1. Introduction. No rational square equals 2. Induction.

The goal of this course is to rigorously study the key ideas in calculus: limits, sequences, continuity, the derivative, and the integral. On one hand, some of the things that we will uncover in the course might appear to be fairly routine. For example, by the end, you will be able give a full proof that the derivative of the function \( f : \mathbb{R} \to \mathbb{R} \) defined by \( f(x) = x^2 \) is \( 2x \), and you will have a rigorous notion of what the word “derivative” means to go alongside the ideas that you learned in your initial exposure to calculus.

On the other hand, there are some subtle issues that require a large amount of development. A key example lies in the study of the real numbers \( \mathbb{R} \). In your initial exposure to calculus, you probably never thought to take the derivative of a function whose domain is \( \mathbb{Z} \) (the integers) or \( \mathbb{Q} \) (the rationals). You only worked with functions on \( \mathbb{R} \). Why is this? Can one prove that working with functions on \( \mathbb{R} \) is the “right setting” for calculus? What is a real number, anyway? You might have an intuitive definition in mind, but we will need a very comprehensive definition in order to develop calculus properly. In order to do this, we need to develop the theory of sequences and limits. Only after we are equipped with the comprehensive properties of \( \mathbb{R} \) and the theory of sequences and limits, we will be able to handily develop the notions of continuity, differentiation, and integration.

What does \( \mathbb{R} \) have that number systems like \( \mathbb{Q} \) and \( \mathbb{Z} \) don’t have? To help motivate this, consider the problem of describing the number \( x \geq 0 \) such that \( x^2 = 2 \). Hopefully, it is clear that \( x \) is not an integer, but perhaps \( x \) might be rational.

**Theorem 1.1.** There is no rational number \( x \) such that \( x^2 = 2 \).

**Proof.** We begin by supposing that there does in fact exist a rational number \( x \) such that \( x^2 = 2 \). We will show that this supposition leads to a conclusion that we know is false, which means that our initial hypothesis (there is a rational number \( x \) such that \( x^2 = 2 \)) is false. This is a classical example of **proof by contradiction**.

Suppose (to the contrary) that \( x \) is a rational number such that \( x^2 = 2 \). We may write \( x \) as a quotient of integers \( a/b \) with \( a, b \neq 0 \) and \( a \) and \( b \) having no common factor. Then \( a^2 = 2b^2 \), which implies that \( a^2 \) is even. Since the square of an odd integer is odd, it follows that \( a \) is even. We may now write \( a \) as \( 2c \), where \( c \) is a nonzero integer. Now, \( 2b^2 = a^2 = (2c)^2 = 4c^2 \), or \( b^2 = 2c^2 \). Hence \( b^2 \) is even, and for the same reason as before, \( b \) is even. Thus \( a \) and \( b \) are nonzero integers which share 2 as a factor. This contradicts our hypothesis that \( a \) and \( b \) have no common factor, and therefore the hypothesis that \( x \) is rational must be false. \( \square \)

This proof indicates something about the layout of the course. We reduce the proof of a statement about the rationals to the proof of a statement about the integers. In this course, we will build the rationals from the integers, and then we will build the real numbers from the rationals. Once we have rigorously developed desirable properties about the reals, we will be able to properly address the usual topics in calculus.

Let us take the time to discuss one property of the **natural numbers** \( \mathbb{N} = \{1, 2, 3, \ldots\} \). We make special mention of \( \mathbb{N} \) because it is the setting for mathematical induction. This is a tool which is ubiquitous throughout mathematics and is incredibly powerful.
Principle of mathematical induction. Let \( P_1, P_2, P_3, \ldots \) be a list of statements or propositions (which may or may not be true) indexed by the natural numbers. Suppose that

I1: \( P_1 \) is true, and

I2: whenever \( P_n \) is true, \( P_{n+1} \) is also true.

Then all of the statements \( P_1, P_2, P_3, \ldots \) are true.

Here is an example of how useful mathematical induction can be.

Example 1.2. For every natural number \( n \), we have that \( 1 + 2 + \cdots + n = \frac{n(n+1)}{2} \).

Proof. For each natural number \( n \), the statement we want to prove is \( P_n \): “\( 1 + 2 + \cdots + n = \frac{n(n+1)}{2} \)”.

