4. Impact of Monetary Shocks in Forward-Looking Models -- Open Economies
Consider a very simple small open economy monetary model ("Dornbusch model")
- Two variables, but only one "jump" variable
- Exchange rate (flexible) and price level (sticky)

\[ m_t - p_t = -\alpha (E_t e_{t+1} - e_t) \]
\[ p_t - p_{t-1} = \beta (e_t - p_t) \]
\[ m_t = \sum_{i=0}^{\infty} \rho^i \epsilon_{t-i} \]
\[ E\epsilon_t = 0 \]
\[ E\epsilon_t \epsilon_s = 0 \quad \text{for } t \neq s \]
\[ E\epsilon_t^2 = \sigma^2 \]
First put model into matrix form:

\[
\begin{pmatrix}
E_t e_{t+1} \\
p_t
\end{pmatrix} = \begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix} \begin{pmatrix}
e_t \\
p_{t-1}
\end{pmatrix} + \begin{pmatrix}
d_1 \\
d_2
\end{pmatrix} m_t
\]

\[A = \begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix} = (1 + \beta)^{-1} \begin{pmatrix}
1 + \beta(1 + 1/\alpha) & 1/\alpha \\
\beta & 1
\end{pmatrix}\]

\[d = \begin{pmatrix}
d_1 \\
d_2
\end{pmatrix} = \begin{pmatrix}
-(1/\alpha) \\
0
\end{pmatrix}\]

\[E_t z_{t+1} = Az_t + dm_t\]
To solve, look for solutions of the form:

\[ e_t = \gamma_{10} e_t + \gamma_{11} e_{t-1} + \gamma_{12} e_{t-2} + \cdots \]
\[ p_t = \gamma_{20} e_t + \gamma_{21} e_{t-1} + \gamma_{22} e_{t-2} + \cdots \]

and plug into basic model

\[
\begin{pmatrix}
E_t e_{t+1} \\
p_t
\end{pmatrix} =
\begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix}
\begin{pmatrix}
e_t \\
p_{t-1}
\end{pmatrix} +
\begin{pmatrix}
d_1 \\
d_2
\end{pmatrix} m_t
\]

\[ e_{t+1} = \gamma_{10} e_{t+1} + \gamma_{11} e_t + \gamma_{12} e_{t-1} + \cdots \]
\[ E_t e_{t+1} = \gamma_{11} e_t + \gamma_{12} e_{t-1} + \cdots \]
\[ p_{t-1} = \gamma_{20} e_{t-1} + \gamma_{21} e_{t-2} + \gamma_{22} e_{t-3} + \cdots \]

to get:
...a deterministic vector difference equation:

\[
\begin{pmatrix}
\gamma_{11} \\
\gamma_{20}
\end{pmatrix} =
\begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix}
\begin{pmatrix}
\gamma_{10} \\
0
\end{pmatrix}
+ \begin{pmatrix}
d_1 \\
d_2
\end{pmatrix}
\]

\[
\begin{pmatrix}
\gamma_{1,i+1} \\
\gamma_{2i}
\end{pmatrix} =
\begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix}
\begin{pmatrix}
\gamma_{1i} \\
\gamma_{2i-1}
\end{pmatrix}
+ \begin{pmatrix}
d_1 \\
d_2
\end{pmatrix}\rho^i \quad i = 1, 2, \ldots
\]

which can be solved for the \( \gamma \)'s
\[ \gamma_i = A\gamma_{i-1} + d\rho^i \quad i = 1, 2, \ldots \]

where \( \gamma_i = \begin{pmatrix} \gamma_{1,i+1} \\ \gamma_{2i} \end{pmatrix} \)

\[ \gamma_i = \gamma_i^{(H)} + \gamma_i^{(P)} \]

\[ \gamma_i^{(H)} = H \Lambda H^{-1} \gamma_{i-1}^{(H)} \]

\[ H^{-1} \gamma_i^{(H)} = \Lambda H^{-1} \gamma_{i-1}^{(H)} \]

or

\[ \mu_i = \Lambda \mu_{i-1} \quad \mu_{1i} = \lambda_1 \mu_{1,i-1} \quad \mu_{2i} = \lambda_2 \mu_{2,i-1} \]

let \[ H^{-1} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \]

then \[ \mu_{1i} = g_{11} \gamma_{1,i+1}^{(H)} + g_{12} \gamma_{2i}^{(H)} \]

and \[ \mu_{2i} = g_{21} \gamma_{1,i+1}^{(H)} + g_{22} \gamma_{2i}^{(H)} \]

Divide into homogeneous and particular solutions

\( \Lambda \) is a diagonal matrix composed of the eigenvalues of A.
Guess form for particular solution

\[ \gamma_{1i}^{(P)} = b_1 \rho^i \]
\[ \gamma_{2i}^{(P)} = b_2 \rho^i \]

and plug into \( \gamma \) – equations to get

\[
\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \rho^2 - a_{11} \rho & -a_{12} \\ -a_{21} \rho & \rho - a_{22} \end{pmatrix}^{-1} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \rho
\]

Particular solution now found from this
For stability we set
\[ \mu_{10} = 0 \]
\[ \Rightarrow g_{11}\gamma_{11} + g_{12}\gamma_{20} = 0 \]
\[ \Rightarrow g_{11}(\gamma_{11} - b_1\rho) + g_{12}(\gamma_{20} - b_2) = 0 \]
from the above equation we can now calculate
\[ \gamma_{10}, \gamma_{11}, \gamma_{20} \]
and finally
\[ \gamma_{2,i+1}^{(H)} = \lambda_2\gamma_{2i}^{(H)} \quad i = 0, 1, \ldots \]
\[ \gamma_{1,i+1}^{(H)} = -(g_{12}/g_{11})\gamma_{2i}^{(H)} \quad i = 1, 2, \ldots \]
with \[ \gamma_{21}^{(H)} = \lambda_2(\gamma_{20} - \gamma_{20}^{(p)}) \]
Example
\[ \alpha = .1, \beta = .1, \rho = 1 \]

\[
\begin{array}{cc}
\gamma_{1i} & \gamma_{2i} \\
6.928 & .623 \\
3.232 & .858 \\
1.840 & .947 \\
1.316 & .980 \\
1.119 & .992 \\
\end{array}
\]

Response to a unanticipated, permanent increase in the money supply

\[
m_t - p_t = -\alpha(E_t e_{t+1} - e_t)
\]

\[
p_t - p_{t-1} = \beta(e_t - p_t)
\]
Larger models can be solved numerically, though economic intuition is easy to lose

\[ f_i(y_t, y_{t-1}, \ldots, E_t y_{t-p}, E_t y_{t+1}, \ldots, E_t y_{t+q}, x_t) = u_{it} \quad i = 1, \ldots, n \]

where

- \( y_t \) is a vector of endogenous variables (like the exchange rate or the price level),
- \( x_t \) is a vector of exogenous variables (like the money supply),
- \( u_{it} \) is a vector of random variables.