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CONDITIONS FOR UNIQUE SOLUTIONS IN STOCHASTIC MACROECONOMIC MODELS WITH RATIONAL EXPECTATIONS

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This paper examines conditions for the uniqueness of an equilibrium price distribution in stochastic macroeconomic models with rational expectations. A model is developed in which many price distributions, each with a finite variance, satisfy the equilibrium requirements of rationality. Hence, the condition that the variance of the equilibrium price distribution be finite, or equivalently, that the conditionally expected price path be stable, does not guarantee uniqueness. In such cases it is shown that an arbitrary random quantity which is widely publicized can become a leading indicator of prices and, consequently, influence the behavior of actual prices. However, by extending the finite variance (stability) condition to a minimum variance condition, these nonuniqueness problems can be avoided. Such stability or minimum variance conditions suggest a kind of collective rationality which, although not unreasonable, has not yet been fully analyzed in rational expectations models.

1. INTRODUCTION

THE RECENT RESEARCH on rational expectations in dynamic stochastic macroeconomic models (see, for example, [**1**, **7**, **9**, and **11**]) has brought focus to a technical problem caused by the self-fulfilling nature of these expectations. In calculating the stochastic price equilibrium in such models one frequently finds that many price distributions satisfy the condition of rationality, and that in order to select one of these as the unique equilibrium further conditions must be imposed. One such condition which has frequently been successfully utilized is the stability of the conditionally expected price path generated by the equilibrium distribution.[2] In stochastic models where the unconditional mean price is well determined and finite this is equivalent to requiring that the equilibrium price distribution have a finite variance. But in some macroeconomic models such stability conditions may not be strong enough to insure uniqueness. Many price distributions, each with a finite variance, may satisfy the self-fulfilling property of rational expectations.

The central purpose of this paper is to present a simple stochastic macroeconomic model in which this stability (finite variance) condition does not result in uniqueness, and to explore some of the properties of this model. One important property is that the variance of the equilibrium price distribution can be permanently altered through random events unrelated to the system. For example, a widely publicized leading indicator of future prices, which is generated in a purely random fashion, can increase the variance of the equilibrium price distribution. If

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[2] Sargent and Wallace [**11**] justify this condition as ruling out speculative bubbles; Lucas [**7**] refers to the work of Brock [**3**] as formal justification, and Phelps and Taylor [**9**] argue that the implied unboundedness of the money rate of interest leads to internal inconsistencies which should be avoided.

everyone expects that everyone else expects that the indicator is leading, then it is impossible to statistically verify that the indicator is a will-o'-the-wisp. It is then individually rational to use the randomly generated indicator to forecast future prices. Previously reported examples giving rise to multiple solutions in stochastic models (after the stability condition is imposed) are due to particular policy formulations. In a recent paper Black [2] gives an example of a money supply policy which generates multiple price distributions.[3] For the analysis of this paper a model is presented in which nonuniqueness is due to the presence of real money balances in the production function. However, the analysis does not depend on this particular source of nonuniqueness. Similar results are obtained when nonuniqueness is due to policy.

A second purpose of this paper is to characterize the resulting multiple equilibria in a way which suggests a further technical condition for uniqueness in those cases where stability is not sufficient. This further condition is placed on the second moment of the equilibrium distribution and requires exclusion of any rational expectations equilibrium which has a price variance larger than some other rational expectations equilibrium. This is a natural extension of the stability condition which rules out all rational expectations equilibria which have an infinite variance in favor of the one with a finite variance. The condition is illustrated in the model introduced in this paper to obtain a unique solution even with real balances in the production function. It seems likely that economic agents would prefer to have expectations which generate the smallest possible price variance (just as they would prefer a finite price variance to an infinite price variance), but until disequilibrium dynamics have been more fully developed for rational expectations models we can offer no rigorous story of how expectations would converge to this collectively rational point.[4] The example of the leading indicator does suggest, however, that collective rationality may not be unreasonable when the choice is obvious. If everyone chooses to ignore the indicator (which is the rational thing to do when everyone expects everyone else to ignore it), then the indicator will not increase the variance of the price level.

2. A RATIONAL EXPECTATIONS MODEL WITH REAL BALANCE EFFECTS

A simple stochastic macroeconomic model[5] with flexible prices, for which it can be shown that the usual stability condition does not provide a unique rational

[3] In Phelps and Taylor [9] such examples are mentioned but are excluded from the admissible class of policy functions. Calvo [5] shows that nonuniqueness can arise in a deterministic utility maximization model with perfect foresight, along the lines of Brock [3], when money is in the production function.