We proceed by induction on \( n \). First we will prove that the base case \( P_1 \) is true. The statement \( P_1 \) reads \( 1 = \frac{1(1+1)}{2} \), which is true. Now, suppose that \( P_n \) is true; in other words, suppose that \( 1 + 2 + \cdots + n = \frac{n(n+1)}{2} \). Using this hypothesis, we want to prove the statement \( P_{n+1} \). To achieve this, we add \( n + 1 \) to both sides to obtain
\[
1 + 2 + \cdots + n + (n + 1) = \frac{n(n + 1)}{2} + n + 1 = \frac{n(n + 1) + 2(n + 1)}{2} = \frac{(n + 1)((n + 1) + 1)}{2}.
\]
Thus \( P_{n+1} \) is true if \( P_n \) is true. By the principle of mathematical induction, \( P_n \) holds for all natural numbers \( n \).

\[\square\]

Note that we did not prove \( P_n \) directly for any \( n \) except for \( n = 1 \). We just proved \( P_1 \), and we proved that if \( P_1 \) is true, so is \( P_2 \) (thus \( P_2 \) is true), and we proved that if \( P_2 \) is true, so is \( P_3 \) (thus \( P_3 \) is true), and we proved that if \( P_3 \) is true, so is \( P_4 \) (thus \( P_4 \) is true), etc.

Example 1.3. Fix \( x > 0 \). For every natural number \( n \), we have that \( (1 + x)^{n+1} > 1 + (n + 1)x \).

Proof. We proceed by induction on \( n \). We first prove the base case \( n = 1 \). We have \( (1 + x)^{1+1} = (1 + x)^2 = 1 + 2x + x^2 \). If \( x > 0 \), then \( 1 + 2x + x^2 > 1 + 2x = 1 + (1 + 1)x \), as desired. Now, suppose that the inequality \( (1 + x)^{n+1} > 1 + (n + 1)x \) has already been proven. We will show that \( (1 + x)^{(n+1)+1} > 1 + ((n + 1) + 1)x \) is true (that is, \( (1 + x)^{n+2} > 1 + (n + 2)x \) is true).

Note that \( (1 + x)^{n+2} = (1 + x)^{n+1}(1 + x) \).

By the inductive hypothesis, we have that \( (1 + x)^{n+2} > (1 + (n + 1)x)(1 + x) \). This expands to \( 1 + (n + 2)x + (n + 1)x^2 \). Since \( x^2 > 0 \) and \( n + 1 > 0 \), we have \( (n + 1)x^2 > 0 \). Hence \( 1 + (n + 2)x + (n + 1)x^2 > 1 + (n + 2)x \), as desired. It follows that if we fix \( x > 0 \), then for every natural number \( N \), we have that \( (1 + x)^{n+1} > 1 + (n + 1)x \).

\[\square\]

Here is a prototype for all future proofs using mathematical induction.

Proposition 1.4 (Induction template). A property \( P_n \) is true for all natural numbers \( n \).

Proof. We proceed by induction on \( n \) (it’s good to specify the variable if there are several variables in the statement you want to prove). We first verify the base case \( n = 1 \); in other words, we prove that \( P_1 \) is true. [Insert the proof of \( P_1 \) here]. Now, suppose that \( P_n \) has already been proven. We will show that \( P_{n+1} \) is true. [Insert the proof of \( P_{n+1} \), assuming \( P_n \) is true]. It follows that \( P_n \) is true for all natural numbers.

\[\square\]
2. The integers.

From this point on, \( \mathbb{Q} \) and \( \mathbb{R} \) do not exist until we construct them. To begin, we will assume some familiarity with the basic properties of the positive integers (the natural numbers) as well as the set of all integers. One can build these properties from the Peano axioms of arithmetic (listed in §1 of Ross), but we will not do this.

We will denote the set of integers \{...,-2,1,0,1,2,...\} by \( \mathbb{Z} \). I am going to assume that you are comfortable with the ideas of addition, subtraction, and multiplication within \( \mathbb{N} \) and \( \mathbb{Z} \). The basic properties of \( \mathbb{Z} \) follow from the following axioms.

**Axiom 2.1** (Addition axioms). A set of numbers \( S \) is said to satisfy the **addition axioms** if the following hold for all members \( x, y, \) and \( z \) of \( S \).

