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Associate Editor's Note: The following 11 articles summarize the efforts to date of a group that has been working on the new and exciting topic of solution strategies for nonlinear rational-expectations models. The members of the group wish to thank the National Bureau of Economic Research, the Institute for Empirical Macroeconomics, and the National Science Foundation for providing financial support for various aspects of the research activities. I wish to thank the many referees whose willingness to review the materials in a careful and timely manner helped improve the work substantially.

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Solving Nonlinear Stochastic Growth Models: A Comparison of Alternative Solution Methods

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The purpose of this article is to report on a comparison of several alternative numerical solution techniques for nonlinear rational-expectations models. The comparison was made by asking individual researchers to apply their different solution techniques to a simple representative-agent, optimal, stochastic growth model. Decision rules as well as simulated time series are compared. The differences among the methods turned out to be quite substantial for certain aspects of the growth model. Therefore, researchers might want to be careful not to rely blindly on the results of any chosen numerical solution method in applied work.

KEY WORDS: Linear-quadratic approximation; Nonlinear models; Numerical solution methods; Optimal growth; Rational expectations.

1. INTRODUCTION

During the last few years, there has been an increased demand for numerical solution methods for nonlinear rational-expectations models. The demand has come from economic researchers with diverse research goals and modeling strategies. In almost all areas of macroeconomics, rational-expectations models are becoming increasingly complex and richer in structure. Empirical researchers studying real business-cycle models are attempting to go beyond simple representative-agent models with convenient, but sometimes unrealistic, functional forms for the utility functions; they are also beginning to study models with distortions and externalities. Researchers focusing on monetary models are finding it necessary to solve large nonlinear stochastic systems to apply rational-expectations techniques to practical problems of monetary policy, including international monetary policy. Finance economists interested in dynamic "consumption-beta" models are finding it necessary to go beyond simple analytical models to confront the theory with the data. As electronic computing power becomes faster and cheaper, numerical solution procedures will enable macroeconomists and financial economists to study these more complex models and apply them to practical policy or other applied problems.

The purpose of this article is to report on a comparison of several alternative numerical solution techniques for nonlinear rational-expectations models. All of the techniques are currently under development and rely on high-speed computer technology or will eventually need this technology when they are moved beyond simple test problems. The comparison is one of the activities of a research group called the Nonlinear Rational Expectations Modelling Group supported by the National Bureau of Economic Research. Participants in the group meetings at Stanford and Minneapolis have included Marianne Baxter, Wilbur John Coleman, Lawrence Christiano, Darrell Duffie, Ray Fair, Joseph Gagnon, Lars Hansen, Beth Ingram, Kenneth Judd, Pamela Labadie, David Luenberger, Rodolfo Manuelli, Albert Marcet, Ellen McGrattan, David Runkle, John Rust, Thomas Sargent, Christopher Sims, Kenneth Singleton, John Taylor, George Tauchen, and Harald Uhlig. The comparison was made by asking individual researchers to apply different solution techniques to a simple representative-agent, optimal, stochastic growth model designed to describe the behavior of aggregate consumption and the capital stock. Though simple, the problem does not have an analytic solution. Hence the solution results are of interest in their own right in addition to enabling a comparison of alternative methods.

Section 2 describes the stochastic growth model. Section 3 very briefly describes the solution methods. More details about each of the techniques are contained in articles by the individual authors that accompany this article. Section 4 presents the comparison of the different solution methods on the test problem. Section 5 considers issues for future research.

2. THE STOCHASTIC GROWTH MODEL

The following problem was proposed by Christopher Sims to be solved by the individual researchers. Let C_t be consumption and K_t be the capital stock. Agents are assumed to maximize

$$E \sum_{t=0}^{\infty} (1 - \tau)^{-1} C_t^{(1-\tau)} \beta^t \tag{1}$$

subject to

$$C_t + K_t - K_{t-1} = \theta_t K_{t-1}^{\alpha} \tag{2}$$

and to the side conditions that $K_t > 0$ and $C_t > 0$ for all t. Note that Equation (2) implies that there is no depreciation of the capital stock. A slightly more general formulation would have some depreciation in which a coefficient less than 1 would multiply the lagged value of the capital stock in Equation (2). Agents at time t choose K_t and C_t. Agents are assumed to know the history of all variables dated t and earlier when they choose variables dated t.

The stochastic process for θ_t is given by

$$\ln \theta_t = \rho \ln \theta_{t-1} + \varepsilon_t, \tag{3}$$

where ε is a serially uncorrelated, normally distributed random variable with mean 0 and constant variance σ_ε^2.

For this problem, decision rules for consumption C_t and the capital stock K_t in any period t are given by the functions $f(K_{t-1}, \theta_t)$ and $g(K_{t-1}, \theta_t)$ of the capital stock in period $t - 1$ and the random shock in period t. Exact solutions for f and g are not known for this problem. If the utility function is logarithmic ($\tau = 1$) and there is full depreciation rather than zero depreciation as in Equation (2), then there is a simple closed-form solution (e.g., see Sargent 1987, p. 122). For the problem in Equations (1) and (2), the functions f and g must be evaluated numerically.

To compare the different solution methods, the stochastic growth problem was solved for 10 cases of parameter values. The parameters for the 10 cases are given in Table 1 with $\alpha = .33$ and $\rho = .95$ for all cases. These values of the coefficient of relative risk aversion (τ) allow for considerable differences in the degree of risk aversion. Note also that the technology shock has a very large variance in cases 1–4, indicating a high degree of uncertainty.

Table 1. Parameter Choices for the 10 Cases

Case	β	τ	σ_ε
1	.95	.5	.1
2	.95	1.5	.1
3	.98	.5	.1
4	.98	1.5	.1
5	.95	.5	.02
6	.95	1.5	.02
7	.95	3.0	.02
8	.98	.5	.02
9	.98	1.5	.02
10	.98	3.0	.02

Individual researchers reported results in two basic forms, decision rules (f and g) for consumption and capital and stochastic simulation paths for consumption and capital. The decision rules f and g were evaluated for a grid of values of capital and the technology shock. For the stochastic simulations, shocks on ε_t were drawn so as to generate a path for C_t and K_t over time.

3. THE SOLUTION METHODS

Ten researchers participated in the solution comparison by submitting decision rules and/or stochastic simulation paths. The names of the researchers, in alphabetical order, along with the type of method that each researcher used, an indication of whether decision rules were submitted, and the number of periods in the simulated time series submitted in each case are listed in Table 2.

A very brief overview of the general features of each method is provided for convenience here. Details of how these methods are implemented in the stochastic growth model can be found in the articles by the individual authors that accompany this article. To use the methods, one, of course, needs to read these articles.

Value-Function Grid. The basic idea here is to approximate the continuous valued-growth problem by a discrete-valued problem over a grid of points. In other words, the values of K and the shocks are discretized. By making the grid finer, the actual solution for K can be approximated arbitrarily closely. These approximations result in a discrete state-space dynamic optimization model that is solved by iterating on the value function. The finer the grid is, the more expensive will be the computation for this method. Higher dimensions for the control variable increase computation time greatly, but for the test problem there is only one dimension, and computing time is not a problem. Christiano used this method to solve the growth problem in Equation (1). See Christiano (1990) for details.

Quadrature Value-Function Grid. This method also discretizes the state space, but it is potentially more efficient than the simple grid in that a quadrature rule is used to discretize the state space. Tauchen has applied this method successfully in several problems. See Tauchen (1987, 1990) for a description of the method and for a discussion of some applications.

Table 2. Summary of the Methods

Researcher	Type of method	Decision rules	Simulation periods	Cases
Baxter	Euler-equation grid	Yes	2,009	All cases
Christiano	Lin-LQ-Normal	Yes	2,000	All cases
Christiano	Lin-LQ-discrete	Yes	2,000	All cases
Christiano	Log-LQ-Normal	Yes	2,000	All cases
Christiano	Log-LQ-discrete	Yes	2,000	All cases
Christiano	Value-function grid	No	2,000	Cases 5–10
Coleman	Euler-equation grid	Yes	1,999	All cases
Gagnon	Extended path	Yes	500	All cases
Ingram	Backsolving	No	1,000	All cases
Labadie	Least square projection	No	680	Cases 1, 5, 7
Marcet	Parameterizing expectations	Yes	1,649	All cases
McGrattan	Lin-LQ-Normal	Yes	2,000	All cases
Sims	Backsolving	No	2,000	All cases
Tauchen	Quadrature value-function grid	Yes	2,000	All cases

Linear-Quadratic (lin-LQ-Normal, lin-LQ-discrete, log-LQ-Normal, log-LQ-discrete). This method approximates the control problem in Equation (1) with a standard linear-quadratic (LQ) control problem to which linear decision rules for K and C are optimal and can be computed easily. The linear decision rules are then treated as approximations to the exact solutions. The approximation is made by first substituting the constraint (2) into the objective function (1) and then making a quadratic approximation of the utility function at each time period. The approximation is taken about the steady-state values of the problem. This method was used by Kydland and Prescott (1982). Its application to the problem considered in this article is described by Christiano (1990) and McGrattan (1990).

In preparing calculations for the LQ method reported in this article, Christiano did four variants of this method. In one variant, log(K) was treated as a control variable, and in another variant, K was treated as a control variable. The two solutions are referred to as log-LQ and lin-LQ respectively. Moreover, for each of these two variants, Christiano drew the shocks in the stochastic simulations either according to a continuous-valued normal distribution or according to a discrete distribution. The identifiers "Normal" and "discrete" are used to indicate these two variants. The latter type of draws were made for comparison with the value-function-grid methods. McGrattan's LQ results are based on treating K as the control variable and drawing normal errors and, therefore, are referred to as the lin-LQ-Normal method in this article.

