The Long and the Short End of the Term Structure of Policy Rules

By

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Abstract

We first document a large secular shift in the estimated response of the entire term structure of interest rates to inflation and output in the United States. The shift occurred in the early 1980s. We then derive an equation that links these responses to the coefficients of the central bank's monetary policy rule for the short-term interest rate. The equation reveals two countervailing forces that help explain and understand the nature of the link and how its sign is determined. Using this equation, we show that a shift in the policy rule in the early 1980s provides an explanation for the observed shift in the term structure. We also explore a shift in the policy rule in the 2002-2005 period and its possible effect on long-term rates.

One of the most debated issues in monetary economics concerns the impact of monetary policy on the term structure of interest rates. Approaching this issue from the perspective of monetary policy rules seems promising because long-term interest rates depend on expectations of future short-term rates, which are determined by the response of the central bank to future developments in the economy, a response most easily captured by a policy rule.

In this paper we investigate how shifts in the central bank's policy rule cause shifts in the term structure of interest rates. We focus on a new representation of the term structure in which long-term interest rates are related to inflation and output, much as a monetary policy rule describes how the short-term interest rate is related to inflation and output. The term structure is thus simply a series of implied "policy rules" for long-term interest rates—one policy rule for each maturity—with "response coefficients" measuring the size of the interest rate reaction. We find that these implied policy rules are very useful for understanding the impact of monetary policy because longer-term interest rates have powerful effects on spending and asset allocation decisions not captured by short-term interest rates.

To begin our analysis, we empirically document a dramatic secular shift over the last several decades in the size of the response coefficients of long-term interest rates to inflation and output in the United States. One important characteristic of this shift is that an increase in inflation has been associated with a larger rise in long-term interest rates in the decades since the mid 1980s than in the 1960s and 1970s. Another is that long-term interest rates have been responding more to real output fluctuations. We then show that a theory of monetary policy based on policy rules can explain and help understand this empirical finding. Using no-arbitrage pricing methods developed by Ang, Dong, and Piazzesi (2005), we derive analytically an equation relating the response coefficients in the implied long-term interest rate rules to the response coefficients of the short-term interest rate rule of the central bank. This equation takes risk premia into account and reveals two countervailing effects of shifts in the policy rule on the long-term yield equations. By differentiating this equation with respect to the response coefficients in the monetary policy rule, we derive our main result: a secular shift in the monetary policy rule in the mid 1980s in United States explains the large shift in the term structure. Previous work exploring the impacts of monetary policy rule shifts on longer-term interest rates by Fuhrer (1996) used the pure expectations model of the term structure and thus did not examine these longer-term response coefficients.

The secular shift in the mid 1980s is not the only possible regime shift in monetary policy in recent years in the United States. During the period from 2003 to 2005, the Federal Funds rate deviated significantly from what would have been predicted by the policy response that was typical over the period since the mid-1980s. Though this two- to three-year period is comparatively short for determining whether market participants interpreted this as a regime shift or a temporary deviation from the post mid-1980s regime, the interest rate response to inflation does seem much lower. A perception of a smaller response coefficient could have led market participants to expect smaller interest rate responses to inflation in the future, and therefore lower long term interest rate responses. If so, our model provides an explanation for the puzzle—coined the “conundrum” by former Federal Reserve Chairman Alan Greenspan—that the rise in the Federal Funds rate starting in 2004 did not bring about an increase in long-term interest rates as would have been expected based on experience over the previous 20 years.

1. The Response Coefficients

We consider a representation of the term structure in which long-term interest rates are linear functions of macroeconomic variables. The specific objects of our investigation are the slopes of these linear functions, which we refer to as the *response coefficients*, because they describe how yields on bonds of different maturities respond to macroeconomic developments. These response coefficients are not individual behavioral parameters; rather they represent the interaction of all participants in the bond markets and other parts of the economy. We are interested in the term structure pattern of these response coefficients and how they change over time. We focus on the responses to two macroeconomic variables—inflation and the real GDP gap—because these are the two variables that are most important in short-term interest rate rules that describe the behavior of central banks.

The simplest linear function occurs when long term rates depend only on inflation and a random error term. Letting $i_t^{(n)}$ be the yield to maturity on bonds with maturity *n*, we then have:

(1) $$i_t^{(n)} = a_n + b_n \pi_t + \eta_t ,$$

where π_t is the inflation rate and η_t is an error term. The coefficients a_n are the intercepts and the coefficients b_n are the response coefficients for maturities $n = 1,\dots, N$. Clearly, the sizes of b_n are important for the overall behavior of the economy. If the b_n are large, then an increase in inflation will bring about an increase in the yields on bonds with those maturities and thereby affect spending by firms or consumers who are borrowing funds at those maturities. A very small value of the response coefficients—say less than one—could lead to such a small increase in yields that the real interest rate (computed with an expected inflation rate corresponding to the maturity of the bonds) could fall with an increase in inflation and thereby exacerbate the rise in inflation.

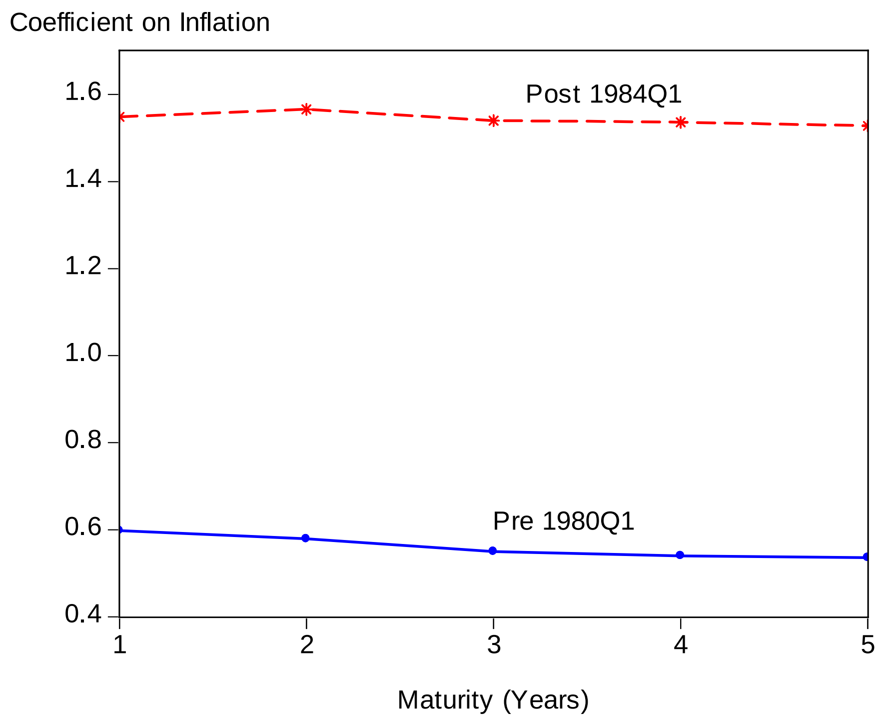
Table 1 presents ordinary least squares estimates of the response coefficients b_n for a range of maturities and for two sample periods: 1960Q1 – 1979Q4 and 1984Q1 – 2006Q4. The zero-coupon bond yields are quarterly averages of monthly CRSP data on U.S. Treasury yields at one- through five-year maturities. We choose these maturities since they are the available maturities in the CRSP database that have been converted to zero-coupon yields. The inflation measure is the four-quarter moving average of the percentage change of the U.S. GDP chain-weighted price index. Figure 1 depicts the pattern of the coefficients graphically.

Note the dramatic secular shift in these coefficients between the two sample periods. For all maturities, the response coefficients are much larger in the second period than in the first period. There is little or no tendency for the coefficients to decline with maturity in either period; tests of the null hypothesis that the coefficients are equal at all maturities cannot be rejected for either period.

Table 1: Estimated Term Structure Response Coefficients for Inflation

Maturity (years)	b_n for 1960Q1 – 1979Q4	b_n for 1984Q1 – 2006Q4
1	0.598	1.549
2	0.579	1.566
3	0.550	1.540
4	0.540	1.536
5	0.536	1.528

Figure 1: Pattern of the Response Coefficients



A similarly large shift is seen if we include a real macroeconomic variable along with the inflation rate in the linear term structure equations. Consider the series of equations:

$$(2) \quad i_t^{(n)} = a_n + b_{1,n} y_t + b_{2,n} \pi_t + \eta_t ,$$

where y_t is the percentage deviation of real GDP from trend (estimated by a Hodrick-Prescott filter). Table 2 reports least squares estimates of the two response coefficients $b_{1,n}$ and $b_{2,n}$ in equation (2), and their term structure pattern is shown graphically in Figures 2A and 2B. Note that the response coefficients for inflation are again much larger in the second period, and a similar upward shift is now evident for the responses of the yields to the output gap. Observe that the response coefficients to the output gap decline with maturity, but, as with the simple regressions, there is no such tendency for the response coefficients to inflation to decline with maturity.