A0: Closure: If \( x \) and \( y \) are in \( S \), then their sum \( x + y \) is an integer.
A1: Associativity: \( (x + y) + z = x + (y + z) \).
A2: Commutativity: \( x + y = y + x \).
A3: Identity: There is a number in \( S \) such that \( 0 + x = x \).
A4: Inverse: To every \( x \) in \( S \) there corresponds a number \( (-x) \) in \( S \) such that \( x + (-x) = 0 \).
A5: Well-definedness: If \( x \) and \( y \) are in \( S \) and \( x = x' \), then \( x + y = x' + y \) and \( (-x) = (-x') \).

**Notation 2.2.** We write \( x - y \) as shorthand for \( x + (-y) \).

**Axiom 2.3** (Multiplication axioms). A set of numbers \( S \) is said to satisfy the **multiplication axioms** if the following hold for all members \( x, y, \) and \( z \) of \( S \).

M0: Closure: If \( x \) and \( y \) are in \( S \), then their product \( xy \) (also written \( x \cdot y \)) is in \( S \).
M1: Associativity: \( (xy)z = x(yz) \).
M2: Commutativity: \( xy = yx \).
M3: Identity: There exists a number in \( S \), denoted \( 1 \), such that \( 1 \cdot x = x \).
M4: Well-definedness: If \( x \) and \( y \) are in \( S \) and \( x = x' \), then \( xy = x'y \).

**Axiom 2.4** (Distributive axiom). A set of numbers \( S \) is said to satisfy the **distribution axiom** if the following holds for all members \( x, y, \) and \( z \) of \( S \).

DL: \( x(y + z) = xy + xz \).

The set \( \mathbb{Z} \) also is equipped with an **ordering** which arises from the usual notion of inequality \(<\). Let’s take some time to spell out the pertinent definitions.

**Definition 2.5.** Let \( S \) be a set of numbers. An **order** on \( S \) is a relation, denoted by \(<\), with the following two properties:

(i) (Trichotomy) If \( x \) and \( y \) are numbers in \( S \), then exactly one of these statements is true:

\[
x < y, \quad x = y, \quad y < x.
\]

(ii) (Transitivity) If \( x, y, \) and \( z \) are numbers of \( S \) such that \( x < y \) and \( y < z \), then \( x < z \).

The statement “\( x < y \)” reads “\( x \) is less than \( y \)”. The statement “\( y > x \)” is interchangeable with “\( x < y \)”.

We use the notation \( x \leq y \) to indicate that \( x < y \) or \( x = y \).

**Theorem 2.6.** (1) The set \( \mathbb{Z} \) is an ordered set of numbers that satisfies the addition axioms, the multiplication axioms, and the distributive axiom.

(2) The set \( \mathbb{N} \) is an ordered set of numbers that satisfies the addition axioms except for A3 and A4, the multiplication axioms, and the distributive axiom.

The order on \( \mathbb{Z} \) is given as follows: For any integers \( x \) and \( y \), we define the statement \( x < y \) to mean that \( y - x \) is a positive integer.
Using Theorem 2.6, we can rigorously prove a wide variety of results that you may have seen before and didn’t prove.

**Lemma 2.7.** If $S$ is a set of numbers satisfying the addition axioms, and $x$, $y$, and $z$ are members of $S$, then:

(i) (Cancellation): if $x + y = x + z$, then $y = z$.

(ii) (Uniqueness of zero): if $x + y = x$, then $y = 0$.

(iii) (Uniqueness of additive inverse): if $x + y = 0$, then $y = -x$.

(iv) (Cancellation of $-$): $-(x) = x$.

**Proof.** Here is a proof for (i). The rest is an exercise.

Suppose that $x + y = x + z$. By A4, there exists $(-x)$ in $S$ such that $x + (-x) = 0$. By A5, we have that $(x + y) + (-x) = (x + z) + (-x)$. By two applications of A2, we have that $(-x) + (x + y) = (-x) + (x + z)$. By two applications of A1, we have that $((-x) + x) + y = ((-x) + x) + z$. By two applications of A2, we have that $(x + (-x)) + y = (x + (z)) + z$. By two applications of A4, we have that $0 + y = 0 + z$. By two applications of A3, we conclude that $y = z$, as desired.

**Proposition 2.8.** If $x$, $y$, $z$ are integers, then:

(i) $0 \cdot x = 0$.

(ii) $(-x) \cdot y = -(x \cdot y)$

(iii) $(-x) \cdot (-y) = x \cdot y$.