Backsolving. This method was proposed by Sims (1984, 1989). The implementation for the stochastic growth problem is described by Ingram (1990) and Sims (1990). The backsolving method is a general approach rather than a specific algorithm, and, in fact, the Ingram and Sims backsolving implementations are considerably different in this application. The backsolving method starts out by solving a problem that is more analytically tractable than the actual problem and then approximates the actual problem at the stage when the stochastic shocks are drawn. For example, in this application Sims solves a linear-quadratic approximation to the original problem and draws shocks for the Euler equation, backsolving for the shocks in the production function. Ingram modifies the original problem by adding another shock with a convenient distribution, thus relaxing the budget constraint.

Extended Path. This method was described in general terms by Fair and Taylor (1983), and its implementation in the stochastic growth problem is described by Gagnon (1990). When applied to the optimal-control problems like the one in Equation (1), it works by solving the nonlinear dynamic first-order conditions that are implied from the discrete-time calculus-of-variations formulation of the problem. These first-order conditions at time t involve conditional expectations of K_{t+1}. These future expectations are solved out iteratively to solve the first-order conditions, thereby obtaining the decision rule solution for K_t. The decision rule for consumption is then computed from the budget identity. Atthough stochastic iterations may improve the accuracy of the method in some cases, only deterministic iterations were performed by Gagnon.

Euler-Equation Grid. Coleman's method and Baxter's method fall into this category. Coleman's method works by approximating the decision rules for consumption and capital (by piecewise linear functions, for example). Using these approximate functions, the method then iteratively solves the Euler equations directly rather than by iterating on the value function. Convergence is checked over a grid of values. [See Coleman (1990) and the references therein.] Baxter's method discretizes the state space and then iterates to find the value for capital, restricted to the grid, that comes closest to solving the Euler equations. [See Baxter, Crucini, and Rouwenhorst (1990) for the implementation of the method in the stochastic growth problem.]

Parameterizing Expectations. This method was originally proposed by Marcet (1988), and its implementation for the stochastic growth problem is described by Den Haan and Marcet (1990). Like the Euler-equation-grid and extended-path methods, this method uses the first-order conditions (Euler equations) for the dynamic-optimization problem. The general idea is to hypothesize a general functional form with undetermined parameters for the conditional expectation of future variables that appear in the first-order conditions. The parameters of this functional form are then "estimated" by least squares using a single set of simulated values. The functional form can then be generalized until convergence of the solution is achieved.

Least Squares Projections. This method was originally proposed by Labadie (1986), and its implementation for the stochastic growth problem is described by Labadie (1990). Like the method of parameterizing expectations, this method focuses on obtaining expressions for the conditional expectations implicit in the first-order conditions (Euler equations). It attempts to "estimate" certain parameters of the conditional expectations functions by using a single simulation of the random shocks in the model.

Counting the LQ methods only once, there are a total of eight different solution methods examined in this article, which reports on 14 different sets of solutions because there are four variants of the LQ method and because the backsolving method, the lin-LQ-Normal method, and the Euler-equation-grid method are each used by two researchers (though in some cases with a very different implementation procedure).

4. A COMPARISON OF THE RESULTS

As indicated previously, researchers reported results both in the form of decision rules and stochastic simulation paths. The stochastic simulation paths were plotted graphically and were also used to calculate several summary statistics to aid in the comparison of the solution algorithms. In the first part of this section, we discuss the plots of the simulation paths and then go on to discuss the decision rules and the summary statistics.

4.1 Plots of the Stochastic Simulations

The reported stochastic simulation paths for all 10 cases are available on request. Due to space limitations, we only report plots of a sample of cases here. These cases were selected with several criteria in mind—to include as many researchers as possible, to demonstrate differences in behavior most clearly, and to illustrate that the differences are not particular to just one case.

4.1.1 Time Series Charts. Figure 1 shows the realizations for consumption and capital for a single stochastic simulation for case 1 for 13 of the different solution methods. (To assist readers in scanning the figures, the charts in each figure are organized in the same order, and in cases in which a solution method is not available, a blank appears in the figure.) Note that each researcher used different sets of draws of the random variable so that the actual realizations will be much different for each method. Even if two methods gave exactly the same accuracy, only the general patterns of the stochastic simulations would appear similar for the different methods. On an absolute basis, the level of consumption is, of course, much less than the level of the capital stock. The fluctuations in consumption are also smaller than the fluctuations in the capital stock. All of the methods show a high degree of contemporaneous correlation between consumption and the level of capital. Most of the variance in both consumption and capital is in the low frequencies (assuming an annual time frame). The discretization of capital in Tauchen's method is quite evident, as is the resulting erratic behavior of consumption. Note also the encounters with 0 in the lin-LQ-Normal simulation and the shock-and-convergence-back behavior in the lin-LQ-discrete simulation. But even aside from this "exotic" behavior, differences among the solution methods may be quite large: compare, for example, the plots for McGrattan's solution and Marcet's solution. Marcet's parameterizing-expectations solution finds a much higher variance for capital and a much lower frequency of fluctuations than does McGrattan's linear-quadratic method. The macroeconomic interpretations of these two simulations would be much different.

Figure 2 shows the time series plots of investment ($K_t - K_{t-1}$) for 12 of the methods for case 10. This case has a much higher coefficent of relative risk aversion and a much lower technology shock than case 1. This comparison also shows considerable differences between the methods. Some of the methods in which the shocks are drawn discretely (Christiano—lin-LQ-discrete, Christiano—value-function grid, and Tauchen) show long periods of no change in the investment series. Note that Ingram's solution appears to have a higher volatility of investment than the other methods.

4.1.2 Empirical Density Functions for Consumption and Investment. In Figure 3, we present empirical density functions for consumption for case 5 (50 grid points), and in Figure 4, we present empirical density functions for investment for case 10 (25 grid points to achieve more smoothness). The density functions all integrate to 1, but notice the different vertical scales. (Frequently, a histogram is drawn as a step function with certain heights for each bin. Note however, that connecting these heights by straight lines, as we do in Figs. 3 and 4, results in a function with the same integral as the original step function if the boundary values are 0.)

As with the time series plots, the differences between the empirical density functions are quite striking. Except for those of Coleman and possibly McGrattan, none of the density functions are particularly smooth. Obviously, even with 2,000 simulated data points the variance on these estimated density functions is quite high. Nonetheless, the differences between the solutions are large with some methods showing very little

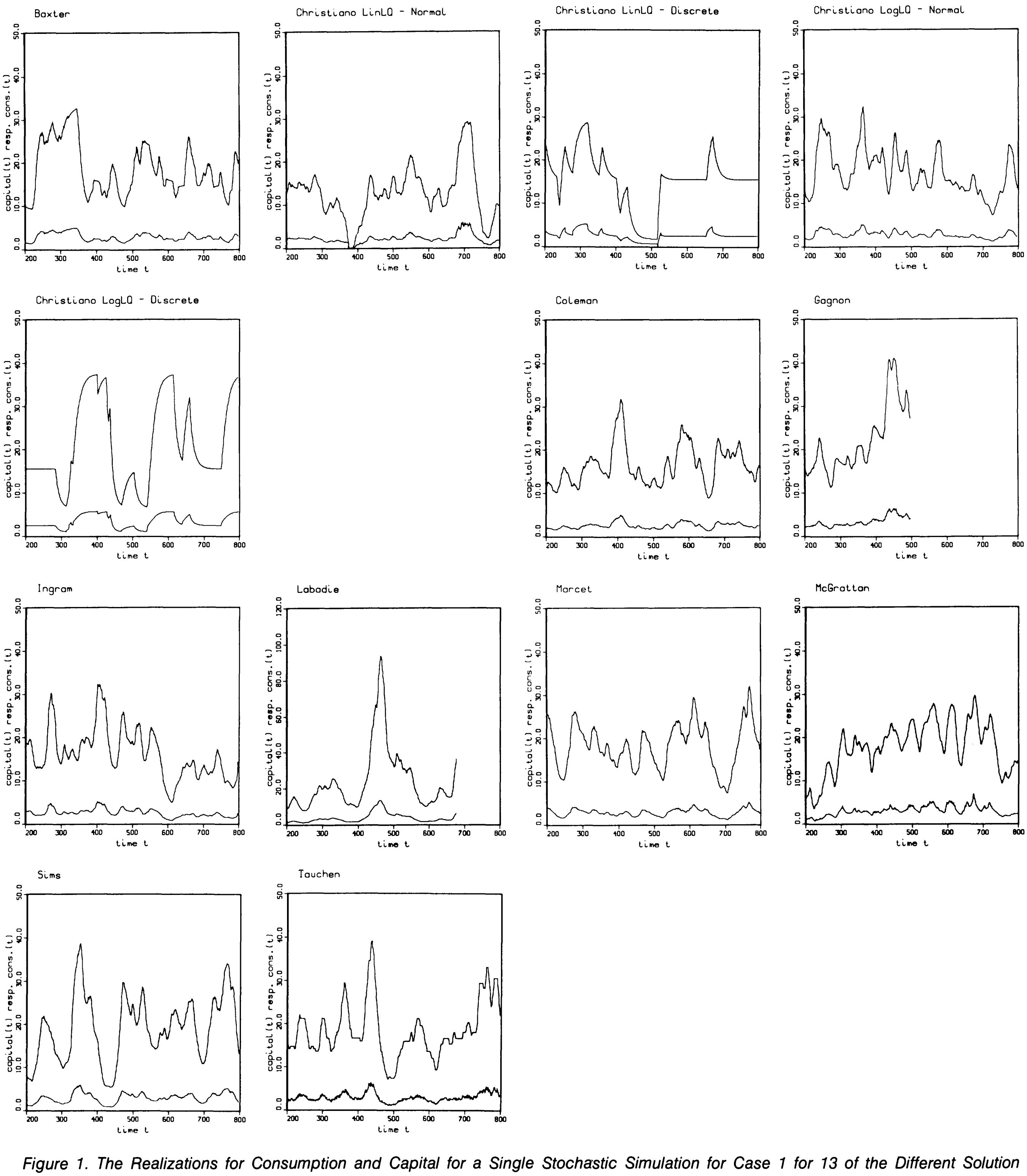

Figure 1. The Realizations for Consumption and Capital for a Single Stochastic Simulation for Case 1 for 13 of the Different Solution Methods. A blank appears here and in Figures 2, 4, 5, and 6 if a solution method is not available. Different cases have been used for different figures to illustrate the points most clearly. Here the time span is 200 to 800. The range for consumption and capital is 0 to 50 for all simulations except Labadie's; in her case the range is 0 to 120. The smaller of the two time series is always the consumption series. Gagnon's and Labadie's series were shorter than the 800 periods and are, therefore, cut off. Note that only patterns can be compared, since different researchers used different random numbers. Note the effect of discretization in the simulations Christiano lin-LQ-discrete, Christiano log-LQ-discrete, and Tauchen.

spread, some showing double peaks, and others showing a very wide spread. In particular, the question of whether investment is sharply peaked cannot be decided from these different methods at this point.

