SYNTHETIC DIFFERENTIAL GEOMETRY

1. Gooey motivational remarks

1.1. The word “synthetic”. The word “synthetic” in “synthetic differential geometry” is an old fashioned word for the axiomatic style of geometry which appears in Euclid’s elements – as opposed to the “analytic” geometry, which uses “Cartesian coordinates”.

In the synthetic approach, one begins with certain “primitive” undefined notions (for example “point”, “line”, “circle”, and “incident”), and then one sets out certain axioms regarding these notions – for example, for each pair of distinct points, there is a unique line incident to both of them. One then proceeds to investigate the consequences of these axioms, being careful not to appeal to any of the intuitive, geometric content of the primitive notions, but to use only those assumptions which were explicitly set out as axioms.

In the analytic approach, one instead defines those notions which were primitive in the synthetic approach; for example, a point is defined to be a pair of numbers, a line is defined to be the set of points satisfying some linear equation, and so on. What were axioms then become theorems on the basis of these axioms.

1.2. The synthetic method today. The analytic method may seem to remove the need for axioms altogether. However, in the analytic approach, the theorems – which were formerly axioms – must now be proven on the basis of some assumptions about whatever objects (such as real numbers) were used to define the formerly primitive notions. If these assumptions are made explicit – as they should be, in the interest of rigor and perspicuity – then we see that we have not removed our axioms, but rather replaced them with a set of more primitive axioms.

In modern mathematics, we have of course performed the amazing feat of producing a single set of axioms – namely, those of axiomatic set theory – which suffice, once and for all, for defining all known mathematical objects and proving all known properties of them. Hence, modern mathematics is in some sense decidedly “analytic”; the objects of interest (besides sets) are always defined and their basic properties proven, rather than merely posited.

However, the “synthetic” method still plays an important role, since – unlike the very natural and canonical Cartesian definitions of line and circle – definitions of mathematical objects in set theory are often rather ad hoc, and we therefore often prefer to set out their fundamental properties and then – having once shown the existence of an object satisfying these properties – proceed to develop the theory on the basis of these properties alone. Thus, once we have established the existence of a complete ordered field, we cease to bother about whether it was defined using Cauchy sequences or Dedekind cuts.

1.3. Synthetic Differential Geometry. Now, what is the situation in differential geometry? Here, we might say that the basic notions are that of, say, smooth manifold and smooth mapping. It must be said that the definitions of these are also somewhat ad hoc. For example, a manifold can be defined using equivalence classes of atlases, or maximal atlases, or in terms of their sheaves of smooth functions, or (as it turns out) as certain subsets of Euclidean space.

The idea in synthetic differential geometry is to set out “smooth manifold” and “smooth mapping” as our “primitive notions”, select certain basic properties of these, and then see
how far we get. Apart from the intrinsic interest of such a task, it turns out that, once carried out, it offers certain additional benefits, in particular a convenient setting for the manipulation of infinite-dimensional spaces (which are rather unwieldy in usual treatments of differential geometry) and infinitesimally small spaces (which are usually non-existent).

The basic insight in this synthetic method is that, since we are making assumptions about a certain collection of spaces and mappings, what we are doing is imposing axioms on a category. At first, we might want to choose axioms which hold in the category of smooth manifolds but, in order to accommodate the above “exotic” spaces, we instead choose axioms which hold in a suitable enlargement

$$\text{Mfld} \hookrightarrow \mathcal{C}$$

of the category of manifolds (into which the latter embeds fully).

2. A sneak peak

We now give a few examples from our limited experience with synthetic differential geometry of the kind of things that once can do with it.

First of all, we suppose that our category $\mathcal{C}$ is cartesian closed. This means that $\mathcal{C}$ possesses along with any two objects $A, B$ their categorical product $A \times B$, as well as an “exponential object” $B^A$ which satisfies the fundamental adjunction

$$\text{Hom}_\mathcal{C}( -, B^A) \cong \text{Hom}_\mathcal{C}( -, A \times B)$$

where this is a natural isomorphism of functors $\mathcal{C}^{\text{op}} \to \text{Set}$.

It is easily seen that $\text{Mfld}$ does not possess this property; only in very special cases (for example, when $A$ is a point) does such an object $B^A$ exist.

Letting $i : \text{Mfld} \to \mathcal{C}$ denote the hypothesized embedding of $\text{Mfld}$ into $\mathcal{C}$, we will have for any manifolds $M, N$ the object $i(N)^{(i(M)}$ in $\mathcal{C}$, which will serve as the space of smooth functions from $M$ to $N$. Rather than defining this as a certain set with a certain topology, as usual, the above adjunction serves as its fundamental property: a “smooth family of maps” $Z \times X \to Y$ “parametrized by $Z$” corresponds to a smooth map $Z \to Y^X$ into the function space.

Next, we observe that the real line $\mathbb{R}$ has the structure of a “ring object” in $\text{Mfld}$. Namely, there are smooth maps

$$+, \cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \quad - : \mathbb{R} \to \mathbb{R} \quad 0, 1 : 1 \to \mathbb{R}$$

(where $1$ is the terminal object, i.e., the one-point space), and these satisfy the usual axioms of a commutative ring (which are purely diagrammatic and can make sense in any category with products).

