T)}.$$

Finally, to find $(\partial^2 S)/(\partial r \partial r)$ we note that

$$\text{(A.8)} \qquad \operatorname{Vec}(A'Z'_{-1}Z_{-1}AR\tilde{\Sigma}^{-1}) = (\tilde{\Sigma}^{-1} \otimes I)(I \otimes A'Z'_{-1}Z_{-1})r$$

and we conclude immediately

$$\frac{\partial^2 S}{\partial r \partial r} = 2(\tilde{\Sigma}^{-1} \otimes I)(I \otimes A'Z'_{-1}Z_{-1}A).$$

Thus, the Hessian of (A.1), setting to zero all elements which converge to zero in probability when divided by T, can be written

$$\text{(A.9)} \qquad \begin{bmatrix} \dfrac{\partial^2 S}{\partial \delta \partial \delta} & \dfrac{\partial^2 S}{\partial \delta \partial r} \\ \dfrac{\partial^2 S}{\partial r \partial \delta} & \dfrac{\partial^2 S}{\partial r \partial r} \end{bmatrix} = 2 \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where

$$A_{11} = [\tilde{Z}^* - (R' \otimes I)Z^*_{-1}]'(\tilde{\Sigma}^{-1} \otimes I)[\tilde{Z}^* - (R' \otimes I)Z^*_{-1}]$$
$$A_{12} = [\tilde{Z}^* - (R' \otimes I)Z^*_{-1}]'(\tilde{\Sigma}^{-1} \otimes I)(I \otimes Z_{-1}A)$$
$$A_{21} = (I \otimes A'Z'_{-1})(\tilde{\Sigma}^{-1} \otimes I)[\tilde{Z}^* - (R' \otimes I)Z^*_{-1}]$$
$$A_{22} = (I \otimes A'Z'_{-1})(\tilde{\Sigma}^{-1} \otimes I)(I \otimes Z_{-1}A).$$

A more illuminating representation of the right member of the Hessian is

$$[\tilde{Z}^* - (R'\otimes I)Z^*_{-1}, I\otimes Z_{-1}A]'(\tilde{\Sigma}^{-1}\otimes I)[\tilde{Z}^* - (R'\otimes I)Z^*_{-1}, I\otimes Z_{-1}A].$$

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