Table 2: Estimated Term Structure Response Coefficients for Inflation and Output

Maturity (years)	$b_{1,n}$ for 1960Q1 – 1979Q4	$b_{2,n}$ for 1960Q1 – 1979Q4	$b_{1,n}$ for 1984Q1 – 2006Q4	$b_{2,n}$ for 1984Q1 – 2006Q4
1	0.454	0.604	1.244	1.224
2	0.335	0.602	1.124	1.273
3	0.259	0.580	1.000	1.279
4	0.222	0.573	0.888	1.304
5	0.191	0.573	0.806	1.318

Figure 2A: Pattern of Output Gap Response Coefficients

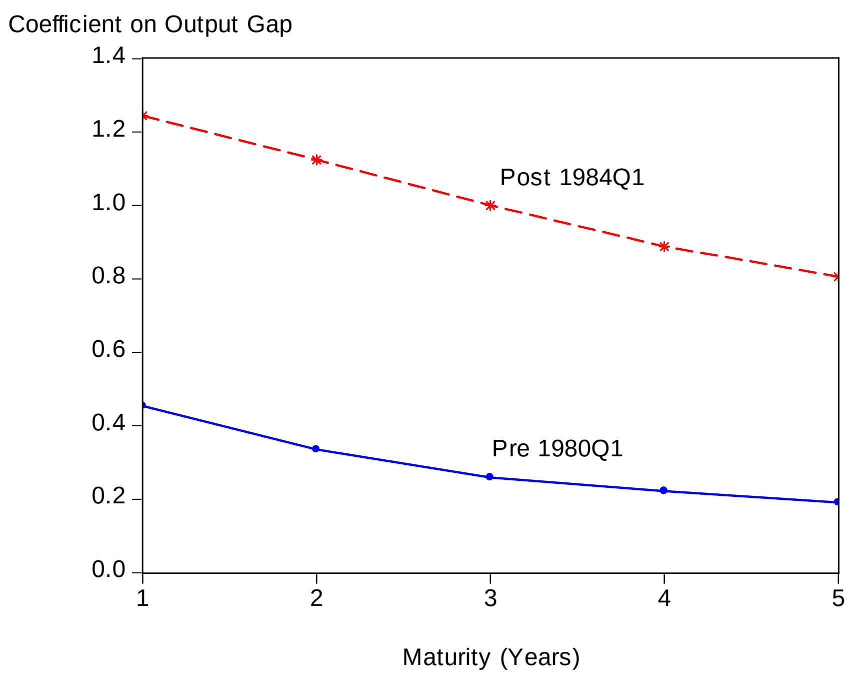
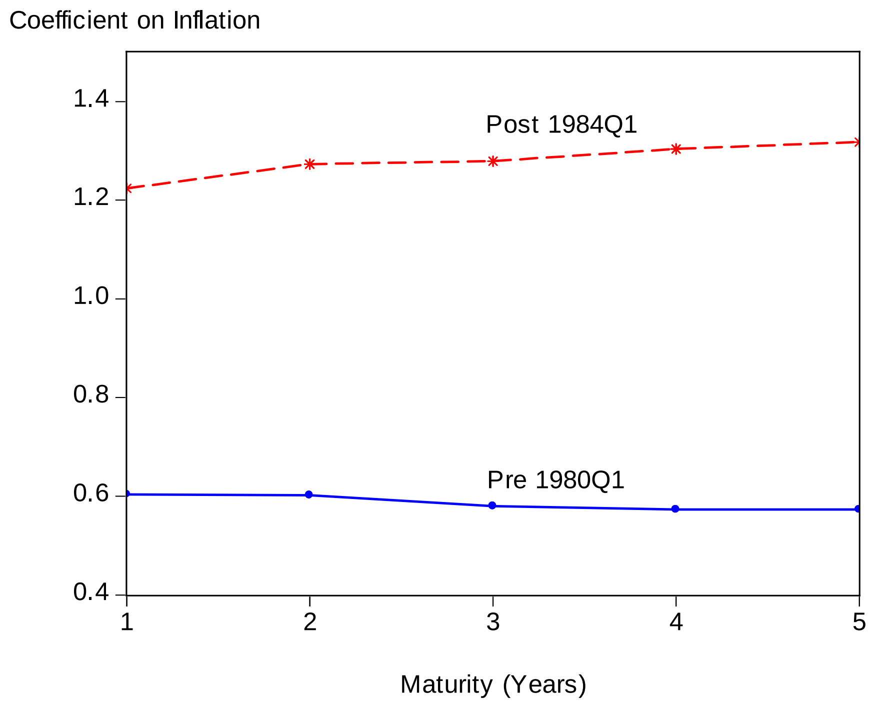


Figure 2B: Pattern of Inflation Response Coefficients



The regressions reported in Tables 1 and 2 only go out to a maturity of five years because of CRSP data availability for zero-coupon Treasury yields. To help assess how longer-term yields might respond, we also estimated equations (1) and (2) using monthly Constant Maturity Treasury (CMT) yields averaged to create a quarterly series for each of the one-, three-, five-, ten-, and twenty-year bond yields. The CMT yields are estimated with a cubic spline which approximates the zero-coupon yields used in (1) and (2), thereby enabling us to examine how the longer-term yields respond to inflation and output. We obtained the CMT data from the FRED database of the St. Louis Federal Reserve Bank. Tables 3 and 4 report the estimated regressions (1) and (2), respectively, using the CMT yields.

Table 3: Estimated Term Structure Response Coefficients for Inflation using CMT Yields

Maturity (years)	b_n for 1960Q1 – 1979Q4	b_n for 1984Q1 – 2006Q4
1	0.637	1.579
3	0.595	1.604
5	0.591	1.583
10	0.584	1.543
20	0.610	1.432

Table 4: Estimated Term Structure Response Coefficients for Inflation and Output using CMT Yields

Maturity (years)	$b_{1,n}$ for 1960Q1 – 1979Q4	$b_{2,n}$ for 1960Q1 – 1979Q4	$b_{1,n}$ for 1984Q1 – 2006Q4	$b_{2,n}$ for 1984Q1 – 2006Q4
1	0.446	0.636	1.233	1.257
3	0.262	0.595	1.031	1.335
5	0.194	0.590	0.817	1.370
10	0.143	0.584	0.591	1.389
20	0.132	0.610	0.745	1.246

We see a similar pattern occurring with the CMT yields as with the zero-coupon yields; the second sub-sample has higher response coefficients for the macro variables, even if we go out as far as twenty years. For the shorter maturities the estimated coefficients with CMT yields are almost identical to those with the zero-coupon yields. It should be noted that the twenty-year yield is not available for 1987Q1 – 1993Q3, thus slightly reducing the accuracy of the estimates of the response coefficients for the second sub-sample.[1]

This evidence of a shift in response coefficients is not sensitive to the exact choice of sample periods. Similar results are obtained, for example, if the observations between 1980Q1 and 1984Q1 are included in the second sample. We have chosen the early- to mid-1980s as a break point because it is around that period that many researchers have documented a regime

[1] We performed similar regressions using the Treasury yield curve derived in Gurkaynak, Sack, and Wright (2006). The empirical shift is robust across these zero-coupon bond yields, and the quantitative difference between the reported coefficient estimates and those using this new zero-coupon bond data is extremely small.

change in monetary policy. The change occurred around the time that Paul Volcker began to pursue a different approach to monetary policy, and has lasted for the next two decades under his successors. Clarida, Gali, and Gertler (2000), Taylor (1999), and Woodford (2003) have previously reported that the response coefficients to inflation and output in a Taylor rule shifted upwards around that time, and such a policy regime shift is also evident in the data and time periods in Tables 1 through 4. Using the federal funds rate, denoted r_t, the policy rule regression results are as follows:

1960Q1 – 1979Q4: $r_t = 2.448 + 0.970\pi_t$

1984Q1 – 2006Q4: $r_t = 1.312 + 1.579\pi_t$

1960Q1 – 1979Q4: $r_t = 2.435 + 0.475 y_t + 0.975\pi_t$

1984Q1 – 2006Q4: $r_t = 2.022 + 1.322 y_t + 1.234\pi_t$.

Note how the monetary policy response coefficients are much larger in the second sub-sample. The reaction coefficient on inflation is less than one in the first sub-sample and shifted to a value substantially greater than one in the second sub-sample. The reaction coefficient on output also shows a much greater responsiveness in the second period.

The obvious question is whether this change in the monetary policy response coefficients can explain the changes in the entire term structure of response coefficients. In the next two sections, we introduce a simple model and use it to derive an equation that shows that there is indeed an intimate connection between the policy rule and the term structure that helps us explain and understand the empirical results for the term structure.

2. The Case where the Monetary Policy Rule Depends Only On Inflation

We begin with a model that enables us to derive a simple equation relating the response coefficient on inflation in equation (1) to the central bank's policy rule coefficients. In this model, the central bank responds to inflation, but not to output. Hence, as we will show, long-term interest rates also respond to inflation, but not independently to output, so that the model implies a relationship of the form of equation (1). The model has the following equations:

$$(3) \quad r_t = \delta \pi_t$$

$$(4) \quad i_t^{(n)} = n^{-1} \log(P_t^{(n)})$$

$$(5) \quad P_t^{(n+1)} = E_t[m_{t+1} P_{t+1}^{(n)}]$$

$$(6) \quad m_{t+1} = \exp(-r_t - 0.5\lambda_t^2 - \lambda_t \varepsilon_{t+1})$$

$$(7) \quad \lambda_t = -\gamma_0 - \gamma_1 \pi_t$$

$$(8) \quad \pi_t = \pi_{t-1} - \phi(r_{t-1} - \pi_{t-1}) + \sigma \varepsilon_t,$$

where the shock $\varepsilon_t \sim iid\ N(0,1)$. Equation (3) is the monetary policy rule in which the short-term nominal interest rate r_t depends on the inflation rate with a policy response coefficient $\delta >$ 0. Equation (4) gives the yield to maturity of a zero-coupon bond with a face value of 1, where $P_t^{(n)}$ is the price of the bond at time t. Equation (5) is a no-arbitrage condition showing that the price of an n+1 period bond at time t must equal the expected present discounted value of the price of an n period bond at time t+1, where m_t is the stochastic discount factor (or pricing kernel). Equation (6) describes this stochastic discount factor, which has the convenient functional form used in the affine term structure literature. Equation (7) shows that the risk term λ_t in equation (6) depends on two coefficients: γ_0, which represents a constant risk premium, and γ_1, which represents the time-varying risk premium attributed to changes in inflation. As we will show the more positive is γ_1, the more long-term yields respond positively to shocks in inflation. Finally, equation (8) describes how monetary policy affects inflation. It is a price adjustment equation in which the change in inflation depends on the lagged real interest rate, which we simply assume depends on the ex-post real interest rate through the parameter $\phi > 0$.