4.1.3 Scatter Diagrams for Consumption and Capital. Figure 5 shows scatterplots of the decision variables capital K_t and consumption C_t for case 4 on a scale common to all researchers. Figure 6 shows scatter dia-

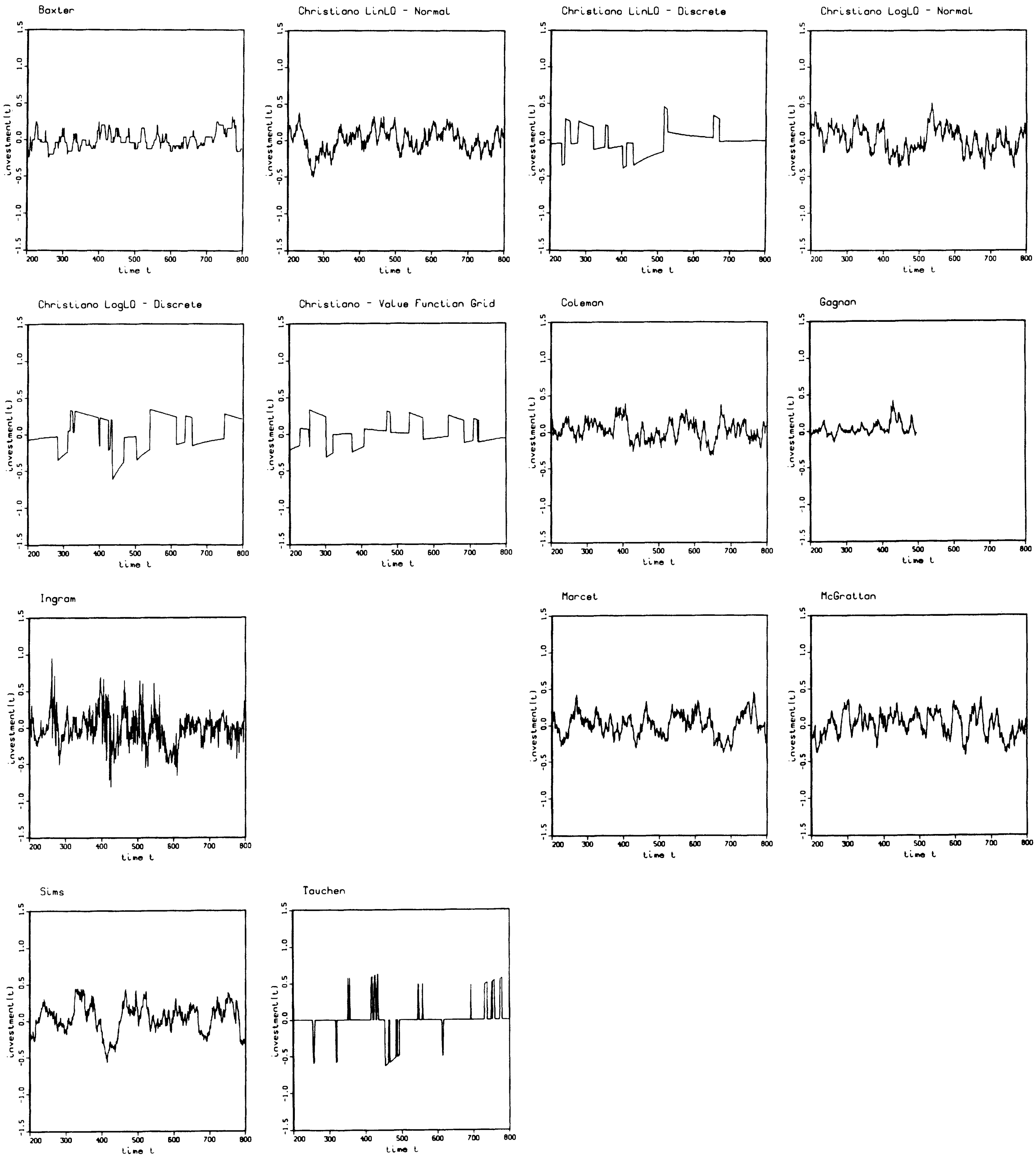

Figure 2. The Time Series Plots of Investment $K_t - K_{t-1}$ for 12 of the Methods for Case 10 and Time Periods 200 to 800. The range for investment is −1.5 to 1.5. Case 10 has a much higher coefficient of relative risk aversion than case 1. Some of the discrete methods show long periods of (almost) no investment (Christiano lin-LQ-discrete, Christiano log-LQ-discrete, Christiano value-function grid, and Tauchen). Ingram's solution is quite volatile compared to others.

grams for a selected sample of points for each method (again case 4) with the points connected to show the general direction of movement. Note that the scales differ in Figure 6. The solutions for Ingram, Sims, and Coleman seem to move along rather large loops, whereas the log-LQ-Normal solution jumps around without ever dropping below a rather rigid boundary. The lin-LQ-discrete solution moves along steadily on apparently parallel lines with "quantum leaps" in between. Moreover, sudden drops in consumption can be observed in Tauchen's solution and Gagnon's solution. Notice in the latter how the values accumu-

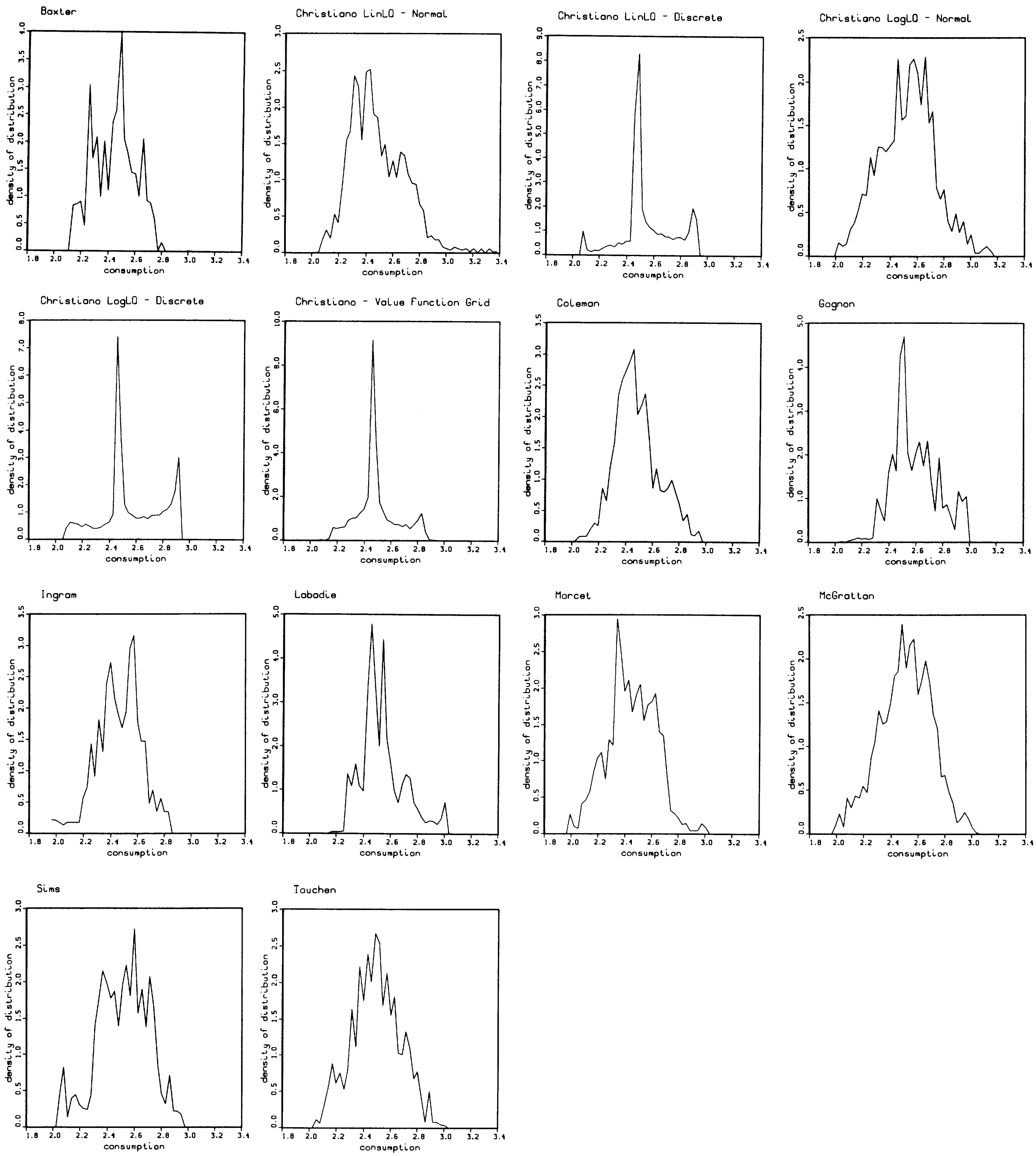

Figure 3. The Empirical Density Functions for Consumption for Case 5 and for all Available Data of a Simulation. The range for consumption is 1.8 to 3.4 (actually, these numbers are rounded from the original bin bounds, which explains the cutoff in Ingram's graph). Fifty bins are used, and the heights are connected by a straight line; this still results in a density integrating to 1 if the boundary values are 0. Most density functions are surprisingly ragged. Some—for example, Christiano's discrete methods—show double peaks. The shapes vary considerably across methods, but it is possible that this is largely due to the rather small length of the simulated time series (mostly 2,000 data points).

late to two "islands." These islands are probably because Gagnon has provided a small sample of points. Gagnon has reported that additional simulations (not reported here) show that more data points begin to fill in the sparse areas and the scatter diagram develops into a simple large scatter, as do most of the other methods.