[4] Some analysis of convergence to rational expectations is found in Taylor [12]. Since the formal utility maximization of Brock [4] has been referred to as justification of the stability condition, perhaps a stochastic version of such a model could provide a formal justification for the minimum variance condition.

[5] Except for the stochastic price dynamics, the equations represent a simple textbook macroeconomic model. Eliminating real balances from equations (1) and (3) will not affect the nonuniqueness results. However, if real balances do not appear in (2), then there is a unique solution.

expectations equilibrium, is given as follows:

(1) $$y_t = -\gamma_1(i_t - E_{t-1}p_{t+1} + E_{t-1}p_t) + \gamma_2(m_t - p_t) + \varepsilon_{1t},$$

(2) $$y_t = \phi_0 + \phi_1(m_t - p_t) + \varepsilon_{2t},$$

(3) $$m_t = y_t + p_t - \alpha_1 i_t + \alpha_2(m_t - p_t) + \varepsilon_{3t},$$

and

(4) $$m_t = m,$$

where y_t is the logarithm of real output, m_t is the logarithm of the stock of nominal money balances, p_t is the logarithm of the price of output, i_t is the nominal rate of interest, and $(\varepsilon_{1t}, \varepsilon_{2t}, \varepsilon_{3t})$ is a serially independent, identically distributed, normal random vector with zero mean and finite nonsingular covariance matrix. It is assumed that all coefficients are positive and that, in addition, $\alpha_2 \leq 1$. Price expectations are rational in the sense that E_{t-1} is the mathematical conditional expectation operatot based on all information observed through time $t-1$.

Equation (1) is the aggregate demand function which depends negatively on the real rate of interest and positively on real money balances. The relevant inflation rate for determining the real rate of interest is the expected first difference in the logarithm of the price level. The analysis which follows does not require that γ_2 be nonzero nor that the conditional expectation of the inflation rate be taken as of date $t-1$ rather than some other date. For example, an alternative formulation of equation (1) would be to define the expected inflation rate as $E_t p_{t+1} - p_t$ rather than as $E_{t-1}p_{t+1} - E_{t-1}p_t$. Such a definition would avoid having both p_t and $E_{t-1}p_t$ in the same equation, and therefore prevent the possible (though not necessary) interpretation that p_t is treated both as unknown (first term) and as known (second term). Because the properties which are derived below are invariant to this choice of definition, we choose to set the conditioning date at $t-1$ for more convenient comparison with other rational expectations models.[6]

Equation (2) is an aggregate supply function which is assumed to depend positively on the level of real money balances but is otherwise fixed. The rationale for this positive effect is that the real money stock is an input into the aggregate production function as described, for example, by Levhari and Patinkin [**6**]. Although empirical and theoretical research has been inconclusive as to the magnitude and sign of this effect, it is nevertheless a convenient way to introduce and study the problem of uniqueness in a simple model. (An alternative source of nonuniqueness is considered in Section 4. Further possibilities may exist in more

[6] Sargent and Wallace [**11**] define the expected inflation rate as in equation (1) of this paper. The only effect which the alternative definition (expectation date equal to t) has on the equilibrium price distribution derived below, is that equation (10) is given by $\pi_1 = \pi_0(1+\delta_1)+1$, with equations (9) and (11) unchanged. Hence, the analysis of Section 3 is the same with further conditions required to determine π_1.

It should also be pointed out that the aggregate demand equation (1) is a "reduced form" of structural equations for investment demand and consumption demand. If investment demand depends on $E_{t-1}p_t$ and consumption demand depends on p_t, then equation (1) does not imply that the price level is treated both as known and unknown in the same behavioral relation.

complex models.) A rational expectations Phillips curve effect could also be included in the supply function by inserting the difference between the actual and the expected price level, but since such a term does not have any bearing on the uniqueness aspects discussed below it is omitted for simplicity.

Equation (3) is the money demand function which depends positively on nominal income, negatively on the nominal rate of interest, and positively on the level of real money balances. The restriction that $\alpha_2 \leq 1$ insures that the real balance effect does not overwhelm the negative interest elasticity of the level of real balances (for a given real income, the *LM* curve in the interest rate-real balance plane is not positively sloped). Finally, in equation (4) it is assumed that the log of the supply of nominal money balances is fixed at m. The uniform notation for y_t in the first pair of equations and m_t in the second pair reflects the assumption that both the goods market and the money market are in equilibrium.