The affine term structure equations (4) through (7) are simplifications of assumptions in Ang, Dong, and Piazzesi (2005). These authors also assume that macroeconomic variables (π in this simple model) evolve according to an autoregression, which does not depend on the policy rule. To answer the questions posed here about the impact of regime shifts on the term structure, it is necessary to describe how the interest rate affects inflation, and for this reason we introduce a simple structure which assumes the interest rate transmission mechanism in equation (8). This effect would be ignored by a vector autoregression model with constant coefficients, leading to errors similar to those pointed out in the Lucas critique. It is possible, of course, to improve on equation (8) and perhaps better account for the Lucas critique by introducing, for example, a forward-looking optimization model, or perhaps staggered price setting, but the simple form of (8) allows us to obtain analytic results and focus on the term structure relations. Examples where more complex structural models have been combined with affine models of the term structure are Bekaert, Cho, and Moreno (2005), Rudebusch and Wu (2006), and Gallmeyer, Hollifield, Palomino, and Zin (2007).

Equations (3) through (8) imply that the yields $i_t^{(n)}$ are linear functions of the inflation rate:

$$(9)\quad i_t^{(n)} = a_n + b_n \pi_t,$$

a functional form which corresponds to the estimated regressions in equation (1). For $i_t^{(1)} = r_t$ this is obvious from the policy rule equation (3), and so a_1=0 and $b_1 = \delta$. For $i_t^{(2)}$ the derivation is as follows: From equation (4), we know that the price of the one-period bond at time *t+1* is simply $P_{t+1}^{(1)} = \exp(-r_{t+1}),$ which can be substituted into equation (5) to obtain $P_t^{(2)} = E_t[m_{t+1} \exp(-r_{t+1})]$.

Now, by substituting for m_{t+1} and r_{t+1} from equations (3), (6), and (8) we get

$$\begin{aligned}
(10)\quad P_t^{(2)} &= E_t[\exp(-\delta\pi_t - 0.5\lambda_t^2 - \lambda_t\varepsilon_{t+1} - \delta(\pi_t - \phi(\delta\pi_t - \pi_t) + \sigma\varepsilon_{t+1}))] \\
&= \exp(-\delta\pi_t - 0.5\lambda_t^2 - \delta(\pi_t - \phi(\delta-1)\pi_t)E_t[\exp(-(\delta\sigma + \lambda_t)\varepsilon_{t+1})] \\
&= \exp(-\delta\pi_t - 0.5\lambda_t^2 - \delta(\pi_t - \phi(\delta-1)\pi_t) + 0.5\delta^2\sigma^2 + \delta\sigma\lambda_t + 0.5\lambda_t{}^2) \\
&= \exp(-\delta\pi_t - \delta(\pi_t - \phi(\delta-1)\pi_t) + 0.5\delta^2\sigma^2 - \delta\sigma(\gamma_0 + \gamma_1\pi_t)) \\
&= \exp(0.5\delta^2\sigma^2 - \delta\sigma\gamma_0 - \delta(2 - \phi(\delta-1) + \sigma\gamma_1)\pi_t).
\end{aligned}$$

where we use the normal distribution assumption for ε_{t+1} to evaluate the expectation in the second line of (10). Since the yield $i_t^{(2)}$ is given by $0.5\log(P_t^{(2)})$ we have that:

$$(11)\quad i_t^{(2)} = 0.5\delta\sigma\gamma_0 - 0.25\delta^2\sigma^2 + 0.5\delta(2 - \phi(\delta-1) + \sigma\gamma_1)\pi_t,$$

which is the linear form of (9), with the response coefficient for the two period yield given by:

$$(12)\quad b_2 = \frac{\delta(2 - \phi(\delta-1) + \sigma\gamma_1)}{2}.$$

A similar result holds for all maturities, as shown in Appendix A. The response coefficient on the current inflation rate for yields to maturity of n periods is:

$$(13) \quad b_n = \frac{\delta \sum_{i=0}^{n-1} (1-\phi(\delta-1)+\sigma\gamma_1)^i}{n}.$$

Observe that the numerator of equation (13) is a geometric series in which each term equals the previous term multiplied by the common ratio $(1-\phi(\delta-1)+\sigma\gamma_1)$. While it is algebraically possible for this common ratio to be negative, this would imply implausible values for the parameters, such as an enormous response coefficient δ for the central bank. Hence we will assume throughout that the common ratio is positive and focus on whether it is greater than one or not.

We noted in Section 1 that the estimated response coefficients are nearly constant as maturity increases. In particular they show no tendency to increase with n. Under what conditions is this empirical observation explained by the simple model? Consider first the case of $n = 2$ and examine equation (12). It shows that $b_2 < \delta$ if

$$(14) \quad \delta > 1 + \frac{\sigma\gamma_1}{\phi}.$$

Condition (14) is closely related to the monetary policy principle that $\delta > 1$ (called the "Taylor principle," by Woodford (2001) and others). This same condition (14) is sufficient to prevent b_n from exploding as n increases. To see this, note that if (14) holds, then the common ratio of terms in the geometric series in the numerator of equation (13) is less than one. For each increase in maturity n, we calculate b_n by (i) adding a term in the numerator which is less than the previous term and (ii) adding a term in the denominator greater than the previous term. Thus we conclude that b_n cannot increase geometrically. Note, however, that while the estimated response coefficients do not explode, they do not significantly decline. This implies that inequality (14) is nearly an equality.

The close connection between condition (14) and the Taylor principle is important because the latter is usually viewed as the *sine qua non* of a good monetary policy. If that principle does not hold (as appears to have been the case in the first sub-sample 1960Q1 – 1979Q4 in Section 1), then the model is not even stable, which is one way to understand why that period showed such macroeconomic turbulence; but it is possible for (14) to hold if γ_I is negative which causes longer-term yields to have a more muted response to inflation. If the Taylor principle holds, it is possible that (14) does not hold if γ_I is sufficiently large and positive. In the case of no risk aversion ($\gamma_I = 0$), the Taylor principle and condition (14) are exactly the same. We note that Ang and Piazzesi (2003) and Ang, Dong, and Piazzesi (2005) estimated the time-varying risk parameter corresponding to inflation to be positive, though in somewhat different set-ups.

3. Impact of Shifts in the Monetary Policy Rule

As shown in Section 1, policy actions have gotten more aggressive in responding to inflation; that is, δ in equation (3) has increased. The question we focus on now is whether that increase may have affected the behavior of the whole term structure. In other words, can such a shift in the response coefficients in the policy rule explain the shift in the response coefficients of longer term interest rates?

First, recall equation (12) for the n = 2 bond yield: $b_2 = \dfrac{\delta(2-\phi(\delta-1)+\sigma\gamma_1)}{2}$. One can easily see that the response coefficient in the policy rule has a direct impact on the response of this longer-term yield to inflation. However, there are countervailing effects. A larger reaction coefficient δ means that expected future short-term interest rates will rise by a larger amount when future inflation rises; this effect is measured by the presence of δ outside the parentheses in (12) and it depends on the risk premium parameter γ_1; but a higher δ also means that inflation is expected to increase by a smaller amount in the future for a given increase in inflation today, because the persistence in inflation declines; this effect is measured by the term $\phi(\delta-1)$ in equation (12).

To sort out these countervailing effects, consider the derivative of (12) with respect to δ:

$$(15) \quad \frac{\partial b_2}{\partial \delta} = \frac{2+\phi+\sigma\gamma_1}{2} - \delta\phi > 0.$$

Note that unless δ is already very large, the derivative is likely to be positive and we will generally make this assumption, as indicated by the inequality sign in (15). Then increasing δ will raise the reaction of the 2-period rate to inflation. Here, the reduced persistence has less effect than the size of the reaction. For high values of δ the derivative could be negative, reflecting that reduced persistence has a larger effect.