Additional scatterplots not reported here show additional anomalies. For example, scatter diagrams of investment versus the change in consumption showed

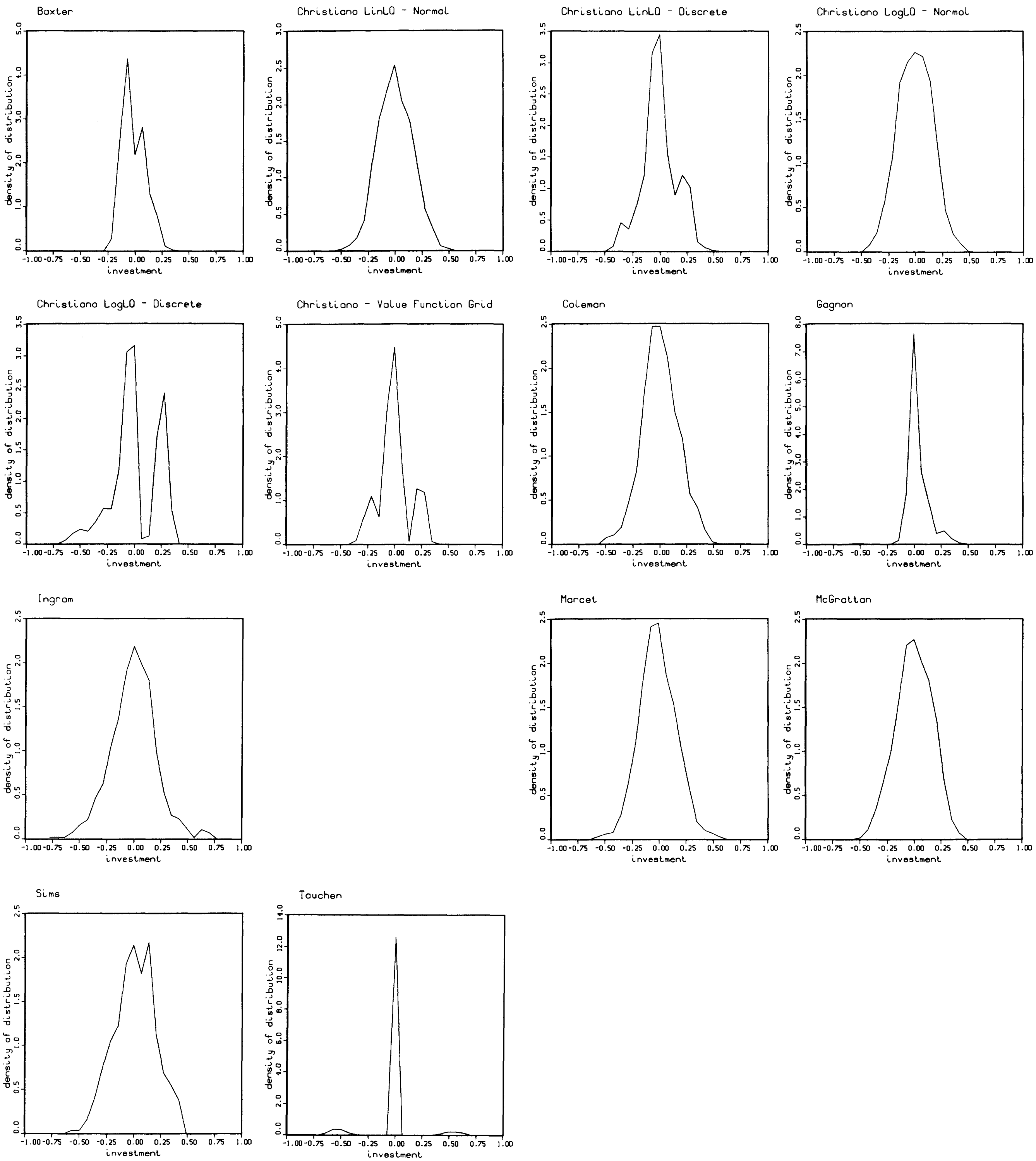

Figure 4. The Density Functions for Investment $K_t - K_{t-1}$ in Case 10, Using 25 Bins to Achieve More Smoothness. Again, for example, Christiano log-LQ-discrete shows double peaks. The density for Gagnon and Tauchen is very narrow and sharply peaked compared with the other methods. It is, therefore, difficult to decide whether this is actually a feature of the true solution. The range for investment is -1.0 to 1.0 in all graphs.

a strikingly curved scatter with a sharp boundary on the inside for the log-LQ-Normal method, and Tauchen's solution showed star-like patterns that were probably the result of discretization. It is not clear to us where the curvature in the log-LQ-Normal solution comes from. It appears to disappear for cases 5 to 10. Since cases 1–4 are parameterized with higher disturbance variance, it is possible that the curvature is a result of

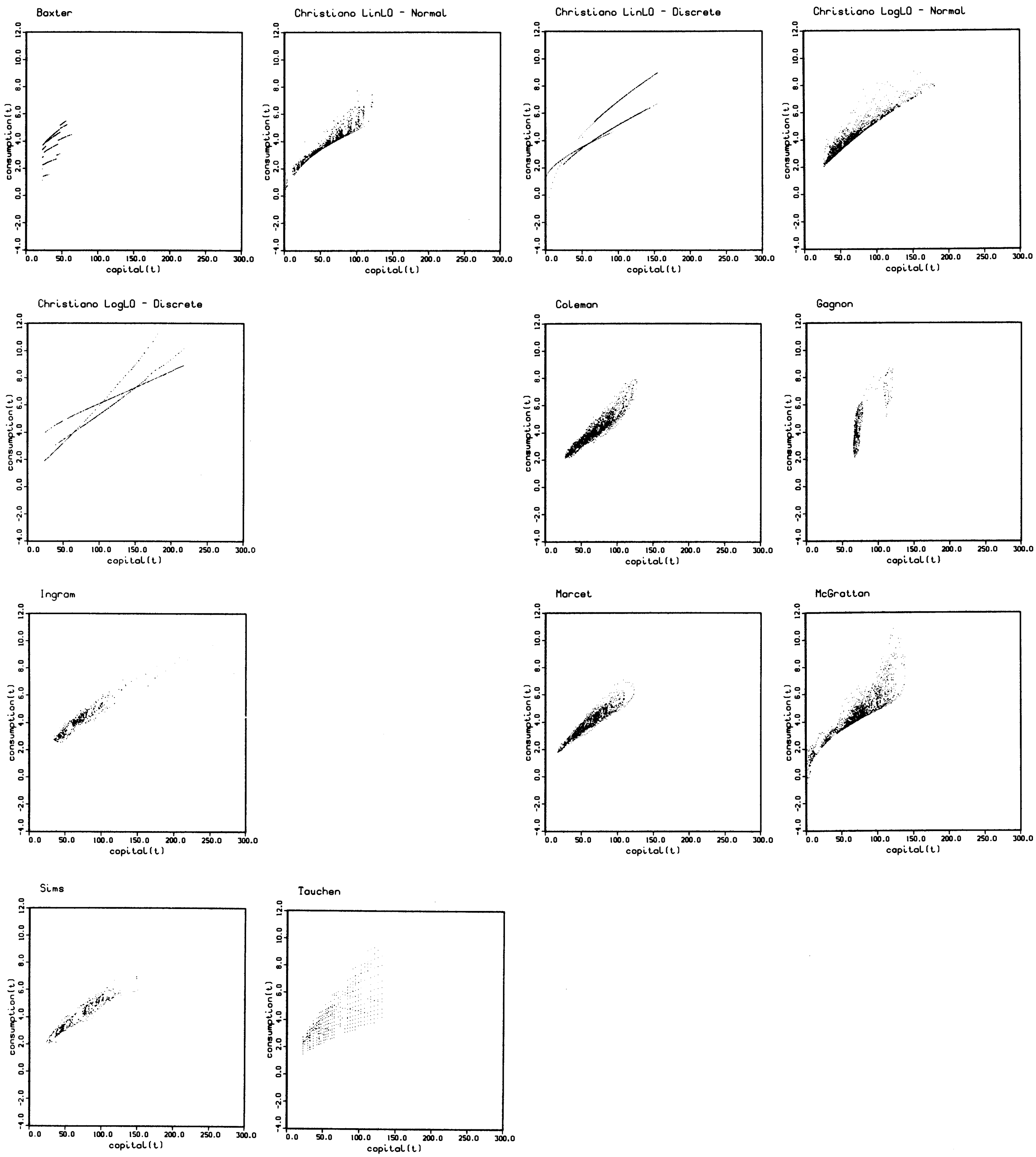

Figure 5. Scatterplots of the Decision Variables, Capital K_t Versus Consumption C_t, in Case 4. A common scale is used for all researchers: 0 to 300 for capital and −4.0 to 12.0 for consumption. Note how a sharp boundary is visible in Christiano lin-LQ-Normal, Christiano log-LQ-Normal, and McGrattan—that is, in the most commonly used linear-quadratic methods. Observe that the points scatter around two "islands" in Gagnon's solution. Gagnon reports that this island structure starts to disappear with longer simulations.

the quadratic approximation. Since the linear-quadratic method is probably one of the most commonly used methods, this is an important issue for future research. These diagrams reveal large differences among the different methods.