We therefore assume the existence of a distinguished ring object $\mathbb{R}$ in $\mathcal{C}$, which is admissible in light of our imagined embedding $i$. Defining the “squaring” and “zero” maps

$$(-)^2 : \mathbb{R} \xrightarrow{\Delta} \mathbb{R} \times \mathbb{R} \xrightarrow{\cdot} \mathbb{R} \quad 0 : \mathbb{R} \to 1 \xrightarrow{0} \mathbb{R}$$

we then define the “space of square-zero elements” to be the equalizer

$$\text{D} \to \mathbb{R} \xrightarrow{(\cdot 0)^{-1}} \mathbb{R}$$

which we might denote by $\text{D} = \{d \in \mathbb{R} \mid d^2 = 0\}$. It is easily seen that in $\text{Mfld}$ (with $\mathbb{R} = \mathbb{R}$), this is just the one-point space. However, we might hope for it to be non-trivial in $\mathcal{C}$. In fact, the main axiom of SDG – the “Kock-Lawvere” Axiom – says that the map

$$\mathbb{R} \times \mathbb{R} \to \mathbb{R}^\text{D}$$

which sends \((a, b)\) to the (restriction to \(D\)) of the map \(x \mapsto ax + b\) (this should – and can easily – be formulated categorically) is an isomorphism, which is clearly false if \(D\) is the terminal object (unless \(R\) is also the terminal object).

In any case, such an object turns out to serve well as an “infinitesimal neighborhood of zero” in \(R\). For example, for an object \(M\) we can define “tangent vector to \(M\)” to simply be a morphism \(D \to M\). Then, the tangent bundle \(TM\) becomes the exponential object \(M^D\), which automatically comes equipped (via the inclusion \(1 \to D\)) with its projection map \(M^D \to M^1 \cong M\).

At this point, since we have provided a new definition of a familiar object, we should make sure that it is consistent with the old definition. That is, in the particular category \(C\) we have in mind, we should make sure that for any manifold \(M\), the image \(i(TM)\) of the tangent bundle of \(M\) in \(C\) is isomorphic to \(i(M)^D\). Hopefully, this will turn out to be the case!

As a final example of something nice which we can do with \(D\), we will consider the space of vector fields on a manifold. Given our definition of the tangent bundle and its projection \(TM \to M\), we can consider sections of this projection and moreover the space \(\Gamma(TM)\) of such sections, which is defined as the pullback

\[
\begin{array}{c}
\Gamma(TM) \\
\downarrow \\
1
\end{array}
\begin{array}{c}
\rightarrow \ 
\downarrow \\
(TM)^M \\
1 \\
\rightarrow \ 
M^M
\end{array}
\]

where the bottom morphism is the one corresponding to the identity morphism \(M \to M\). This corresponds to the usual definition of the space of tangent vectors.

However, we often think of vector fields on \(M\) as “infinitesimal transformations”; we might like to say in light of this that the space of vector fields is the tangent space \(T_{id} \text{Diff}(M)\) to the diffeomorphism group at the identity. We can make sense of this. Given an object \(X\) and a point \(p : 1 \to X\), we define the tangent space at that point to be the pullback

\[
\begin{array}{c}
T_pM \\
\downarrow \\
1
\end{array}
\begin{array}{c}
\rightarrow \\
TX \\
p \\
\rightarrow \\
X
\end{array}
\]

We can then take \(X\) to be the function space \(M^M\) and \(p\) to be the map \(id : 1 \to M\) corresponding to the identity morphism \(M \to M\); the resulting object \(T_{id}M^M\) is then easily seen to be isomorphic to \(\Gamma(TM)\) as defined above.

3. The game plan

Our goal for the seminar is to prove some of the basic theorems of differential geometry using synthetic methods. This will require us first to set up the appropriate axioms and definitions and – since we want to give proofs of the “traditional statements” and not just of analogues of them – to verify that these correspond to the usual definitions.

Here is an outline of the talks, followed by descriptions of them.

(1) Axioms
(2) Models
   (a) Naive models
   (b) \(C^\infty\)-rings
   (c) Dubuc topos
(3) Topics
First (1), we will take a survey of all the various axioms that people have thought to impose on categories for doing SDG.

Then (2), we will look at the various categories people have constructed which satisfy these axioms. We will begin (2a) with the most easily constructed categories, but which have certain serious deficiencies (i.e., fail to satisfy the most important axioms) – for example, the categories $\text{Set}^{\text{Mfld}^{op}}$ and $\text{Set}^{\mathbb{R}-\text{Alg}}$ of presheaves on the category of manifolds or $\mathbb{R}$-algebras. Then (2b), we will learn about $C^\infty$-rings – a special class of rings which generalize the rings of smooth functions on a manifold – which are necessary for the development of the better-behaved models. Lastly (2c), we will define the Dubuc topos, which seems to be the best-behaved category people have come up with.

Finally (3), we will choose as many topics in differential geometry as we have time for, and try to develop them in the framework which we have set up.