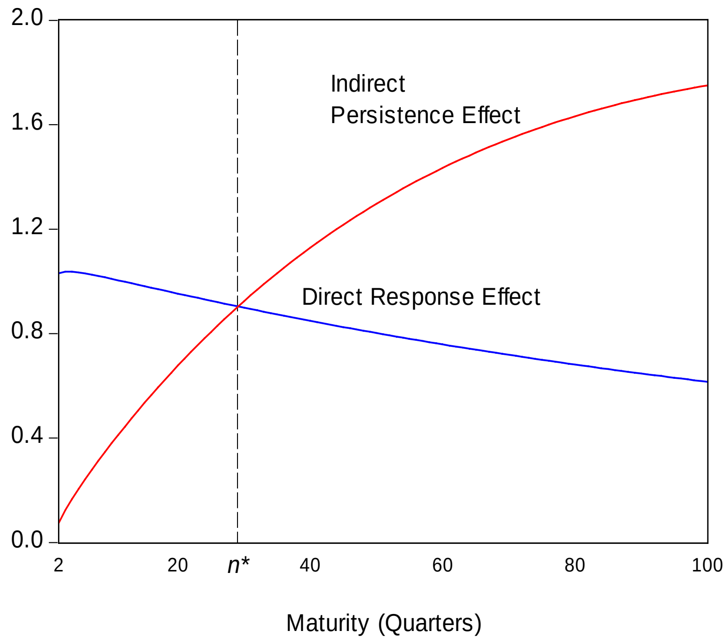
Now consider the general case of maturity n and differentiate equation (13) with respect to δ. As shown in Appendix A, the derivative for n = 2, 3,…,N is:

$$(16) \quad \frac{\partial b_n}{\partial \delta} = \frac{1}{n}(1 + \sum_{i=0}^{n-2}(1+\phi+\sigma\gamma_1)(1-\phi(\delta-1)+\sigma\gamma_1)^i - \phi\delta\sum_{i=0}^{n-2}(i+2)(1-\phi(\delta-1)+\sigma\gamma_1)^i).$$

Observe that equation (16), much like equation (15), is composed of two countervailing terms. Both terms are strictly monotonically increasing and multiplied by 1/n. The first term is a geometric sum with common factor equal to $1-\phi(\delta-1)+\sigma\gamma_1$, while the second term, which is subtracted from the first, is an arithmetic-geometric sum with the same common factor. Much as in equation (15), there are two countervailing effects: the first term is the direct effect of policy while the second term is the indirect persistence effect.

Figure 3 depicts these two terms as functions of the maturity length n and for some example parameter values. Note how the curves cross at a particular maturity, which we call n*. The derivative is the difference between the two curves, shown by the distance between them in Figure 3. For all n < n*, the first term is larger so equation (16) is positive and $\frac{\partial b_n}{\partial \delta} > 0$, but the second term is larger starting at n* so that equation (16) becomes positive and $\frac{\partial b_n}{\partial \delta} < 0$.

Figure 3: The Countervailing Effects of a More Responsive Policy



Note : Parameters are set at $\phi = \gamma_1 = 0.05, \delta = 1.5, \sigma = 0.25$.

To be precise we summarize this argument with the following:

Proposition 1: *Suppose conditions (14) and (15) holds and that* $\gamma_1 > 0$. *Then there exists a unique* n^* *such that for all* $n < n^*$, $\frac{\partial b_n}{\partial \delta} > 0$, and for all $n > n^*$, $\frac{\partial b_n}{\partial \delta} < 0$.

Proof: First, multiply equation (16) by n and consider the two resulting series which we denote as:

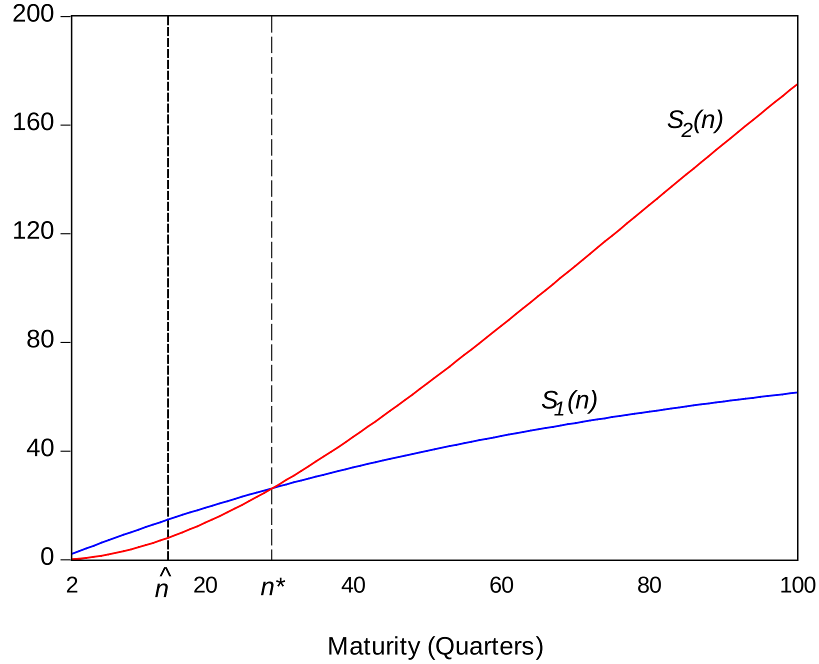
$$S_1(n) = (1 + \sum_{i=0}^{n-2} (1+\phi+\sigma\gamma_1)(1-\phi(\delta-1)+\sigma\gamma_1)^i$$

$$S_2(n) = (\phi\delta\sum_{i=0}^{n-2}(i+2)(1-\phi(\delta-1)+\sigma\gamma_1)^i)$$

$$= \phi\delta\sum_{i=0}^{n-2} i(1-\phi(\delta-1)+\sigma\gamma_1)^i + 2\phi\delta\sum_{i=0}^{n-2}(1-\phi(\delta-1)+\sigma\gamma_1)^i .$$

for $n = 2,3,\ldots, N$. Figure 4 illustrates $S_1(n)$ and $S_2(n)$ for the same parameter values used in Figure 3. It is clear that these two series are both monotonically increasing in n.

Figure 4: Graphical Illustration Proof of Proposition 1



Note : Parameters are set at $\phi = \gamma_1 = 0.05, \delta = 1.5, \sigma = 0.25$.

Now, note that:

$$S_1(2) - S_2(2) = 2 + \sigma\gamma_1 - \phi(2\delta - 1),$$

and since (15) holds, we have $S_1(2) - S_2(2) > 0$.

We next consider maturities longer than $n = 2$ and define $D_1(n) = S_1(n) - S_1(n-1)$ and $D_2(n) = S_2(n) - S_2(n-1)$, which are simply the terms in the two sums and are given by

$$D_1(n) = (1+\phi+\sigma\gamma_1)(1-\phi(\delta-1)+\sigma\gamma_1)^{n-2}$$

$$D_2(n) = \phi\delta n(1-\phi(\delta-1)+\sigma\gamma_1)^{n-2}.$$

Note that for all $n < \hat{n} = \dfrac{1+\phi+\sigma\gamma_1}{\phi\delta}$, $D_1(n)$ is larger than $D_2(n)$ and for all $n > \hat{n}$, $D_2(n)$ is larger than $D_1(n)$. Unless $\phi\delta = 0$, which would only occur under the case of inflation following a simple AR(1) process, there must exist some finite $\hat{n}$. In fact, if $\phi\delta = 0$, then it would always be the case that $\dfrac{\partial b_n}{\partial \delta} > 0$.

Now consider $n = 3 < \hat{n}$. We know that:

$S_1(3) = S_1(2) + D_1(3)$ and

$S_2(3) = S_2(2) + D_2(3)$.

Thus $S_1(3) - S_2(3) = (S_1(2) - S_2(2)) + (D_1(3) - D_2(3)) > 0$.

Similarly, using induction on n for all $n < \hat{n}$,

$$S_1(n) - S_2(n) = (S_1(n-1) - S_2(n-1)) + (D_1(n) - D_2(n)) = S_1(n-1) - S_2(n-1)$$

so that $S_1(n) - S_2(n) > 0$.

Finally, we let n approach infinity, and use the formula for the limits of a geometric and arithmetic-geometric series to compute that:

$$S_1(\infty) = \frac{(1+\phi\delta)(\phi(\delta-1)-\sigma\gamma_1)}{(\phi(\delta-1)-\sigma\gamma_1)^2}$$

$$S_2(\infty) = \frac{\phi\delta(1+\phi(\delta-1)-\sigma\gamma_1)}{(\phi(\delta-1)-\sigma\gamma_1)^2},$$

and therefore given the assumptions on our parameters,

$$S_2(\infty) - S_1(\infty) = \frac{\phi+\sigma\gamma_1}{(\phi(\delta-1)-\sigma\gamma_1)^2} > 0.$$

Now we have that

(i) $S_1(2) - S_2(2) > 0$,

(ii) $S_2(n) - S_1(n) > 0$ for n sufficiently large, and

(iii) $D_2(n) > D_1(n)$ and for all $n > \hat{n}$,

So there must be a single point at which these two curves cross, given by $n*$. Thus we have proved that $\frac{\partial b_n}{\partial \delta} > 0$ for all $n < n*$ and $\frac{\partial b_n}{\partial \delta} < 0$ for $n > n^*$.

We illustrate the proof in Figure 4. The slopes of $S_1(n)$ and $S_2(n)$ are graphical representations of $D_1(n)$ and $D_2(n)$, respectively. Note that the number $\hat{n}$ lies at the point where these two slopes are equal. The proof first considers values of n greater than 2 but to the left of $\hat{n}$ and then goes on to consider values of n to the right of $\hat{n}$. In the first region $S_1(n) > S_2(n)$ because $S_1(2) - S_2(2) > 0$ and, as n increases, a smaller term is added to the smaller sum than to the larger sum. In the second region, which begins with a value of n where $S_1(n) > S_2(n)$, there is a single crossing point $n*$ at which the two curves are shown to intersect.

We feel that the assumptions underlying Proposition 1 are empirically realistic. It is reasonable to assume that (14) holds because the estimated response coefficients in Section 1 do not explode. Condition (15) holds unless δ is extremely large. The assumption of a positive γ_1 falls in line with previous empirical estimates, and even if it is not positive and thus n^* is infinite, this does not contradict our empirical findings. Hence, the model does explain the facts: a monetary policy that reacts more aggressively against inflation implies that bond yields respond more aggressively to inflation.

Recall that the regressions reported in Section 1 went out to a five-year or twenty- quarter maturity. These regressions showed no indication that the response coefficients declined with the large increase in the policy response parameter δ we saw in the two sub-samples. This suggests that all these maturity lengths are less than n^*. It would be interesting to see if the sign reversed for longer maturity U.S. zero-coupon Treasuries, as well as for zero-coupon bond issued by other countries.