4.2 Decision Rules

For 10 of the 14 methods, researchers reported decision rules $K_t = f(K_{t-1}, \theta_t)$ and $C_t = g(K_{t-1}, \theta_t)$ for consumption and capital. The results for cases 1 and 2

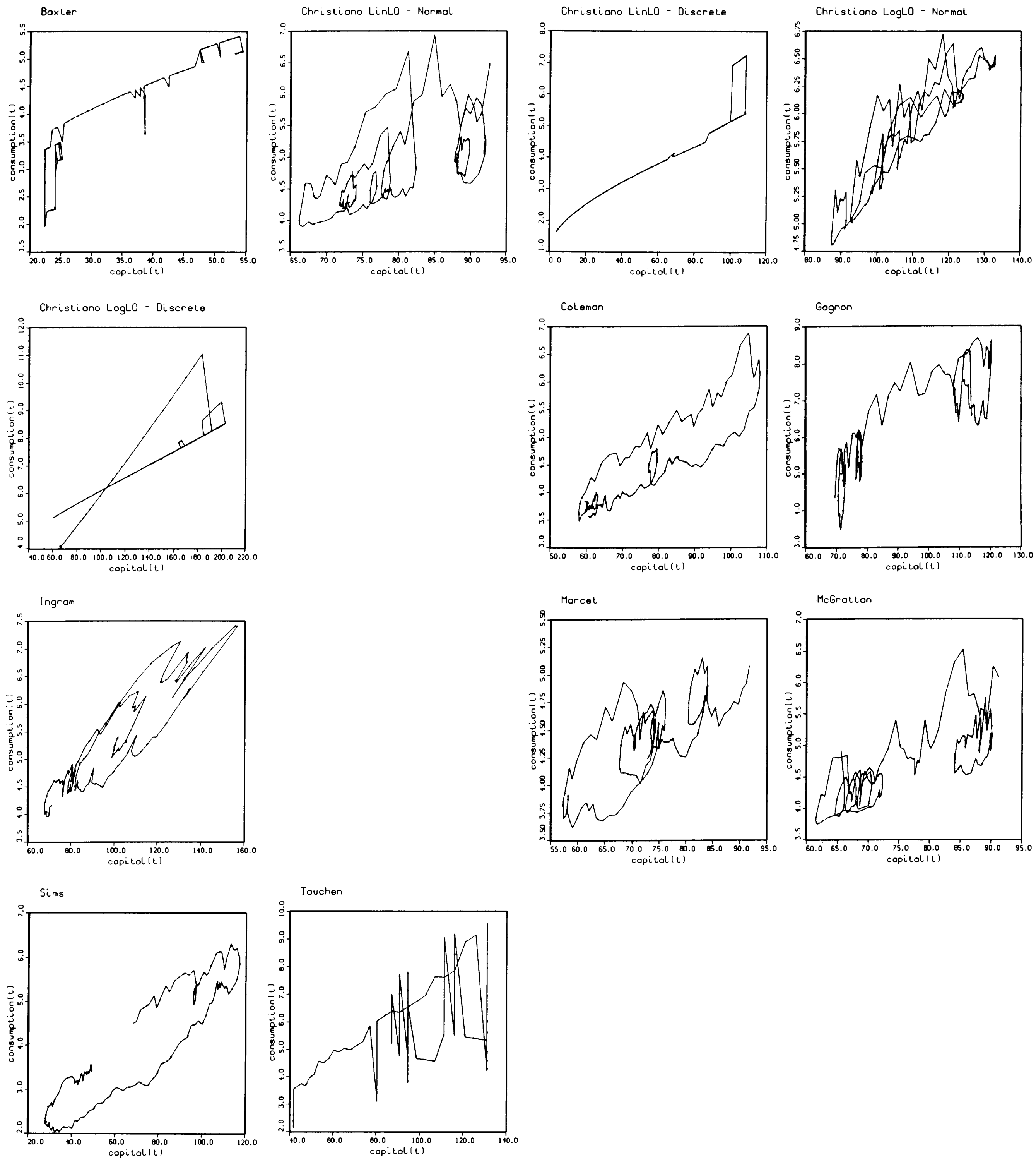

Figure 6. Scatterplots Similar to Figure 5 Except That Subsequent Data Points Are Connected by a Straight Line and the Scale Varies Across Methods. Subsamples of 100 data points were used for these diagrams. Note the sharp adjustments in, for example, Baxter, Christiano lin-LQ-discrete and log-LQ-discrete, and Tauchen. The simulations of Coleman, Ingram, and Sims show rather large loops. Shorter loops can be seen in the simulations of, for example, Christiano lin-LQ-Normal and Marcet. Again the features and the behavior of the methods are quite different. This can be relevant in economic applications.

are reported in Tables 3–10. The decision rules were evaluated for a grid of values of K_{t-1} and θ_t. The grid of values for the tabulation of the function f and g in cases 1 and 2 were θ_1 = .4, .7, 1.0, 1.3, and 1.6 and K_0 = 5, 10, 15, 20, and 25.

Note first that the results for the two independent calculations of the lin-LQ are identical. (For the decision rules there is, of course, no difference between lin-LQ-Normal and lin-LQ-discrete.) This is, of course, not surprising, but it provides a useful check on the results.

Table 3. *Decision Rules for Euler-Equation Grid (Baxter)*

K_0	θ_1 .40	.70	1.00	1.30	1.60
			Capital: case 1		
5.0	5.02	5.19	5.62	6.05	6.48
10.0	9.84	10.01	10.36	10.88	11.48
15.0	14.15	14.41	15.01	15.62	16.22
20.0	18.63	18.98	19.67	20.18	20.87
25.0	23.19	23.54	24.23	25.01	25.70
			Consumption: case 1		
5.0	.66	1.00	1.08	1.16	1.24
10.0	1.46	1.49	1.78	1.90	1.94
15.0	1.83	2.30	2.43	2.56	2.69
20.0	2.44	2.90	3.02	3.31	3.43
25.0	2.97	3.48	3.66	3.75	3.93
			Capital: case 2		
5.0	5.02	5.02	5.19	5.53	5.88
10.0	9.93	9.93	10.01	10.44	10.88
15.0	14.84	14.84	14.93	15.27	15.79
20.0	19.84	19.84	19.92	20.18	20.78
25.0	24.75	24.75	24.84	25.18	25.78
			Consumption: case 2		
5.0	.66	1.17	1.51	1.68	1.84
10.0	.93	1.57	2.13	2.34	2.54
15.0	1.14	1.87	2.51	2.91	3.12
20.0	1.23	2.04	2.77	3.31	3.52
25.0	1.41	2.27	3.05	3.58	3.85

Table 5. *Decision Rules for Log-LQ (Christiano)*

K_0	θ_1 .40	.70	1.00	1.30	1.60
			Capital: case 1		
5.0	4.88	5.22	5.45	5.63	5.77
10.0	9.25	9.90	10.34	10.68	10.95
15.0	13.45	14.40	15.04	15.53	15.92
20.0	17.54	18.78	19.61	20.25	20.77
25.0	21.55	23.08	24.10	24.88	25.52
			Consumption: case 1		
5.0	.80	.97	1.25	1.58	1.95
10.0	1.61	1.60	1.80	2.10	2.47
15.0	2.53	2.31	2.41	2.65	2.99
20.0	3.53	3.10	3.07	3.24	3.53
25.0	4.60	3.95	3.79	3.88	4.11
			Capital: case 2		
5.0	4.80	5.05	5.22	5.34	5.45
10.0	9.36	9.84	10.17	10.41	10.61
15.0	13.82	14.54	15.02	15.38	15.67
20.0	18.23	19.18	19.81	20.28	20.67
25.0	22.60	23.77	24.55	25.14	25.62
			Consumption: case 2		
5.0	.88	1.14	1.48	1.87	2.28
10.0	1.50	1.65	1.97	2.37	2.81
15.0	2.15	2.17	2.43	2.80	3.24
20.0	2.84	2.70	2.88	3.21	3.63
25.0	3.56	3.25	3.34	3.62	4.01

The log-LQ results are somewhat different from the lin-LQ results. The log-LQ results are very similar to the quadrature-value-function-grid solution (Tauchen) or the Euler-equation-grid solution (Coleman). Assuming that these grid solutions are fairly accurate, this shows the advantages of choosing functional forms when using the linear-quadratic method. The values for the quadrature method reported in the table are interpolated

Table 4. *Decision Rules for Lin-LQ (Christiano)*

K_0	θ_1 .40	.70	1.00	1.30	1.60
			Capital: case 1		
5.0	4.07	5.13	5.80	6.30	6.69
10.0	8.69	9.75	10.42	10.91	11.31
15.0	13.31	14.36	15.04	15.53	15.92
20.0	17.93	18.98	19.65	20.15	20.54
25.0	22.54	23.60	24.27	24.77	25.16
			Consumption: case 1		
5.0	1.61	1.06	.90	.91	1.03
10.0	2.16	1.75	1.72	1.86	2.11
15.0	2.67	2.35	2.41	2.64	2.99
20.0	3.15	2.90	3.03	3.34	3.76
25.0	3.61	3.43	3.62	3.99	4.47
			Capital: case 2		
5.0	4.11	4.90	5.40	5.76	6.06
10.0	8.92	9.71	10.21	10.58	10.87
15.0	13.74	14.52	15.02	15.39	15.68
20.0	18.55	19.33	19.83	20.20	20.49
25.0	23.36	24.14	24.64	25.01	25.30
			Consumption: case 2		
5.0	1.57	1.29	1.30	1.45	1.67
10.0	1.93	1.79	1.93	2.20	2.55
15.0	2.24	2.19	2.43	2.79	3.23
20.0	2.53	2.55	2.86	3.30	3.81
25.0	2.80	2.88	3.25	3.75	4.33