Determining a rational expectations solution for this model involves finding a distribution function for p_t which satisfies the three equations. To do this it is convenient to reduce these structural equations to a reduced form equation involving $E_{t-1}p_{t+1}, E_{t-1}p_t$, and p_t. By substituting for i_t in equation (1) using equation (3), and subsequently substituting for y_t using equation (2) and m_t using equation (4) we obtain

$$E_{t-1}p_{t+1} = E_{t-1}p_t + \delta_1 p_t + \delta_0 + u_t \tag{5}$$

where

$$\delta_1 = \alpha_1^{-1}(1-\alpha_2) + \gamma_2\gamma_1^{-1} - \phi_1(\alpha_1^{-1} + \gamma_1^{-1}),$$

$$\delta_0 = \phi_0(1+\alpha_1\gamma_1^{-1}) - \delta_1 m,$$

and

$$u_t = \varepsilon_{1t} - \alpha_1\gamma_1^{-1}\varepsilon_{2t} + (1+\alpha_1\gamma_1^{-1})\varepsilon_{3t}.$$

We note that the parameter δ_1 must be nonzero for the following reason. The random variable u_t is normally distributed with a positive variance, because it is given by a nontrivial linear transformation of $(\varepsilon_{1t}, \varepsilon_{2t}, \varepsilon_{3t})$ which has a nonsingular variance-covariance matrix. If δ_1 were equal to zero then, by equation (5), the variance of u_t must equal zero. Hence, $\delta_1 \neq 0$. This restriction is used below.

3. MULTIPLE PRICE DISTRIBUTION EQUILIBRIA

There are several methods of obtaining distributions for p_t which satisfy equation (5) and thus qualify as rational expectations solutions. The method first utilized by Muth [**8**] is most convenient for the purposes of this paper, however, because it leads to a characterization of the multiple equilibria in terms of a single free parameter.[7] By this method the aggregate price level is represented in the general form

$$p_t = \bar{p} + \sum_{i=0}^{\infty} \pi_i u_{t-i} \tag{6}$$

where $\bar{p}$ and π_i are parameters to be determined and where u_t is the zero mean serially uncorrelated random variable introduced in equation (5). From this expression for p_t we have that

$$E_{t-1}p_t = \bar{p} + \sum_{i=1}^{\infty} \pi_i u_{t-i} \tag{7}$$

and

$$E_{t-1}p_{t+1} = \bar{p} + \sum_{i=2}^{\infty} \pi_i u_{t+1-i} \tag{8}$$

which along with (6) can be substituted back into (5) to give a set of identities in the parameters $\bar{p}$ and π_i. Solving these identities results in

$$\bar{p} = -\delta_0\delta_1^{-1}, \tag{9}$$

$$\pi_0 = -\delta_1^{-1}, \tag{10}$$

and

$$\pi_{i+1} = (1+\delta_1)\pi_i \qquad (i = 1, 2, \ldots). \tag{11}$$

Therefore, $\bar{p}$ and π_0 are determined (recall that $\delta_1 \neq 0$) but π_1 and therefore π_i, $i \geq 2$ are left undetermined. Any representation for p_t through (6) is not unique without further specification of π_1, which is a free parameter. Substituting these values into (6) and subtracting $(1+\delta_1)p_{t-1}$ from p_t gives a first-order autoregressive first-order moving average representation for p_t

$$p_t = (1+\delta_1)p_{t-1} + \delta_0 - \delta_1^{-1}u_t + (\pi_1 - (1+\delta_1)\delta_1^{-1})u_{t-1} \tag{12}$$

where each value of π_1 gives a different distribution for p_t.

In order to examine methods of determining π_1 consider first the stability condition, which in this stochastic context is equivalent to the requirement that p_t have a finite variance because the mean price $\bar{p}$ is determined and finite. From

[7] One method of solution involves taking expectations in equation (5) and writing it as a backward difference equation in the multiperiod price forecasts. Under certain conditions this leads to a value for the expected price level which can be substituted back into (5) to obtain a stochastic difference equation for the price level. This procedure is used in Sargent and Wallace [**11**] and in a multivariate context for a sticky price model in Phelps and Taylor [**9**]. Another method is to guess a particular functional form for the rational stochastic difference equation in prices, and determine the coefficients of this equation using the identities found from substitution in equation (5). This method is a little trickier than Muth's method and requires some intuitive feel for the model in order to guess the appropriate functional form. One disadvantage is that it is possible to obtain a solution but not recognize it as one of many solutions, the others being obscured in the functional forms that are ruled out a priori. This latter method has been successfully employed by Lucas [**7**] and Barro [**1**].