4. A Model with Both Inflation and Output in the Policy Rules

Now, consider the following model which includes real output as well as inflation. Equation (17) is the policy rule; it replaces the simpler policy rule of equation (3), incorporating a measure of the real output gap in the interest rate rule of the central bank. We specify bond prices analogous to (6), but the pricing kernel m_{t+1} now has the matrix form shown in equation (18). Equation (19) shows how inflation and real output affect risk aversion in the pricing kernel, a generalization of equation (7). The risk term λ_t is now two-dimensional. The first element corresponds to the risk term associated with real output, whereas the second element corresponds to the risk term associated with inflation. Equations (20) and (21) replace equation (8). Equation

(20) shows how real output depends on its own lag and on the ex post real interest rate. Equation (21) is a price adjustment equation in which inflation depends on lagged inflation and lagged real output:

(17) $r_t = \boldsymbol{\delta}'\mathbf{z}_t$

(18) $m_{t+1} = \exp(-r_t - 0.5\boldsymbol{\lambda}_t'\boldsymbol{\lambda}_t - \boldsymbol{\lambda}_t'\mathbf{v}_{t+1})$

(19) $\boldsymbol{\lambda}_t = \boldsymbol{\gamma} + \boldsymbol{\Gamma}\mathbf{z}_t$

(20) $y_t = -\alpha_1(r_{t-1} - \pi_{t-1}) + \alpha_2 y_{t-1} + \sigma_\eta \eta_t,$

(21) $\pi_t = \pi_{t-1} + \phi y_t + \sigma_\varepsilon \varepsilon_t,$

where $\mathbf{z}_t = \begin{pmatrix} y_t \\ \pi_t \end{pmatrix}, \mathbf{v}_t = \begin{pmatrix} \eta_t \\ \varepsilon_t \end{pmatrix}, \boldsymbol{\delta} = \begin{pmatrix} \delta_y \\ \delta_\pi \end{pmatrix}, \boldsymbol{\gamma} = \begin{pmatrix} \gamma_{01} \\ \gamma_{02} \end{pmatrix}, \boldsymbol{\Gamma} = \begin{pmatrix} -\gamma_{11} & \gamma_{12} \\ \gamma_{21} & -\gamma_{22} \end{pmatrix}$, and where the shocks

$\eta_t \sim$ *iid* $N(0,1)$ and $\varepsilon_t \sim$ *iid* $N(0,1)$ are independent of each other.

Analogously with the model of Section 2, the yield on an *n*-period bond is a linear function of inflation and output:

(22) $i_t^{(n)} = a_n + \mathbf{b}_n'\mathbf{z}_t,$

with the *n*-period yield intercept term given by a_n and the *n*-period response coefficient vector given by $\mathbf{b}_n = \begin{pmatrix} b_{1,n} \\ b_{2,n} \end{pmatrix}$. $b_{1,n}$ corresponds to the real output gap, and the second element $b_{2,n}$ corresponds to the inflation rate.

To show that (22) holds, we first consider the two-period bond yield and derive a_2 and $\mathbf{b}_2$. Observe that we can rewrite (20) – (21) as a VAR(1) in the following way:

(23) $\mathbf{z}_t = \mathbf{\Phi}\mathbf{z}_{t-1} + \mathbf{\Sigma}\mathbf{v}_t,$

where

$$\mathbf{\Phi} = \begin{pmatrix} \alpha_2 - \alpha_1\delta_y & -\alpha_1(\delta_\pi - 1) \\ \phi(\alpha_2 - \alpha_1\delta_y) & 1 - \phi\alpha_1(\delta_\pi - 1) \end{pmatrix}, \quad \mathbf{\Sigma} = \begin{pmatrix} \sigma_\eta & 0 \\ \phi\sigma_\eta & \sigma_\varepsilon \end{pmatrix}.$$

Using the same method we used in the univariate model, we first write the two-period bond as: $P_t^{(2)} = E_t[m_{t+1}P_{t+1}^{(1)}] = E_t[m_{t+1}\exp(-r_{t+1})]$. Substituting in for m_{t+1} and r_{t+1}, we find:

$$\begin{aligned}
(24) \quad P_t^{(2)} &= E_t[\exp(-\boldsymbol{\delta}'\mathbf{z}_t - 0.5\boldsymbol{\lambda}_t'\boldsymbol{\lambda}_t - \boldsymbol{\lambda}_t'\mathbf{v}_{t+1} - \boldsymbol{\delta}'\mathbf{z}_{t+1})] \\
&= \exp(-\boldsymbol{\delta}'\mathbf{z}_t - 0.5\boldsymbol{\lambda}_t'\boldsymbol{\lambda}_t - \boldsymbol{\delta}'\mathbf{\Phi}\mathbf{z}_t)E_t[\exp(-(\boldsymbol{\lambda}_t' + \boldsymbol{\delta}'\mathbf{\Sigma})\mathbf{v}_{t+1})] \\
&= \exp(-\boldsymbol{\delta}'\mathbf{z}_t - 0.5\boldsymbol{\lambda}_t'\boldsymbol{\lambda}_t - \boldsymbol{\delta}'\mathbf{\Phi}\mathbf{z}_t + 0.5\boldsymbol{\lambda}_t'\boldsymbol{\lambda}_t + \boldsymbol{\delta}'\mathbf{\Sigma}\boldsymbol{\lambda}_t + 0.5\boldsymbol{\delta}'\mathbf{\Sigma}\mathbf{\Sigma}'\boldsymbol{\delta}) \\
&= \exp(-\boldsymbol{\delta}'\mathbf{z}_t - \boldsymbol{\delta}'\mathbf{\Phi}\mathbf{z}_t + \boldsymbol{\delta}'\mathbf{\Sigma}(\boldsymbol{\gamma} + \mathbf{\Gamma}\mathbf{z}_t) + 0.5\boldsymbol{\delta}'\mathbf{\Sigma}\mathbf{\Sigma}'\boldsymbol{\delta}) \\
&= \exp(0.5\boldsymbol{\delta}'\mathbf{\Sigma}\mathbf{\Sigma}'\boldsymbol{\delta} + \boldsymbol{\delta}'\mathbf{\Sigma}\boldsymbol{\gamma} - \boldsymbol{\delta}'(\mathbf{I} + \mathbf{\Phi} - \mathbf{\Sigma}\mathbf{\Gamma})\mathbf{z}_t).
\end{aligned}$$

Therefore, the two-period bond yield is given by:

(25) $i_t^{(2)} = -(0.25\boldsymbol{\delta}'\mathbf{\Sigma}\mathbf{\Sigma}'\boldsymbol{\delta} + 0.5\boldsymbol{\delta}'\mathbf{\Sigma}\boldsymbol{\gamma}) + 0.5\boldsymbol{\delta}'(\mathbf{I} + \mathbf{\Phi} - \mathbf{\Sigma}\mathbf{\Gamma})\mathbf{z}_t,$

where we see that $a_2 = -(0.25\boldsymbol{\delta}'\boldsymbol{\Sigma\Sigma}'\boldsymbol{\delta} + 0.5\boldsymbol{\delta}'\boldsymbol{\Sigma\gamma})$ and $\mathbf{b}_2' = 0.5\boldsymbol{\delta}'(\mathbf{I} + \boldsymbol{\Phi} - \boldsymbol{\Sigma\Gamma})$. Note the similarities and differences between equation (25) which includes real output and equation (11) which does not. In both equations the policy parameters affect the coefficient in predictable ways.

More generally (see Appendix B), the *n*-maturity bond yield response coefficient vector is:

$$(26) \quad \mathbf{b}_n = \frac{1}{n}\left(\sum_{i=0}^{n-1}(\boldsymbol{\Phi} - \boldsymbol{\Sigma\Gamma})^i\right)'\boldsymbol{\delta}.$$

which an obvious generalization of equation (13).

5. Impacts of Shifts in the Policy Rule on the Term Structure Response Coefficients

In our simple, univariate model, we saw that the inflation response coefficient in the simple policy rule had a direct impact on the response of longer-term yields to inflation, yet the direction of the reaction had two countervailing forces. We can also examine how longer-term yields are impacted by the policy reaction coefficients in this two-dimensional model. Empirical estimates of monetary policy rules have shown that both δ_y and δ_π significantly increased in the 1980s, and our regressions from Section 1 indicate that when output and inflation are both used as factors determining bond yields, more aggressive policy is associated with substantial increases in bond yield reaction coefficients. We want to see how such a policy change might impact the response coefficients of the output gap and inflation for bond yields in our analytical model, in order to reconcile the observed empirics. To simplify the analytics we set $\gamma_{12} = \gamma_{21} = 0$. The existing empirical literature does not provide much guidance about the values of these off-diagonal elements, though we can compute the response coefficients numerically for any values.