(iv) If $x \cdot y = 0$, then $x = 0$ or $y = 0$ (or both).

(v) If $z \neq 0$ and $x \cdot z = y \cdot z$, then $x = y$.

**Proof.**

(i) By A3, A4, and DL, $0 + 0 \cdot x = 0 \cdot x = (0 + 0) \cdot x = 0 \cdot x + 0 \cdot x$. Thus $0 + 0 \cdot x = 0 \cdot x + 0 \cdot x$. Then the cancellation proved in Lemma 2.7(i) now tells us that $x \cdot 0 = 0$.

(ii) Observe that $(-x)y + xy = ((-x) + x)y$ by DL. This equals $0y$ by A4, which equals $0$ by Part (a). The result now follows by Lemma 2.7(iii). The other equality is proved similarly (but you should work it out!).

(iii) Exercise.

(iv) We will prove that if $x \neq 0$ and $y \neq 0$, then $xy \neq 0$. (Convince yourself that this is enough! This is the contrapositive of a statement we want to prove.)

First, suppose that $x > 0$ and $y > 0$. Then $x$ and $y$ are natural numbers, which are closed under multiplication. Thus $x \cdot y$ is a natural number, and zero is not.

Second, suppose that $x < 0$ and $y < 0$. Then $x = -m$ and $y = -n$ for certain positive integers $m$ and $n$. Thus $x \cdot y = (-m) \cdot (-n)$, which equals $m \cdot n$ by Part (iii). By our initial argument, $m \cdot n \neq 0$. Thus $x \cdot y \neq 0$, as desired.

Finally, suppose that one of $x > 0$ and $y < 0$. (You can switch these up if you’d like; the proof will be the same.) Then $x$ is a natural number, and $y = -n$ for some natural number $n$. Now, $x \cdot y = x \cdot (-n) = -(x \cdot n)$ by (ii). By our initial argument $x \cdot n \neq 0$. Thus $-(x \cdot n) \neq 0$.

(v) Exercise.
3. Equivalence relations

We now proceed to the construction of the rationals from the integers. But first, what should the rationals look like? For instance, integers should be rationals. Rationals should capture our usual notion of division. The addition, multiplication, and distribution axioms should hold. The rationals should be ordered.

Here is a trickier task: We want to properly ensure that $1/2 = 2/4 = 4/8 = (-1)/(-2) = (-2)/(-4) = (-4)/(-8) = \cdots$. We have infinitely many ways to write $1/2$. How do we rigorously say that they are all equal? How do we know that working with one “version” of $1/2$ is exactly the same as working with another “version” of $1/2$? This is a practical problem: If $1/2 = 2/4$, then how do we properly guarantee that $1/2 + 6/7 = 2/4 + 6/7$? How do we properly guarantee that $-(1/2) = -(2/4)$? How do we properly guarantee that $(1/2) \cdot (6/7) = (2/4) \cdot (6/7)$? Thus we need a very robust notion of “equality”.

We will introduce equivalence relations and equivalence classes in order to address this issue. Our ideas will be robust enough to accommodate the reals when we come to them. We begin with some convenient shorthand notation (largely to help my hand at the board!).

**Notation 3.1.** Let $A$ be any set (whose members may be numbers or other objects). We write $x \in A$ to indicate that “$x$ is in $A$” or “$x$ is a member of $A$” or “$x$ is an element of $A$”. If $x$ is not in $A$, then we write $x \notin A$.

**Notation 3.2.** The statement “$x := y$ is shorthand for “$x$ is defined to equal $y$.”

**Notation 3.3.** Given a set $X$, we let $X \times X$ denote the ordered pairs $(x_1, x_2)$, where $x_1$ and $x_2$ are members of $X$. (Think back to linear algebra; $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ is the set of ordered pairs of real numbers.) Order matters here!

**Definition 3.4.** A binary relation on a set $X$ is a subset of $X \times X$. If $\sim$ is a binary relation on $X$ and $x, y \in X$, we say that $x \sim y$ (“$x$ is related to $y$”) if $(x, y) \in \sim$.

**Example 3.5.** Let $X$ be the set of all people in Indonesia; then $X \times X$ is the set of all ordered pairs of people in Indonesia. The set $\sim = \{(x, y) : x, y \in X$ and $x, y$ are on the same island$\}$ is a binary relation.