Table 6. *Decision Rules for Euler-Equation Grid (Coleman)*

K_0	θ_1 .40	.70	1.00	1.30	1.60
			Capital: case 1		
5.0	4.88	5.27	5.67	6.08	6.49
10.0	9.43	9.90	10.40	10.90	11.41
15.0	13.96	14.49	15.04	15.60	16.18
20.0	18.47	19.04	19.65	20.26	20.89
25.0	22.99	23.59	24.23	24.89	25.56
			Consumption: case 1		
5.0	.80	.92	1.03	1.13	1.23
10.0	1.42	1.59	1.74	1.88	2.01
15.0	2.02	2.23	2.41	2.57	2.73
20.0	2.60	2.84	3.04	3.23	3.41
25.0	3.17	3.43	3.66	3.87	4.07
			Capital: case 2		
5.0	4.82	5.08	5.36	5.65	5.94
10.0	9.54	9.86	10.21	10.58	10.96
15.0	14.26	14.63	15.03	15.46	15.90
20.0	19.00	19.39	19.84	20.31	20.80
25.0	23.74	24.16	24.64	25.15	25.68
			Consumption: case 2		
5.0	.86	1.11	1.34	1.57	1.78
10.0	1.32	1.64	1.93	2.20	2.46
15.0	1.71	2.09	2.41	2.72	3.01
20.0	2.08	2.49	2.85	3.19	3.50
25.0	2.42	2.86	3.25	3.61	3.95

Table 7. Decision Rules for Extended Path (Gagnon)

	θ_1				
K_0	.40	.70	1.00	1.30	1.60
			Capital: case 1		
5	4.88	5.27	5.68	6.07	6.48
10	9.45	9.89	10.38	10.88	11.39
15	13.98	14.53	15.00	15.57	16.16
20	18.51	19.07	19.69	20.24	20.83
25	23.05	23.63	24.24	24.94	25.56
			Consumption: case 1		
5	.80	.92	1.04	1.14	1.24
10	1.40	1.60	1.75	1.89	2.03
15	2.00	2.19	2.44	2.61	2.76
20	2.56	2.81	3.00	3.26	3.47
25	3.10	3.39	3.62	3.82	4.07
			Capital: case 2		
5	4.82	5.06	5.34	5.62	5.91
10	9.55	9.87	10.18	10.54	10.91
15	14.30	14.66	15.00	15.40	15.84
20	19.07	19.45	19.90	20.23	20.71
25	23.87	24.26	24.73	25.09	25.57
			Consumption: case 2		
5	.86	1.13	1.36	1.59	1.81
10	1.31	1.62	1.96	2.24	2.51
15	1.68	2.05	2.44	2.78	3.07
20	2.00	2.43	2.79	3.27	3.59
25	2.28	2.77	3.16	3.67	4.06

Table 9. Decision Rules for Lin-LQ (McGrattan)

	θ_1				
K_0	.40	.70	1.00	1.30	1.60
			Capital: case 1		
5.0	4.07	5.13	5.80	6.30	6.69
10.0	8.69	9.75	10.42	10.91	11.31
15.0	13.31	14.36	15.04	15.53	15.92
20.0	17.93	18.98	19.65	20.15	20.54
25.0	22.54	23.60	24.27	24.77	25.15
			Consumption: case 1		
5.0	1.61	1.06	.90	.91	1.03
10.0	2.16	1.75	1.72	1.86	2.11
15.0	2.67	2.35	2.41	2.64	2.99
20.0	3.15	2.90	3.03	3.34	3.76
25.0	3.61	3.43	3.62	3.99	4.47
			Capital: case 2		
5.0	4.11	4.90	5.40	5.76	6.06
10.0	8.92	9.71	10.21	10.58	10.87
15.0	13.74	14.52	15.02	15.39	15.68
20.0	18.54	19.33	19.83	20.20	20.49
25.0	23.36	24.14	24.64	25.01	25.30
			Consumption: case 2		
5.0	1.57	1.29	1.30	1.45	1.67
10.0	1.93	1.70	1.93	2.20	2.55
15.0	2.24	2.19	2.43	2.79	3.22
20.0	2.53	2.55	2.86	3.30	3.81
25.0	2.80	2.88	3.25	3.75	4.3

from the grid values that automatically emerge from the method, so there is some question about the accuracy of these numbers as estimates of the exact solution. Given the small computation time for the linear-quadratic approximations, these preliminary results are very promising for the log-LQ method.

One puzzle about the linear-quadratic method (especially the lin-LQ version) is that the response of

Table 8. Decision Rules for Parameterizing Expectations (Marcet)

	θ_1				
K_0	.40	.70	1.00	1.30	1.60
			Capital: case 1		
5	4.84	5.27	5.70	6.15	6.61
10	9.39	9.88	10.39	10.93	11.47
15	13.95	14.47	15.03	15.61	16.20
20	18.51	19.05	19.64	20.25	20.88
25	23.09	23.64	24.24	24.88	25.54
			Consumption: case 1		
5	.84	.92	1.00	1.06	1.12
10	1.46	1.62	1.74	1.85	1.95
15	2.03	2.24	2.42	2.57	2.71
20	2.56	2.83	3.05	3.24	3.42
25	3.07	3.39	3.65	3.89	4.10
			Capital: case 2		
5	4.67	5.01	5.36	5.72	6.08
10	9.38	9.77	10.17	5.72	6.08
15	14.13	14.54	14.98	15.43	15.90
20	18.91	19.34	19.80	20.28	20.77
25	23.70	24.15	24.62	25.12	25.63
			Consumption: case 2		
5	1.01	1.18	1.34	1.49	1.64
10	1.48	1.73	1.97	2.19	2.41
15	1.85	2.17	2.46	2.74	3.01
20	2.17	2.54	2.89	3.22	3.53
25	2.45	2.88	3.27	3.64	3.99

Table 10. Decision Rules for Quadrature Grid (Tauchen)

	θ_1				
K_0	.40	.70	1.00	1.30	1.60
			Capital: case 1		
5	4.96	5.26	5.65	6.06	6.49
10	9.60	9.97	10.43	10.85	11.42
15	13.52	14.40	15.00	15.63	16.16
20	18.43	19.20	19.67	20.18	20.83
25	23.04	23.69	24.00	25.00	25.52
			Consumption: case 1		
5	.72	.93	1.05	1.15	1.23
10	1.26	1.54	1.70	1.93	2.00
15	2.46	2.31	2.44	2.55	2.75
20	2.64	2.68	3.02	3.32	3.47
25	3.12	3.33	3.89	3.76	4.11
			Capital: case 2		
5	4.69	5.05	5.37	5.63	5.96
10	9.60	9.95	10.24	10.57	10.98
15	14.40	14.40	15.00	15.48	15.81
20	19.20	19.20	20.00	20.18	20.83
25	24.00	24.00	24.91	25.00	25.65
			Consumption: case 2		
5	.99	1.14	1.33	1.58	1.76
10	1.26	1.55	1.90	2.21	2.44
15	1.58	2.31	2.44	2.70	3.10
20	1.87	2.68	2.69	3.32	3.47
25	2.16	3.02	2.99	3.76	3.98

consumption to the technology shock is surprisingly nonmonotonic; over some regions, lower values of the technology shock actually increase consumption, and over other regions, lower values more plausibly decrease consumption. This result may reflect the inaccuracies of the method that could lead to theoretical misconceptions. Note that the grid methods do not have this property. One exception is the rise in the value of C_t in case 1 for Tauchen's method when $K_{t-1} = 15$ and θ_t falls from .7 to .4. This does not occur for Coleman's solution at this point.

There is also a broad similarity between the results for the extended-path method and the two grid methods of Tauchen and Coleman. Given the relatively low cost of the extended-path method, these results are promising, especially for application in higher dimension problems or in problems that are mixtures of optimization equations and other equations. Note that the extended-path method does not have the nonmonotonicity property mentioned in the previous paragraph. The decision rule for consumption shows that consumption is a positive function of the technology shock over the entire region of initial capital stocks and technology shocks.

4.3 Summary Statistics

From the stochastic simulations, the contemporaneous covariance matrix of $(C_t, K_t)'$, univariate autoregressions of C_t and K_t [AR(1), AR(2), AR(3), and AR(4)] and bivariate autoregressions of $(C_t, K_t)'$ [VAR(1), VAR(2), VAR(3), and VAR(4)] were computed. These statistics are available on request. All of the statistics reveal a high degree of serial dependence for consumption and capital and a high degree of correlation between consumption and capital. These properties were also evident from the time series charts.

In addition, four other summary statistics were computed and are reported and discussed hereafter. These include the following:

1. The statistic

$$m = \hat{a}'(\Sigma x_t' x_t)(\Sigma x_t' x_t \eta_t^2)^{-1}(\Sigma x_t' x_t)\hat{a}, \tag{4}$$

where

$$\hat{a} = (\Sigma x_t' x_t)^{-1}(\Sigma x_t' \eta_t) \tag{5}$$

is the usual ordinary least squares estimator in a regression of the Euler-equation residual

$$\eta_t = \beta C_t^{-\tau}(1 + \alpha\theta_t K_{t-1}^{\alpha-1})C_{t-1}^{\tau} - 1 \tag{6}$$

on a list x_t of a constant and five lags of consumption and θ. The statistic m provides a test for the martingale-difference property $E_{t-1}\eta_t = 0$, a property that is satisfied by the theoretical solution. Focusing on η_t and the statistic m was suggested by Den Haan and Marcet (1989) as a way to overcome the fact that an analytical solution to this problem was not available. We call m the Den Haan–Marcet statistic in the sequel. The statistic is closely related to the statistic suggested by White (1980).