Expanding the coefficient vector we derived for the two-period yield in (27) gives:

$$(27)\quad \mathbf{b}_2 = 0.5\begin{pmatrix} \delta_y(1+\alpha_2-\alpha_1\delta_y+\sigma_\eta\gamma_{11})+\delta_\pi(\phi(\alpha_2-\alpha_1\delta_y+\sigma_\eta\gamma_{11})) \\ -\alpha_1\delta_y(\delta_\pi-1)+\delta_\pi(2-\phi\alpha_1(\delta_\pi-1)+\sigma_\varepsilon\gamma_{22}) \end{pmatrix}.$$

Consider the derivative of the response coefficient vector for the two-period yield with respect to the elements of $\boldsymbol{\delta}$:

$$(28)\quad \frac{\partial \mathbf{b}_2}{\partial \delta_y} = 0.5\begin{pmatrix} 1+\alpha_2+\sigma_\eta\gamma_{11}-\alpha_1(2\delta_y+\phi\delta_\pi) \\ -\alpha_1(\delta_\pi-1) \end{pmatrix}$$

$$(29)\quad \frac{\partial \mathbf{b}_2}{\partial \delta_\pi} = 0.5\begin{pmatrix} \phi(\alpha_2-\alpha_1\delta_y)+\phi\sigma_\eta\gamma_{11} \\ 2+\sigma_\varepsilon\gamma_{22}-\alpha_1\delta_y-\phi\alpha_1(2\delta_\pi-1) \end{pmatrix}.$$

The first element of each of these expressions corresponds to the real output gap coefficient in the bond yield equation (22), and the second element of each of these expressions corresponds to the inflation coefficient.

If we examine the element $b_{2,2}$ corresponding to inflation, we see that $\frac{\partial b_{2,2}}{\partial \delta_\pi} = \frac{1}{2}\left(2+\phi\alpha_1+\sigma_\varepsilon\gamma_{22}-\alpha_1\delta_y-2\phi\alpha_1\delta_\pi\right)$. Recall from (14), the derivative from the univariate model at $n = 2$: $\frac{\partial b_2}{\partial \delta} = \frac{1}{2}\left(2+\phi+\sigma\gamma_1-2\phi\delta\right)$. Now compare $\frac{\partial b_{2,2}}{\partial \delta_\pi}$ and $\frac{\partial b_2}{\partial \delta}$. We now have $\phi\alpha_1$ rather than ϕ. If we let $\delta_y = 0$ to allow for only inflation as in the univariate model and let $\sigma_\varepsilon \gamma_{22} = \sigma\gamma_1$, we see the similarities between the two models. The only addition compared to the simpler model is the term $\alpha_1\delta_y$. Unless this term is very large, a reasonable assumption is that $\dfrac{\partial b_{2,2}}{\partial \delta_\pi} > 0$.

We can also examine how the coefficient vector of the n-period bond yield responds to policy changes. Let $\mathbf{H} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} = \mathbf{\Phi} - \mathbf{\Sigma\Gamma}$. Using this notation, the derivative of $\mathbf{b}_n$ with respect to each of the elements of $\boldsymbol{\delta}$ is:

$$(30)\quad \frac{\partial \mathbf{b}_n}{\partial \delta_y} = \frac{1}{n}\begin{pmatrix} 1 - \alpha_1 \sum_{i=1}^{n-1}\left(i(\delta_y h_{11}^{i-1} + \phi\delta_\pi h_{21}^{i-1})\right) + \sum_{i=1}^{n-1} h_{22}^{i} \\ \sum_{i=1}^{n-1} h_{12}^{i} \end{pmatrix}$$

$$(31)\quad \frac{\partial \mathbf{b}_n}{\partial \delta_\pi} = \frac{1}{n}\begin{pmatrix} \sum_{i=1}^{n-1} h_{21}^{i} \\ 1 - \alpha_1 \sum_{i=1}^{n-1}\left(i(\delta_y h_{12}^{i-1} + \phi\delta_\pi h_{22}^{i-1})\right) + \sum_{i=1}^{n-1} h_{22}^{i} \end{pmatrix}.$$

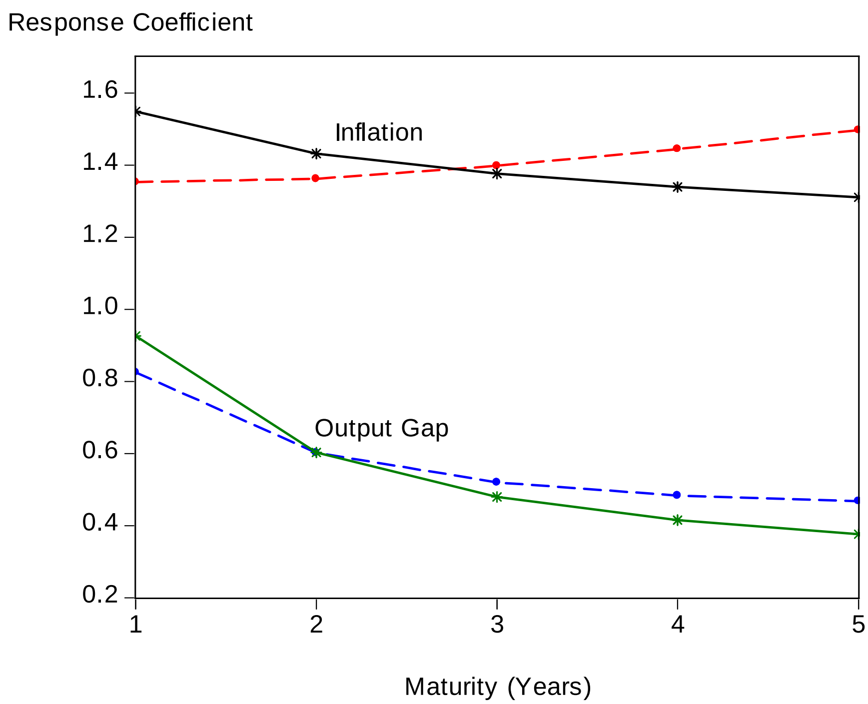
as shown in Appendix B.

Now consider the derivative: $\dfrac{\partial b_{2,n}}{\partial \delta_\pi} = \dfrac{1}{n}\left(1 + \sum_{i=1}^{n-1} h_{22}^{i}\right) - \dfrac{1}{n}\left(\alpha_1 \sum_{i=1}^{n-1} i(\delta_y h_{12}^{i-1} + \phi\delta_\pi h_{22}^{i-1})\right)$.

Observe that this derivative is very similar to (16); we again have countervailing forces at work. We have two sums, one is geometric and the other is arithmetic-geometric. The geometric term represents the direct effect of policy, while the arithmetic-geometric term is a combination of the persistence of inflation and the output gap. And as in the simpler case we can calculate how the response coefficients for the longer yields are affected by the policy response coefficients.

Figure 5 shows how the response coefficients shift with a change in policy parameters. The dashed lines represents a regime with $\delta_y = 1.3$ and $\delta_\pi = 1.4$, while the solid represents a more responsive regime with $\delta_y = 1.7$ and $\delta_\pi = 1.8$. These policy parameters are much greater than those estimates in the pre-1980 regime.

Figure 5: Behavior of Response Coefficients with a More Aggressive Policy Regime



Note : Parameters are set at $\alpha_1 = 0.2, \alpha_2 = 0.7, \phi = 0.2, \sigma_\eta = 0.75, \sigma_\varepsilon = 0.36, \gamma_{11} = 0.15, \gamma_{22} = 0.15$.

First, notice how the output gap coefficients decline much more rapidly than the inflation coefficients, much as in the empirical estimates. This is due to the persistence of output being smaller than the persistence of inflation, a common stylized fact we see in the data. Even more intriguing, however, is how the slope of the response coefficient curves change as the policy parameters change. As the monetary policy rule becomes more responsive, shorter-term yields respond more than longer-term yields to both the output gap and inflation. This is the coutervailing forces at work; as the indirect persistence effect grows larger than the direct effect of higher policy reaction coefficients, the bond yield response coefficients change and longer-term response coefficients adjust downward.

6. An Explanation of the Term Structure Conundrum

The above results also shed light on a famous asset pricing puzzle which first arose when the Federal Reserve started raising the Federal Funds rate in 2004. That increase in the short-term interest rate was not associated with nearly as large an increase in long-term interest rates as would have been expected based on experience over the previous 20 years. The puzzle was coined the "conundrum" by former Federal Reserve Chairman Alan Greenspan; it was a great concern for policymakers for it appeared that the tightening of monetary policy would not have had the bite that it had in previous periods of tightening. There have been many explanations for the conundrum, including the idea of a global saving glut that drove down the world real interest rate. But that explanation has been challenged because world saving as a share of world GDP had actually fallen during this period.

An alternative explanation naturally emerges from the theory in this paper. During this period, the Federal Funds rate deviated significantly from what would have been predicted by the Fed's typical response as exemplified by the estimates we reported in Section 1 of this paper for the sample period from 1984Q1 to 2006Q4. While it is difficult to determine whether this was a shift in the policy response coefficients or simply an additive deviation from the rule, there is econometric evidence that it may have been interpreted as a regime shift. To see this, we estimated the following regression over the 1984Q1 – 2006Q4 period:

$$r_t = 2.056 + 1.016 y_t + 1.428 \pi_t - 1.327 \pi_t^D,$$

where π_t^D is a multiplicative dummy variable in which the actual inflation rate is multiplied by a dummy variable that equals one from 2002Q4 through 2005Q4, and zero elsewhere. This is the period when, by all accounts, the actual Federal Funds rate deviated significantly from the estimated Taylor rule. All the coefficients in this equation are statistically significant, and in particular the inflation terms and the multiplicative dummy are highly significant. The equation clearly suggests the possibility that the response coefficient on inflation dropped significantly during this period. The short-term interest rate response to inflation (δ_π) would seem much lower to market participants trying to assess Federal Reserve policy. To investigate other possibilities, we also included an additive dummy along with the multiplicative dummy to the regression; we still found a significant downward shift in the inflation response. We note that Davig and Leeper (2006) found a similar shift using an estimated Markov switching model.