equation (6) the variance of p_t can be written as:

$$\text{(13)} \qquad \operatorname{var}(p_t)=\sigma^2 \sum_{i=0}^{\infty} \pi_i^2=\sigma^2\left[\pi_0^2+\pi_1^2 \sum_{i=0}^{\infty} (1+\delta_1)^{2i}\right]$$

where σ^2 is the variance of u_t. Now, if $\delta_1>0$ then the only value of π_1 which gives a finite variance for p_t is $\pi_1=0$. Thus, when $\delta_1>0$ the finite variance condition gives a unique distribution for p_t. Using (6) p_t then has the representation

$$\text{(14)} \qquad p_t=-\delta_0\delta_1^{-1}-\delta_1^{-1}u_t.$$

But, alternatively, if $-2<\delta_1<0$, then equation (13) implies that the variance of p_t is finite for any arbitrary finite π_1 so that even after imposition of the stability condition multiple equilibria remain.

What can be said about the sign of δ_1 in the model of this paper? Examining the expression following equation (5) it is clear that $\delta_1>0$ if $\phi_1=0$ (if real balances do not appear in the production function) because, by assumption, $\alpha_2\leqslant 1$. Thus, under these parametric specifications the usual stability condition will lead to a unique solution even if real balances appear in the aggregate demand and money demand functions so long as they do not appear in the production function. (The Sargent and Wallace **[11]** result is the special case where real balances appear nowhere.)

But if one cannot rule out real balances in the production function then there is a possibility that ϕ_1 is large enough to make δ_1 negative[8] and, consequently, that multiple equilibria exist corresponding to different values of π_1. The stability condition is not enough for uniqueness in the general version of this model.

The nature of this nonuniqueness can be further illustrated with the following modification of the model. Suppose that in each period the government publishes a leading indicator v_t of the price level p_{t+1} in the next period, but that v_t is generated randomly (presumably by accident). The variable v_t could actually be published for some legitimate purpose other than predicting prices, but might come to be thought of as a leading indicator by market participants, as is shown below. Assume that v_t is independently and normally distributed with zero mean and finite nonzero variance, and that v_t is independent of u_t. We wish to show that it is rational for v_t to be a leading indicator, even though it is generated randomly. To do this it must be shown that the equilibrium price level can be written as

$$\text{(15)} \qquad p_t=\bar{p}+\sum_{i=0}^{\infty} \pi_i u_{t-i}+\sum_{i=1}^{\infty} \beta_i v_{t-i}$$

where β_1 is nonzero. Following the same procedures as above, but with equation

[8] Of course, it might be possible to impose further conditions similar to the one placed on the coefficient of real money balances in the money demand function. One condition of interest is obtained by constructing modified *LM-IS* lines (in the interest rate-real balance plane) which incorporate the real balance effect in production. It can then be shown that if the modified *LM* line cuts the modified *IS* line from above (slope $LM<$ slope IS) then δ_1 must be positive. One might not wish to place such a condition on the model, however.

(15) replacing equation (6), we find that

$$\beta_{i+1}=(1+\delta_1)\beta_i \qquad (i=1,2,\ldots), \tag{16}$$

with $\bar{p}$ and π_i as given in equations (9) through (11). Now, if $|1+\delta_1|<1$, then β_1 need not be zero for the variance of p_t to be finite. Thus, there exist rational expectations equilibrium distributions for which $\beta_1 \neq 0$. If people expect v_t to have an effect on the price level, then it will. Of course, the variance of p_t is larger when $\beta_1 \neq 0$, so that the indicator has increased the variance of prices.

One might argue that it would be very easy for people to agree not to believe in the indicator, so as to avoid these undesirable and unnecessary price fluctuations. But this requires some kind of collective rationality which is not usually assumed in rational expectations models. Such an argument can, however, lead to a general condition for uniqueness as is discussed below.

4. UNIQUE EQUILIBRIUM THROUGH THE MINIMUM VARIANCE CONDITION

The structural restrictions which must be imposed to obtain a unique solution using the stability condition in rational expectation models may be unsatisfactory in some analyses (one may be particularly interested in examining a model with $\phi_1 \neq 0$). Similar policy restrictions made in the interest of uniqueness may also be unsatisfactory in analyses of stabilization or other policy issues under rational expectations. It would therefore be useful to have a condition stronger than the finite variance condition which gives unique solutions without such structural or policy restrictions.