**Definition 3.6.** A binary relation $\sim$ on a set $X$ is an equivalence relation if:

1. (Reflexivity) For all $x \in X$, $x \sim x$.
2. (Symmetry) For all $x, y \in X$ such that $x \sim y$, we have $y \sim x$.
3. (Transitivity) For all $x, y, z \in X$ such that $x \sim y$ and $y \sim z$, we have $x \sim z$.

**Example 3.7.** Referring to our previous example, suppose that $x, y, z$ are people in Indonesia. We have that $x \sim y$ precisely when $x$ is on the same island of Indonesia as $y$. If $x$ is on a given island, then $x$ is on the same island as $x$. Thus $(x, x)$ is in $\sim$, so $\sim$ is reflexive.

If $x$ and $y$ are on the same island, then $(x, y)$ is in $\sim$. But since $y$ is on the same island as $x$, $(y, x)$ is in $\sim$ as well. Thus $\sim$ is symmetric.

Suppose $x$ and $y$ are on the same island, and suppose that $x$ and $z$ are on the same island. Then $(x, y)$ and $(y, z)$ are in $\sim$. But since $x$ shares the island with $y$ and $y$ shares the island with $z$, $(x, z)$ is in $\sim$ as well. Thus $\sim$ is transitive, hence $\sim$ is an equivalence relation. Thus being on the same island of Indonesia is an equivalence on the people in Indonesia.

**Definition 3.8.** Let $\sim$ be an equivalence relation on a set $X$. We define the equivalence class of $x \in X$ (relative to the equivalence relation $\sim$) to be the set $[x] := \{y \in X : y \sim x\}$.
Example 3.9. Keeping with our example, for any person \( x \) on the island of Sumatra, the set \( \{ \text{people on Sumatra} \} \) is the equivalence class \( [x] \) of \( x \in X \), the set of all people in Indonesia.

Lemma 3.10. Let \( X \) be the set of all ordered pairs of integers with the second entry being non-zero. In set-builder notation, \( X = \{(a,b) : a, b \in \mathbb{Z} \text{ and } b \neq 0\} \). The relation \( \sim \) on \( X \) defined so that \( (a,b) \sim (c,d) \) if \( ad = cb \) is an equivalence relation. If \( a, b \in \mathbb{Z} \) and \( b \neq 0 \), the equivalence class of \( (a,b) \) (relative to \( \sim \)) is

\[
\{(c,d) : c, d \in \mathbb{Z}, d \neq 0, (a, b) \sim (c, d)\} = \{(c,d) : c, d \in \mathbb{Z}, d \neq 0, ad = cb\}.
\]

Proof. Let \( a, b, c, d, r, s \in \mathbb{Z} \), and assume that \( b, d, r \) are all nonzero. First, since \( ab = ba \), we have that \( (a,b) \sim (a,b) \), so \( \sim \) is reflexive. Second, if \( (a,b) \sim (c,d) \), then \( ad = cb \). But then \( cb = ad \) by commutativity, so \( (c,d) \sim (a,b) \). Finally, suppose \( (a,b) \sim (c,d) \) and \( (c,d) \sim (r,s) \). Then \( ad = cb \) and \( cs = rd \). We do not have “division” yet, but since \( b, d, s \) are non-zero, we have that \( ads = bcs \) (multiply \( ad = bc \) by \( s \) on both sides) and \( bcs = rbd \) (multiply \( cs = rd \) by \( b \) on both sides). Thus \( ads = rbd \). Since \( d \) is nonzero, it follows from Proposition 2.8 that \( as = rb \). Thus \( (a,b) \sim (r,s) \), and \( \sim \) is an equivalence relation. \( \square \)

Example 3.11. The binary relation on \( \mathbb{Z} \) given by \( a \sim b \) if \( a \leq b \) is not an equivalence relation. We have that \( 4 \leq 7 \), but \( 7 \) is not less than or equal to \( 4 \). Thus \( \sim \) is not reflexive. Similar problems exist for the binary relations given by \( \geq, <, \) and \( > \).

Definition 3.12. Let \( X \) be a set. A partition of \( X \) is a grouping of the members of \( X \) into non-empty subsets in such a way that each element of \( X \) is in exactly one of the subsets.

Example 3.13. Let \( X \) be the set of all people in Indonesia. Everyone in Indonesia must be on one of its islands (the water line forms the border of the state for our discussion). One cannot be on two of these islands at the same time. Thus if we group the people in Indonesia by which island you are on, we establish a partition of the people in Indonesia.