Now, according to the theory presented in this paper, a perception of a smaller response coefficient in the policy rule could well have led market participants to expect smaller interest rate responses to inflation in the future, and therefore lower long term interest rate responses. That is, we would predict that the lower response δ_π would have lowered the response coefficient $b_{2,n}$ for inflation. Hence, our model provides a simple consistent explanation for the conundrum. While the regime shift was clearly temporary when viewed from the perspective of today, it would have been difficult to assess at the time whether the Federal Reserve would have returned to the typical rule followed during the post 1984Q1 period.

7. Conclusion

In this paper, we have shown that the theory of monetary policy rules helps explain and understand the dramatic changes in the comovement of note and bond yields and macroeconomic variables over the past several decades. We showed that a more aggressive policy regime has a substantial impact on the entire term structure of interest rates: a more responsive monetary policy increases the response of longer-term yields to macroeconomic variables. Our model also helps explain the behavior of the longer-term interest rates during the recent spell in 2002 – 2005, where standard models predicted that longer-term yields would increase far more than they did.

There are several directions that future research might take. Using models of stochastic regime shifting, such as Davig and Leeper (2007), along with no-arbitrage restrictions may help pin down the exact patterns of bond yields and macroeconomic variables. Including longer-term interest rates directly in the macro model would be an important check for robustness. Applying the model to policy changes in other countries—even more dramatic changes than studied here—would also be valuable. For example, do we see the same shift in the reaction coefficients for Brazil when comparing the 1980s and 1990s to the 2000s? All of these topics are worth pursuing and will help shed light upon how the decisions of monetary policymakers at the short end of the yield curve affect behavior at the long end.

Appendix A

Derivation of the Relationship between the Policy Rule and Response Coefficients

We use the method of undetermined coefficients, and begin by conjecturing that the *n*-period bond prices have the following form:

$$(A1)\quad P_t^{(n)} = \exp(A_n + B_n \pi_t).$$

The continuously-compounded yields on *n*-period bonds are then:

$$(A2)\quad \begin{aligned} i_t^{(n)} &= \log(P_t^{(n)})/n \\ &= a_n + b_n \pi_t, \end{aligned}$$

where $a_n = -n^{-1} A_n$ and $b_n = -n^{-1} B_n$

Using the same approach we used for the case of n=2 in the text we have:

$$(A3)\quad \begin{aligned} P_t^{(n+1)} &= E_t[m_{t+1} P_{t+1}^{(n)}] \\ &= E_t[\exp(-r_t - 0.5\lambda_t^2 - \lambda_t \varepsilon_{t+1} + A_n + B_n \pi_{t+1})] \\ &= E_t[\exp(-\delta\pi_t - 0.5\lambda_t^2 - \lambda_t \varepsilon_{t+1} + A_n + B_n((1+\phi)\pi_t - \phi\delta\pi_t + \sigma\varepsilon_{t+1}))] \\ &= \exp(-\delta\pi_{\mathrm{t}} - 0.5\lambda_t^2 + A_n + B_n(1+\phi-\phi\delta)\pi_t)\, E_t[\exp((\sigma B_n - \lambda_t)\varepsilon_{t+1})] \\ &= \exp(-\delta\pi_{\mathrm{t}} - 0.5\lambda_t^2 + A_n + B_n(1-\phi(\delta-1))\pi_t + 0.5\sigma^2 B_n^2 + 0.5\lambda_t^2 - \sigma\lambda_t B_n) \\ &= \exp(A_n + \sigma\gamma_0 B_n + 0.5\sigma^2 B_n^2 + (B_n(1-\phi(\delta-1)+\sigma\gamma_1)-\delta)\pi_t). \end{aligned}$$

Then, it must be the case that:

$$(A4)\quad \exp(A_{n+1} + B_{n+1}\pi_t) = \exp(A_n + \sigma\gamma_0 B_n + 0.5\sigma^2 B_n^2 + (B_n(1-\phi(\delta-1)+\sigma\gamma_1)-\delta)\pi_t).$$

Matching coefficients, we find that:

$$(A5)\quad A_{n+1} = A_n + \sigma\gamma_0 B_n + 0.5\sigma^2 B_n^2 \quad \text{with } A_1 = 0,\ \ B_1 = -\delta$$

and

$$(A6)\quad B_{n+1} = B_n(1-\phi(\delta-1)+\sigma\gamma_1) - \delta.$$

We can then derive the yield response coefficients in a recursive fashion. For $n = 2$:

$$(A7) \quad B_2 = -\delta(2 - \phi(\delta - 1) + \sigma\gamma_1))$$

and

$$(A8) \quad A_2 = -\sigma\gamma_0\delta + 0.5\sigma^2\delta^2.$$

For $n = 3$:

$$(A9) \quad B_3 = -\delta(1 + (2 - \phi(\delta - 1) + \sigma\gamma_1))(1 - \phi(\delta - 1) + \sigma\gamma_1))$$

and

$$(A10) \quad A_3 = -\sigma\gamma_0\delta + 0.5\sigma^2\delta^2 - \sigma\gamma_0\delta(2 - \phi(\delta - 1) + \sigma\gamma_1)) + 0.5\sigma^2\delta^2(2 - \phi(\delta - 1) + \sigma\gamma_1))^2.$$

And in general:

$$(A11) \quad B_n = -\delta\sum_{i=0}^{n-1}(1 - \phi(\delta - 1) + \sigma\gamma_1)^i$$

$$\begin{aligned}(A12) \quad A_n &= \sigma\gamma_0\sum_{i=1}^{n-1}B_i + 0.5\sigma^2\left(\sum_{i=1}^{n-1}B_i^2\right) \\ &= \sigma\gamma_0\sum_{i=1}^{n-1}\left(-\delta\sum_{j=0}^{i-1}(1 - \phi(\delta - 1) + \sigma\gamma_1)^j\right) + 0.5\sigma^2\sum_{i=1}^{n-1}\left(-\delta\sum_{j=0}^{i-1}(1 - \phi(\delta - 1) + \sigma\gamma_1)^j\right)^2 \\ &= -\sigma\gamma_o\delta\sum_{i=1}^{n-1}\left(\sum_{j=0}^{i-1}(1 - \phi(\delta - 1) + \sigma\gamma_1)^j\right) + 0.5\sigma^2\delta^2\sum_{i=1}^{n-1}\left(\sum_{j=0}^{i-1}(1 - \phi(\delta - 1) + \sigma\gamma_1)^j\right)^2.\end{aligned}$$

A simple transformation gives us the bond yield response coefficients:

$$(A13) \quad b_n = \frac{\delta\sum_{i=0}^{n-1}(1 - \phi(\delta - 1) + \sigma\gamma_1)^i}{n}$$

$$(A14) \quad a_n = \frac{\sigma\gamma_o\delta\sum_{i=1}^{n-1}\left(\sum_{j=0}^{i-1}(1 - \phi(\delta - 1) + \sigma\gamma_1)^j\right) - 0.5\sigma^2\delta^2\sum_{i=1}^{n-1}\left(\sum_{j=0}^{i-1}(1 - \phi(\delta - 1) + \sigma\gamma_1)^j\right)^2}{n}.$$

Finding the Derivative of the Response Coefficient to Inflation

We find $\frac{\partial b_n}{\partial \delta}$ from equation (A13) by differentiating each element in (A13) separately.

Consider the first few elements and ignoring n:

$$(A15)\quad \frac{\partial(\delta(1-\phi(\delta-1)+\sigma\gamma_1))}{\partial\delta} = 1-\phi(2\delta-1)+\sigma\gamma_1$$

$$(A16)\quad \frac{\partial(\delta(1-\phi(\delta-1)+\sigma\gamma_1)^2)}{\partial\delta} = (1-\phi(3\delta-1)+\sigma\gamma_1)(1-\phi(\delta-1)+\sigma\gamma_1)$$

$$(A17)\quad \frac{\partial(\delta(1-\phi(\delta-1)+\sigma\gamma_1)^3)}{\partial\delta} = (1-\phi(4\delta-1)+\sigma\gamma_1)(1-\phi(\delta-1)+\sigma\gamma_1)^2$$

So that in general the emerging pattern is given by:

$$(A18)\quad \frac{\partial b_n}{\partial\delta} = \frac{1}{n}(1+\sum_{i=0}^{n-2}(1+\phi+\sigma\gamma_1)(1-\phi(\delta-1)+\sigma\gamma_1)^i - \phi\delta\sum_{i=0}^{n-2}(i+2)(1-\phi(\delta-1)+\sigma\gamma_1)^i).$$

Appendix B

The Relation between the Policy Rule and Response Coefficient Vector

Following Ang and Piazzesi (2003) we have

$$(B1)\quad A_{n+1} = A_n + \mathbf{B}_n^{'}(\mathbf{\Sigma\gamma}) + 0.5\mathbf{B}_n^{'}\mathbf{\Sigma\Sigma}'\mathbf{B}_n \quad \text{with } A_1 = 0,\quad \mathbf{B}_1^{'} = -\boldsymbol{\delta}'$$

and

$$(B2)\quad \mathbf{B}_{n+1}^{'} = \mathbf{B}_n^{'}(\mathbf{\Phi} - \mathbf{\Sigma\Gamma}) - \boldsymbol{\delta}'.$$

From these we find the response coefficient vector $\mathbf{b}_n$. Using (B2) and the fact that $a_n = -n^{-1}A_n$ and $\mathbf{b}_n = -n^{-1}\mathbf{B}_n$ we have:

$$(B3)\quad -(n+1)\mathbf{b}_{n+1}^{'} = -n\mathbf{b}_{n}^{'}(\mathbf{\Phi} - \mathbf{\Sigma\Gamma}) - \boldsymbol{\delta}'$$

and

$$(B4)\quad -(n+1)a_{n+1} = -n(a_n + \mathbf{b}_n^{'}(\mathbf{\Sigma}\boldsymbol{\gamma}) + 0.5\mathbf{b}_n^{'}\mathbf{\Sigma\Sigma}'\mathbf{b}_n).$$