2. TR^2 from the regression of the productivity shock ε_t on five lags of consumption, capital, and θ. The idea is to test for the martingale-difference property $E_{t-1}\varepsilon_t = 0$.

3. R^2 from the regressions of the first difference of consumption on both lagged consumption and capital. This is a test of the random-walk hypothesis for consumption; note that in general the random-walk hypothesis will not hold with the utility function in this simple growth model, but the differences in the test statistic are a useful way to assess the different solution methods.

4. Ratios of the variance of investment to the variance of the change in consumption. This ratio is a measure of the relative volatility of consumption and investment, a frequently discussed feature of economic fluctuations. (Note that this ratio has a flow variable in the numerator and a change in a flow in the denominator but still is a useful measure of relative volatility.)

The differences among the methods turned out to be quite substantial for some of these statistics. The results for the statistic m (for η_t) are found in Table 11. Under the null hypothesis of a martingale difference, this statistic has approximately a $\chi^2(11)$ distribution asymptotically (see Den Haan and Marcet 1989). A two-sided test at a significance level of 2.5% for each side would be $3.82 < m < 21.92$, using the asymptotic distribution. Unless, of course, a solution method works directly to enforce the Euler equation (like the backward-solution methods, in which case the statistic m must be $\chi^2(11)$ by construction), the Euler-equation residual is likely to have a predictable component, which will be picked up by this statistic.

The same approach can be used for ε_t, although we do not have to correct for heteroscedasticity here. Thus the statistic TR^2 suffices. The test statistics are reported in Table 12. Since there are 15 regressors plus a constant term in each regression, TR_2 has an asymptotic $\chi^2(15)$ distribution. Observe that this test does not detect a deviation from 0 for the mean of the residual. A solution method would probably not generate a systematic bias without being linked to past data in the model, however. The majority of the methods generated the technology shocks directly from a random number generator, in which case the test statistic is $\chi^2(15)$ by construction. But several methods do not, or they generate the shocks for a slightly modified problem. In these cases, Table 12 provides a genuine accuracy check. The two-sided test at the significance level of 2.5% for each side is given by $6.26 < TR^2 < 27.49$.

Table 13 shows the significance of a regression of the first difference in consumption on past data, which is a test for the random-walk hypothesis for consumption in the simulated data. We report the R^2 statistic. An R^2 close to 0 supports the random-walk hypothesis.

Table 11. The Den Haan–Marcet Statistic m for the Martingale Difference on η_t

Method	Case 1	2	3	4	5	6	7	8	9	10
Baxter	21	390	303	527	321	595	187	781	1,172	290
Christiano										
Log-LQ-Normal	56	32	34	44	17	10	18	28	16	12
Log-LQ-discrete	128	158	110	112	67	59	69	74	76	78
Lin-LQ-Normal	82	73	42	61	64	16	11	65	23	13
Lin-LQ-discrete	174	64	78	29	64	44	43	57	24	99
Value-function grid					30	27	26	39	22	18
Coleman	18	14	18	24	18	13	11	15	12	12
Gagnon	71	24	50	31	61	20	18	60	27	23
Ingram	11	8	11	10	10	11	12	8	11	12
Labadie	122				85		161			
Marcet	27	22	22	25	18	18	12	30	7	9
McGrattan	84	55	69	22	96	22	17	62	26	21
Sims	11	14	12	12	12	12	12	11	12	10
Tauchen	584	396	284	153	704	558	502	322	234	215

The two bottom lines of the table report the total range and the range for those simulations that were within the confidence range for both the statistics m and TR^2. Since the random-walk hypothesis might be considered an important issue in this model, the finding that the different solution techniques seem to be rather far apart are disturbing. The different solution methods are delivering different answers to the same question.

Restricting the comparison to those models that passed the preceding tests narrows the range substantially, however. In case 4 this narrowing may occur simply because the range is smaller for a much smaller number of models (in case 4, only Sims's method), but in other cases there are a fairly large number of methods and the range is small. In unpublished work (and using a TR^2 statistic instead of the Den Haan–Marcet statistic m also for η_t), Sims demonstrated that discrimination based on the TR^2 test increases agreement among the methods. He showed this by using a weighted regression approach in which methods with high TR^2 values in both the η_t and the ε_t test are given less weight.

Note also that there is at least one general pattern of some economic interest that emerges from all of the methods with few exceptions; as the coefficient of relative risk aversion rises, the tabulated R^2 for the random walk declines. The exceptions are the Gagnon extended-path results in cases 3–4 and cases 8–10, the Christiano value-function grid for cases 9–10, and the Coleman Euler-equation grid for cases 9–10. There is no evidence either way on this issue for the methods of Baxter and Labadie.

In Table 14, we report the ratio of the variance of investment to the variance of the first difference of consumption. As noted previously, this ratio is meant to measure the relative volatilities of investment and consumption. The four lines at the bottom of Table 12 report the total range of the methods, as well as those that were within the stated range for both the m statistic and the TR^2 statistic. Again the results show large differences among the different methods. For methods that only allow discrete choices for some or all of their variables, differences can arise if one variable is bearing relatively too much of the adjustment burden either because the grid is much finer for that variable or because the variable is chosen in a continuum to begin with. This makes Tauchen's numbers, in particular,

Table 12. TR^2 Statistic for the Martingale Difference on ε_t

Method	Case 1	2	3	4	5	6	7	8	9	10
Baxter	30	673	317	214	254	187	189	66	230	267
Christiano										
Log-LQ-Normal	31	16	12	25	10	10	19	20	25	16
Log-LQ-discrete	30	31	31	34	33	32	33	32	34	35
Lin-LQ-Normal	16	18	17	13	14	8	11	6	20	15
Lin-LQ-discrete	14	16	16	14	15	15	15	16	16	15
Value-function grid					12	12	17	16	14	17
Coleman	22	19	19	19	24	21	21	20	21	23
Gagnon	17	19	16	16	21	19	19	19	18	17
Ingram	46	123	75	172	17	165	394	15	203	381
Labadie	122				167		107			
Marcet	12	11	11	12	15	14	13	15	14	14
McGrattan	21	20	18	14	19	19	19	19	17	16
Sims	27	26	22	19	24	24	22	19	16	14
Tauchen	8	9	19	12	11	9	14	16	13	10

Table 13. R^2 Tests for Random Walk for Consumption

Method	Case 1	2	3	4	5	6	7	8	9	10
Baxter	.36	.01	.02	.02	.07	.03	.02	.07	.02	.05
Christiano										
Log-LQ-Normal	.36	.04	.12	.04	.43	.05	.02	.24	.03	.01
Log-LQ-discrete	.46	.16	.26	.12	.42	.08	.05	.37	.06	.04
Lin-LQ-Normal	.16	.06	.08	.04	.34	.05	.02	.28	.04	.01
Lin-LQ-discrete	.14	.03	.05	.02	.33	.03	.01	.20	.01	.01
Value-function grid					.37	.04	.02	.29	.02	.02
Coleman	.40	.05	.29	.03	.41	.05	.02	.30	.02	.01
Gagnon	.07	.05	.05	.06	.15	.02	.02	.05	.04	.04
Ingram	.35	.18	.26	.10	.44	.06	.03	.33	.04	.02
Labadie	.91				.98		.99			
Marcet	.40	.04	.33	.03	.42	.06	.03	.35	.04	.02
McGrattan	.13	.04	.07	.04	.34	.04	.02	.21	.02	.01
Sims	.41	.06	.32	.04	.44	.07	.04	.36	.04	.02
Tauchen	.50	.38	.34	.27	.50	.38	.33	.34	.27	.27
Max/min for group										
Min	.07	.01	.02	.02	.07	.02	.01	.05	.01	.01
Max	.91	.38	.34	.27	.98	.38	.99	.37	.27	.27
Max/min for subgroup										
Min	.40	.04	.29	.04	.41	.02	.02	.30	.02	.01
Max	.41	.06	.33	.04	.44	.07	.04	.36	.04	.02

NOTE: The subgroup consists of solution methods that are within the symmetric 95% confidence bands in Table 11 and Table 12.

very small: in his method, it is mainly the consumption series that adjusts (look also at the time series plots discussed previously). Note also the dependencies of the results on the parameters of each case.

4.4 Computing Times

In Table 15, we compare the computing times in seconds. The data were reported to us by the individual researchers. Time 1 refers to the computation of the decision rules, whereas time 2 is the time needed to compute the simulations. The numbers are hard to compare because they certainly vary strongly with the machine and the software used, as well as with the precision desired and the number of grid points, for example. It is desirable to perform all calculations on the same machine with the same software and with some common standard for precision in future comparisons.

Still it is probably fair to state that the methods of, for example, Baxter, Gagnon, Tauchen, and Christiano's value-function grid—that is, grid methods and

Table 14. Ratios of the Variance of Investment to the Variance of the First Difference in Consumption

Method	Case 1	2	3	4	5	6	7	8	9	10
Baxter	30	5	9	3	5	1	1	2	3	8
Christiano										
Log-LQ-Normal	24	9	45	36	29	11	8	132	59	45
Log-LQ-discrete	25	23	80	74	29	12	12	167	79	83
Lin-LQ-Normal	8	4	23	13	25	10	8	136	50	48
Lin-LQ-discrete	8	3	21	11	24	9	7	114	43	37
Value-function grid					28	10	8	149	53	47
Coleman	29	12	139	37	29	10	8	155	53	47
Gagnon	10	3	5	2	17	5	3	8	4	3
Ingram	29	170	155	490	30	12	20	162	66	98
Labadie	56				61		481			
Marcet	28	9	168	55	30	13	10	178	78	74
McGrattan	6	3	17	10	24	9	7	112	44	38
Sims	30	12	165	64	31	13	11	171	66	59
Tauchen	3	2	2	2	3	2	2	2	2	2
Max/min for group										
Min	3	2	2	2	3	1	1	2	2	2
Max	56	170	168	490	61	13	481	178	79	98
Max/min for subgroup										
Min	29	9	139	64	29	5	3	155	53	38
Max	30	12	168	64	31	13	11	171	78	74

NOTE: The subgroup consists of solution methods that are within the symmetric 95% confidence bands in Table 11 and Table 12.