Equivalence classes have the following useful and important property:

Lemma 3.14. Suppose that \( \sim \) is an equivalence relation on a set \( X \). Then the set of equivalence classes in \( X \) (relative to \( \sim \)) form a partition of \( X \). In other words, each \( x \in X \) must lie in exactly one of the equivalence classes in \( X \) (relative to \( \sim \)).

Proof. Homework. \( \square \)

Another way to state the same thing is to say that the union of all the equivalence classes in \( X \) (relative to \( \sim \)) is \( X \), and the equivalence classes are pairwise disjoint. Once we have this property, we are prepared to define the quotient space \( X/\sim \) as follows.

Definition 3.15. Let \( \sim \) be an equivalence relation on a set \( X \). The quotient space \( X/\sim \) is the set \( \tilde{X} := \{[x] : x \in X\} \), the set of all equivalence classes of \( X \) (relative to \( \sim \)).

Quotients...this sounds like a great direction. (Think about where the notation \( \mathbb{Q} \) comes from.) Note that in Lemma 3.10, we have that \( (1,2) \sim (2,4) \sim (−1, −2) \sim (−2, −4) \sim \ldots \)
4. The rationals

**Definition 4.1** (The rationals). Let $X = \{(a,b) : a, b \in \mathbb{Z}, b \neq 0\}$. Let $\sim$ be the equivalence relation from Lemma 3.10. A **rational number** is an equivalence class in $X/\sim$. If $(a, b) \in X$, then we write $a/b$ (instead of $[(a,b)]$) for the equivalence class of $(a,b)$. We denote the set of rational numbers by $\mathbb{Q}$. If $a/b$ and $c/d$ are rational numbers, we define their sum

$$(a/b) + (c/d) := (ad + bc)/(b \cdot d),$$

their product

$$(a/b) \cdot (c/d) := (a \cdot c)/(b \cdot d),$$

and the negation $-(a/b) := (-a)/b$.

We have defined the rationals, but we have no idea how the arithmetic of $\mathbb{Q}$ works yet. We have to prove everything! The notation $a/b$ is meant to be suggestive of the fact that we’ll eventually get to the usual $a/b$, but we’re not there yet! But we can take advantage of our pre-existing knowledge about the arithmetic of fractions to guide us. This guided our definitions for adding/multiplying/negating rationals.

We already proved that if $b$ and $d$ are non-zero, then $bd$ is also non-zero. Thus the sum and product of two rational numbers remains a rational number. But we encounter the subtle problem of *well-definedness*. Remember, $a/b = [(a,b)]$, which is an entire equivalence class of pairs of integers. Moreover, we can see that $[(1,2)] = [(2,4)]$. Does this necessarily mean that $1/2 + 6/7 = 2/4 + 6/7$? Does this necessarily mean that $(1/2) \ast (6/7) = (2/4) \ast (6/7)$? Does this necessarily mean that $-(1/2) = -(2/4)$? In particular, since there are infinitely many ways to represent a rational number,

$$(***)$$

does the way we represent a rational number affect the arithmetic of $\mathbb{Q}$?

**Lemma 4.2.** The sum, product, and negation operations on rational numbers are well-defined. That is, if $a/b$ and $c/d$ are rational numbers and $a'/b' = a/b$ (under the equivalence relation in Lemma 3.10), then $a'/b + c'/d = a'/b' + c'/d$, and similarly for products and negation. (So the answer to the question $(***)$ is NO.)

**Proof.** I’ll work out the proof for sums; I leave products and negation as homework.

Let $a/b$, $a'/b'$, and $c/d$ be rationals (but they are still equivalence classes!); thus $a, b, a', b', c, d$ are integers, and $b, b', d$ are non-zero. Suppose that $a/b = a'/b'$, in which case $ab' = a'b$. We shall show that

$$a/b + c/d = a'/b' + c'/d.$$

By definition, the left-hand side is $(ad + bc)/bd$ and the right-hand side is $(a'd + b'c)/b'd$, so upon unravelling the definition of the equivalence relation, we have to show that $(ad + bc)b'd = (a'd + b'c)b d$. This expands to $ab'd^2 + bb'cd = a'bd^2 + bb'cd$. By additive cancellation for integers, it remains to prove that $ab'd^2 = a'bd^2$. Since $ab' = a'b'$, we have the desired conclusion. □

Observe that the rational numbers $a/1$ behave just like the integers $a$:

$$(a/1) + (b/1) = (a + b)/1, \quad (a/1) \cdot (b/1) = (a \cdot b)/1, \quad -(a/1) = (-a)/1.$$

Also, $a/1 \sim b/1$ exactly when $a = b$. Because of these, we will identify each integer $a$ with $a/1$; the above observation guarantees that the arithmetic of the integers is consistent with the arithmetic of the rationals, and we can think of the integers as being embedded (sitting inside) the rationals. We identify 0 with 0/1 and 1 with 1/1.