Since $\mathbf{b}_1 = \boldsymbol{\delta}$, we can derive the recursive relationship for the response coefficient vector as we did in Appendix A:

$$(B5)\quad \mathbf{b}_n = \frac{1}{n}\left(\sum_{i=0}^{n-1}(\mathbf{\Phi} - \mathbf{\Sigma\Gamma})^i\right)^{'}\boldsymbol{\delta}.$$

Finding the Derivative of the Response Coefficient Vectors

We assume that $\gamma_{12} = \gamma_{21} = 0,$ so that:

$$\begin{aligned}(B6)\quad \mathbf{H} &= \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \\ &= \mathbf{\Phi} - \mathbf{\Sigma\Gamma} \\ &= \begin{pmatrix} \alpha_2 - \alpha_1\delta_y + \sigma_\eta\gamma_{11} & -\alpha_1(\delta_\pi - 1) \\ \phi(\alpha_2 - \alpha_1\delta_y) + \sigma_\eta\gamma_{11} & 1 - \alpha_1\phi(\delta_\pi - 1) + \sigma_\varepsilon\gamma_{22} \end{pmatrix}.\end{aligned}$$

We differentiate each term of the sum, as we did in Appendix A. Ignoring n, we see the following pattern emerging when we differentiate with respect to δ_y:

$$(B7)\qquad \frac{\partial(\boldsymbol{\delta}^{'}\mathbf{H})}{\partial\delta_y}=\begin{pmatrix} h_{11}-\alpha_1(\delta_y+\phi\delta_\pi) \\ h_{12}\end{pmatrix}^{'}$$

$$(B8)\qquad \frac{\partial(\boldsymbol{\delta}^{'}\mathbf{H}^2)}{\partial\delta_y}=\begin{pmatrix} h_{11}^2-2\alpha_1(\delta_y h_{11}+\phi\delta_\pi h_{21}) \\ h_{12}^2\end{pmatrix}^{'}$$

$$(B9)\qquad \frac{\partial(\boldsymbol{\delta}^{'}\mathbf{H}^3)}{\partial\delta_y}=\begin{pmatrix} h_{11}^3-3\alpha_1(\delta_y h_{11}^2+\phi\delta_\pi h_{21}^2) \\ h_{12}^3\end{pmatrix}^{'}.$$

Similarly, when we differentiate with respect to δ_π:

$$(B10)\qquad \frac{\partial(\boldsymbol{\delta}^{'}\mathbf{H})}{\partial\delta_\pi}=\begin{pmatrix} h_{21} \\ h_{22}-\alpha_1(\delta_y+\phi\delta_\pi)\end{pmatrix}^{'}$$

$$(B11)\qquad \frac{\partial(\boldsymbol{\delta}^{'}\mathbf{H}^2)}{\partial\delta_\pi}=\begin{pmatrix} h_{21}^2 \\ h_{22}^2-2\alpha_1(\delta_y h_{12}+\phi\delta_\pi h_{22})\end{pmatrix}^{'}$$

$$(B12)\qquad \frac{\partial(\boldsymbol{\delta}^{'}\mathbf{H}^3)}{\partial\delta_\pi}=\begin{pmatrix} h_{21}^3 \\ h_{22}^3-3\alpha_1(\delta_y h_{12}^2+\phi\delta_\pi h_{22}^2)\end{pmatrix}^{'}.$$

Combining the above and recognizing the patterns, we thus have:

$$(\mathrm{B16})\qquad \frac{\partial\mathbf{b}_n}{\partial\delta_y}=\frac{1}{n}\begin{pmatrix} 1-\alpha_1\sum_{i=1}^{n-1}\left(i(\delta_y h_{11}^{i-1}+\phi\delta_\pi h_{21}^{i-1})\right)+\sum_{i=1}^{n-1}h_{22}^i \\ \sum_{i=1}^{n-1}h_{12}^i\end{pmatrix}$$

$$(B17)\qquad \frac{\partial\mathbf{b}_n}{\partial\delta_\pi}=\frac{1}{n}\begin{pmatrix} \sum_{i=1}^{n-1}h_{21}^i \\ 1-\alpha_1\sum_{i=1}^{n-1}\left(i(\delta_y h_{12}^{i-1}+\phi\delta_\pi h_{22}^{i-1})\right)+\sum_{i=1}^{n-1}h_{22}^i\end{pmatrix}.$$

Appendix C

Standard Errors of the Regressions Estimates

Tables C1, C2, C3 and C4 contain additional details about the regressions reported in Tables 1, 2, 3, and 4, respectively. Under each point estimate, the standard error is reported in parenthesis. We also report the R^2 for each regression.

Table C1

Maturity (Years)	a_n for 1960Q1 – 1979Q4	b_n for 1960Q1 – 1979Q4	R^2 for 1960Q1 – 1979Q4	a_n for 1984Q1 – 2006Q4	b_n for 1984Q1 – 2006Q4	R^2 for 1984Q1 – 2006Q4
1	2.913	0.605	0.681	1.487	1.549	0.279
	(0.244)	(0.047)		(0.698)	(0.263)	
2	2.989	0.603	0.758	1.790	1.566	0.286
	(0.201)	(0.039)		(0.693)	(0.261)	
3	3.188	0.580	0.797	2.114	1.540	0.291
	(0.173)	(0.033)		(0.674)	(0.253)	
4	3.282	0.573	0.816	2.338	1.536	0.297
	(0.161)	(0.031)		(0.663)	(0.249)	
5	3.319	0.573	0.832	2.480	1.528	0.305
	(0.152)	(0.029)		(0.646)	(0.243)	

Table C2

Maturity (Years)	a_n for 1960Q1 – 1979Q4	$b_{1,n}$ for 1960Q1 – 1979Q4	$b_{2,n}$ for 1960Q1 – 1979Q4	R^2 for 1960Q1 – 1979Q4	a_n for 1984Q1 – 2006Q4	$b_{1,n}$ for 1984Q1 – 2006Q4	$b_{2,n}$ for 1984Q1 – 2006Q4	R^2 for 1984Q1 – 2006Q4
1	2.874	0.454	0.604	0.847	2.155	1.244	1.224	0.513
	(0.170)	(0.050)	(0.033)		(0.586)	(0.190)	(0.223)	
2	2.960	0.335	0.602	0.860	2.394	1.124	1.273	0.478
	(0.154)	(0.045)	(0.030)		(0.605)	(0.196)	(0.230)	
3	3.166	0.259	0.580	0.865	2.652	1.000	1.279	0.451
	(0.142)	(0.041)	(0.027)		(0.606)	(0.197)	(0.230)	
4	3.263	0.222	0.572	0.868	2.815	0.888	1.304	0.426
	(0.137)	(0.040)	(0.026)		(0.611)	(0.198)	(0.232)	
5	3.303	0.191	0.573	0.872	2.913	0.806	1.318	0.416
	(0.134)	(0.039)	(0.026)		(0.605)	(0.196)	(0.230)	

Table C3

Maturity (Years)	a_n for 1960Q1 – 1979Q4	b_n for 1960Q1 – 1979Q4	R^2 for 1960Q1 – 1979Q4	a_n for 1984Q1 – 2006Q4	b_n for 1984Q1 – 2006Q4	R^2 for 1984Q1 – 2006Q4
1	2.840	0.637	0.695	1.449	1.579	0.282
	(0.249)	(0.048)		(0.707)	(0.266)	
3	3.246	0.595	0.796	2.017	1.604	0.296
	(0.178)	(0.034)		(0.693)	(0.261)	
5	3.370	0.591	0.831	2.391	1.583	0.313
	(0.157)	(0.030)		(0.658)	(0.247)	
10	3.468	0.584	0.844	2.868	1.543	0.326
	(0.148)	(0.028)		(0.622)	(0.234)	
20	3.354	0.610	0.877	3.477	1.432	0.211
	(0.135)	(0.026)		(0.832)	(0.349)	

Table C4

Maturity (Years)	a_n for 1960Q1 – 1979Q4	$b_{1,n}$ for 1960Q1 – 1979Q4	$b_{2,n}$ for 1960Q1 – 1979Q4	R^2 for 1960Q1 – 1979Q4	a_n for 1984Q1 – 2006Q4	$b_{1,n}$ for 1984Q1 – 2006Q4	$b_{2,n}$ for 1984Q1 – 2006Q4	R^2 for 1984Q1 – 2006Q4
1	2.802	0.446	0.636	0.842	2.112	1.233	1.257	0.506
	(0.180)	(0.053)	(0.035)		(0.599)	(0.194)	(0.227)	
3	3.224	0.262	0.595	0.863	2.571	1.031	1.335	0.456
	(0.147)	(0.043)	(0.028)		(0.622)	(0.202)	(0.236)	
5	3.353	0.194	0.590	0.870	2.830	0.817	1.370	0.421
	(0.139)	(0.040)	(0.027)		(0.617)	(0.200)	(0.234)	
10	3.455	0.143	0.584	0.866	3.186	0.591	1.389	0.388
	(0.139)	(0.040)	(0.027)		(0.605)	(0.196)	(0.230)	
20	3.343	0.132	0.610	0.895	3.789	0.745	1.246	0.295
	(0.126)	(0.037)	(0.024)		(0.801)	(0.273)	(0.340)	

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