Table 15. Computing Times

Method	Machine	Co-chip	Megahertz	Software	Time 1	Time 2
Baxter	IBM PS2-80	80287	16	FORTRAN 3.31	1,188.0	
				Matlab 3.13		164.0
Christiano						
Log-LQ-Normal	Amdahl 5860		244	RATS	Total .6	
Log-LQ-discrete	Amdahl 5860		244	RATS	Total 1.2	
Lin-LQ-Normal	Amdahl 5860		244	RATS	Total .6	
Lin-LQ-discrete	Amdahl 5860		244	RATS	Total 1.2	
Value-function grid	Amdahl 5860			IBM FORTRAN 1.4.1	Total 5 hours	
Coleman	Amdahl 5890-300			VS-FORTRAN 2.3	110.92	.32
Gagnon	Amdahl 5850			TROLL 13.0	396	5,320
Ingram	HP Vectra ES/12	80287	12/10	GAUSS 2.0	Total 72.01	
Labadie	IBM Model 30	Yes		GAUSS 1.496	Total 4 hours	
Marcet	Compaq 386/25	Weitek	25	R. McFarland FORTRAN	Total 60	
McGrattan	Compaq 386/20	Weitek 1167	20	Matlab 3.25	Total 0.7	
Sims	Dell System 310	80386	20	MICROSOFT C	20	107
Tauchen	Compaq 386/25	80387-25	25	GAUSS 1.49b	Total 2,768	

NOTE: Time 1 refers to the central processing unit (CPU) time in seconds to compute the decision rule for one case (typically case 1). Time 2 refers to the CPU time in seconds to compute a simulation of 2,000 data points for one case (typically case 1). The term total indicates that the sum of time 1 and time 2 is given. In Christiano's value-function grid, 20,000 grid points were used.

the extended-path method—are computationally quite involved, whereas linear-quadratic methods are typically quite fast for the simple stochastic growth model.

One should recognize that differences in computing costs can be enormous once the problem at hand goes beyond only a few dimensions and the "curse of dimensionality" starts to matter. It might be quite impossible to compute the solution for a model with 15 state variables, say, using some grid method. Methods that work with linear-quadratic approximation or parameterizing expectations (including backsolving) or extended-path methods will still be available at reasonable costs for these problems, however.

5. CONCLUSION

The conclusions from this comparison of different solution techniques for nonlinear rational-expectations models can be summarized briefly as follows.

1. The simulated sample paths generated by the different solution methods have significantly different properties. Although certain common time series features of the behavior of consumption and investment emerge from time series plots for all the methods, other features show up in the empirical density functions and scatter diagrams that reveal quite different behavior even though the same model is being solved by each method.

2. The decision rules indicate that some of the easily computed rules—the linear-quadratic (log-LQ) method and the extended-path method—are fairly close to the "exact" decision rule as represented here by the quadrature-value-function-grid method of Tauchen or the Euler-equation grid method of Coleman. Given the relatively low computation times for these methods and their relatively easy generalization to higher dimensions, it is important to establish whether this property holds up in other problems. Neither the log-LQ nor the extended-path method performs particularly well in the martingale-difference tests for the Euler-equation residual, however.

3. Summary statistics, which researchers might typically examine to test theoretical hypotheses, are significantly different for many of the solution methods, even though the theoretical problem solved is exactly the same for each method. For example, the solution methods give very different answers to basic questions concerning the relative volatility of investment and consumption. There is some similarity among the methods in detecting the effects of risk aversion on random-walk consumption behavior, however, and the methods that satisfy both the Den Haan–Marcet test for the accuracy of the Euler equation and the TR^2 test for the distribution of the disturbance term—Sims's backsolving implementation, Marcet's parameterizing-expectations method, and Coleman's Euler-equation iteration method—produce similar summary statistics and plots.

Given these large differences in the solution methods, the most obvious question is who won? Unfortunately, this question is still very difficult to answer, the criteria of success for the solution methods are different. For some researchers, the appropriate measuring stick might be the closeness of the numerical solution to the true decision rule. Grid methods are likely to do very well here, and we noted that the log-LQ and the extended-path methods come close to the grid methods in terms of the decision rules. For others, it is computing time that is most important, as long as the results are within reason. This might be the case for estimation applications or with applications with a large number of state variables. Applications of this type can potentially exhibit financially significant savings in computing time when solved with methods that work with linear-quadratic approximations or parameterization of the expectations or extended-path methods instead of one of the grid methods. In other applications, it might be important to be accurate with respect to first-order conditions to test, for example, asset-pricing relationships;

Sims's backsolving method or Marcet's parameterizing-expectations method are likely to perform very well in this respect. Finally, the level of difficulty and the judgment required to implement a particular method can be of great importance to the practitioner.

The comparisons performed previously did not single out one or several of the methods as performing at the very top in every respect. For a researcher who wants to select one of the techniques, it seems important to consider the particular problem and the budget constraint. Researchers might want to be careful not to use any solution method blindly hoping that the results are within acceptable bounds. An article that relies primarily on one method could include at least a partial set of results using an alternative, preferably unrelated method as an accuracy check and a diagnostic of potential areas where results or inference might be distorted. For example, a researcher who uses linear-quadratic methods might want to compare the results to those from some grid method for a few simple cases. Tests like the Den Haan–Marcet statistic seem reasonable as an additional diagnostic device. More checks of this type are desirable.

Even in a simple model such as that considered in this article, the different solution methods can yield quite different econometric results. It is essential to get a better understanding of where these differences come from and how big they can be in a particular application before relying too much on conclusions drawn from these solution methods.

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REFERENCES

Baxter, M., Crucini, M. J., and Rouwenhorst, K. G. (1990), "Solving the Stochastic Growth Model by a Discrete-State-Space, Euler-Equation Approach," *Journal of Business and Economic Statistics*, 8, 19–21.

Christiano, L. J. (1990), "Solving the Stochastic Growth Model by Linear-Quadratic Approximation and by Value-Function Iteration," *Journal of Business and Economic Statistics*, 8, 23–26.

Coleman, W. J., II (1990), "Solving the Stochastic Growth Model by Policy-Function Iteration," *Journal of Business and Economic Statistics*, 8, 27–29.

Den Haan, W. J., and Marcet, A. (1989), "Accuracy in Simulations," unpublished manuscript, Carnegie-Mellon University, Graduate School of Industrial Administration.

——— (1990), "Solving the Stochastic Growth Model by Parameterizing Expectations," *Journal of Business and Economic Statistics*, 8, 31–34.

Fair, R. C., and Taylor, J. B. (1983), "Solution and Maximum Likelihood Estimation of Dynamic Nonlinear Rational Expectations Models," *Econometrica*, 51, 1169–1185.

Gagnon, J. E. (1990), "Solving the Stochastic Growth Model by Deterministic Extended Path," *Journal of Business and Economic Statistics*, 8, 35–36.

Ingram, B. F. (1990), "Solving the Stochastic Growth Model by Backsolving With an Expanded Shock Space," *Journal of Business and Economic Statistics*, 8, 37–38.

Kydland, F., and Prescott, E. (1982), "Time to Build and Aggregate Fluctuations," *Econometrica*, 50, 1345–1370.

Labadie, P. (1986), "Solving Nonlinear Rational Expectations Models for Estimation: An Application to an Asset Pricing Model With Currency," unpublished manuscript, Columbia University, Graduate School of Business.

——— (1990), "Solving the Stochastic Growth Model by Using a Recursive Mapping Based on Least Squares Projection," *Journal of Business and Economic Statistics*, 8, 39–40.

Marcet, A. (1988), "Solving Non-linear Models by Parameterizing Expectations," unpublished manuscript, Carnegie-Mellon University, Graduate School of Industrial Administration.

McGrattan, E. (1990), "Solving the Stochastic Growth Model by Linear-Quadratic Approximation," *Journal of Business and Economic Statistics*, 8, 41–44.

Sargent, T. J. (1987), *Dynamic Macroeconomic Theory*, Cambridge, MA: Harvard University Press.

Sims, C. A. (1984), "Solving Nonlinear Stochastic Equilibrium Models 'Backwards'," unpublished manuscript, University of Minnesota, Dept. of Economics.

——— (1989), "Solving Nonlinear Stochastic Optimization and Equilibrium Problems Backwards," Discussion Paper 15, Institute for Empirical Macroeconomics, Federal Reserve Bank of Minneapolis.

——— (1990), "Solving the Stochastic Growth Model by Backsolving With a Particular Nonlinear Form for the Decision Rule," *Journal of Business and Economic Statistics*, 8, 45–47.

Tauchen, G. (1987), "Quadrature-Based Methods for Obtaining Approximate Solutions to Nonlinear Asset Pricing Models," technical report, Duke University, Dept. of Economics.

——— (1990), "Solving the Stochastic Growth Model by Using Quadrature Methods and Value-Function Iterations," *Journal of Business and Economic Statistics*, 8, 49–51.

White, H. (1980), "A Heteroskedasticity-Consistent Covariance Matrix Estimator and a Direct Test for Heteroskedasticity," *Econometrica*, 48, 817–839.