One such condition, which is suggested here, requires that any price distribution which satisfies the conditions of rationality be ruled out as a candidate for the equilibrium distribution if it has a variance which is larger than that of some other price distribution which satisfies the condition of rationality. Viewed as a natural extension of the finite variance condition it seems no less unreasonable as a constraint on expectations. People are normally assumed to prefer less price uncertainty. If they collectively choose a finite variance path rather than an infinite variance path, then why would they not choose the smallest of all the finite variances? It seems then that the reasonability of the minimum variance condition rises and falls with that of the usual finite variance condition.

But perhaps neither condition is reasonable. Until the dynamics of disequilibrium rational expectations (transitional expectations) are more fully developed we cannot say much about how people's expectations converge to any rational expectations equilibrium, let alone how they converge to one with finite variance, or to that finite variance distribution which has minimum variance. Awaiting such further development, perhaps it is best to argue in terms of a theory of collective rationality which leads to those self-fulfilling expectations which have highest expected utility.

In any case it can be shown that the minimum variance condition does provide a unique solution in the model of this paper even when δ_1 is negative. From equation (15) it is clear that the minimum price variance occurs when π_1 and β_1

both equal zero. Of course, these are the same values which are required when δ_1 is positive, so that the unique representation for p_t is still given by equation (14), with the mean and variance of p_t still given by $-\delta_0\delta_1^{-1}$ and $\delta_1^{-2}\sigma^2$, respectively.

In order to focus on structural (rather than policy) reasons for multiple rational price distributions we have thus far assumed a passive monetary policy with a fixed nominal supply of money. In order to see some of the *policy* implications of these uniqueness conditions, consider for example the policy rule by which the money supply is set so that[9]

$$m_t = m + \mu E_{t-1}p_t. \tag{17}$$

For $\mu = 1$ this rule pegs the expected money rate of interest to a constant as was discussed by Sargent and Wallace [**11**]. Replacing equation (4) by equation (17) we obtain a reduced form equation

$$E_{t-1}p_{t+1} = (1-\delta_1\mu)E_{t-1}p_t + \delta_1 p_t + \delta_0 + u_t \tag{18}$$

in place of equation (5). Of concern now is what values of μ are admissible in the sense that they lead to unique rational price distributions.

Using the same solution procedure as in Section 3 we find that parameters of equation (6) must satisfy

$$\bar{p} = -\delta_0\delta_1^{-1}(1-\mu)^{-1}, \tag{19}$$

$$\pi_0 = -\delta_1^{-1}, \tag{20}$$

and

$$\pi_{i+1} = [1+\delta_1(1-\mu)]\pi_i \qquad (i = 1, 2, \ldots). \tag{21}$$

When $\mu = 0$ these equations correspond to equations (9) through (11). Note that if we are to have a determinate finite mean for the price level we must require that $\mu \neq 1$; otherwise, $\bar{p}$ would be undefined. This corresponds to the result of Sargent and Wallace [**11**] regarding policies which peg the interest rate. Using only the finite variance condition we must also rule out all μ for which $|1+\delta_1(1-\mu)| > 1$. But with the minimum variance condition imposed, the admissible class of policies may contain all μ not equal to one. The resulting unique representation for p_t is then

$$p_t = -\delta_0\delta_1^{-1}(1-\mu)^{-1} - \delta_1^{-1}u_t \tag{22}$$

and for all $\mu \neq 1$ the variance[10] of the price level is given by $\delta_1^{-2}\sigma^2$. Thus, the admissible class of policy functions is enlarged by the imposition of the minimum variance condition.

[9] Since prices are perfectly flexible in the model, monetary policy has no effect on the distribution of output, as was emphasized by Sargent and Wallace [**11**]. Thus, for purposes of illustration this functional form will serve as well as a more complex one.

[10] This rule does not affect the variance of prices because of the serial independence of the disturbances. With a more general error structure the variance would be affected.

5. CONCLUDING REMARKS

The main purpose of this paper has been to present a simple stochastic macroeconomic model with rational expectations in which multiple price distribution equilibria exist even when all rational price distributions which are unstable (have infinite variance) are excluded. The economic source of this multiplicity is the presence of real balances in the production function. This multiplicity of equilibrium price distributions implies that random events unrelated to the economic system can increase the variance of the price level. However, the multiplicity can be avoided if one is willing to extend the usual finite variance condition to a minimum variance condition which then eliminates all but one equilibrium price distribution. These results point out the need for further research to explain how expectations might converge to equilibria which have a finite or, more strongly, a minimum variance.

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