Observe that a rational number $a/b$ is equal to 0 = 0/1 if and only if $a \cdot 1 = b \cdot 0$, i.e., if the numerator $a$ is equal to 0. Thus if $a$ and $b$ are non-zero then so is $a/b$. 

Definition 4.3. If \( x = a/b \) is a non-zero rational number, (so \( a, b \neq 0 \)), then the reciprocal \( x^{-1} \) of \( x \) is the rational number \( x^{-1} := b/a \). (Check that the reciprocal is well-defined: \( a/b = c/d \), then they have the same reciprocal.)

We now finally get to prove the usual arithmetic laws for \( \mathbb{Q} \).

Proposition 4.4. The set of numbers \( \mathbb{Q} \) satisfies the addition axioms (Axiom 2.1), the multiplication axioms (Axiom 2.3), and distributive axiom (Axiom 2.4). Additionally, if \( x \) is a non-zero rational number, then \( x \cdot x^{-1} = x^{-1} \cdot x = 1 \).

Proof. To give you an idea of what is needed, we will prove the longest part, namely that addition is associative. The rest will be left as homework.

Let \( x, y, z \) be rational numbers. We write \( x = a/b, y = c/d, \) and \( z = e/f \) for certain integers \( a, c, e \) and certain nonzero integers \( b, d, f \). Now, we compute

\[
(x + y) + z = ((a/b) + (c/d)) + e/f = (ad + bc)/(bd) + e/f = (adf + bcf + bde)/bdf = (a/b) + ((c/d) + e/f) = x + (y + z).
\]

Thus we see that \((x + y) + z \) and \( x + (y + z) \) are equal.

Definition 4.5. If \( x, y \) are rational numbers and \( y \neq 0 \), we define the quotient of \( x \) and \( y \) by the formula \( x/y := x \cdot y^{-1} \).

Example 4.6. \( (3/4)/(5/6) = (3/4) \cdot (6/5) = 18/20 = 9/10 \).

Using the definition of the quotient and viewing the integer \( a \) as the rational \( a/1 \), we find that \( a/b = (a/1) \cdot (b/1)^{-1} \) corresponds naturally with the usual notion of fraction \( a/b \) that you are familiar with. Thus we can (finally!) discard the // notation and use the usual \( a/b \) notation instead. Similarly, we use the shorthand \( x - y \) to denote \( x + (-y) \).

Definition 4.7. A rational number \( a/b \) is defined to be positive when \( x = a/b \) for some positive integers \( a \) and \( b \) and negative if \( x = (-a)/b \) for some positive integers \( a \) and \( b \).

Note that since \( ab = (-a)(-b) \) (HW), we have that \( (a, b) \sim (-a, -b) \). One now can appeal to Lemma 3.14 to see that \( a/b = (-a)/(-b) \). In our new shorthand, this reads as \( a/b = (-a)/(-b) \). It now follows from our definition of positivity that \( a/b \) is positive if and only if \( a \) and \( b \) are both negative. Thus our definition of positivity is comprehensive, and the same can be shown for our definition of negativity.

Definition 4.8. Let \( a/b \) and \( c/d \) be rational numbers. We say that \( x > y \) precisely when \( x - y \) is a positive rational. We say that \( x < y \) precisely when \( x - y \) is a negative rational number. We say that \( x \geq y \) when either \( x > y \) or \( x = y \), and we similarly define \( x \leq y \).

Proposition 4.9. The relation \( < \) in Definition 4.8 on \( \mathbb{Q} \) makes \( \mathbb{Q} \) an ordered set (recall Definition 2.5).

Proof. Exercise.

5. Oct. 3: Absolute value and sequences

A key concept in analysis is that of the absolute value, which measures closeness to zero.
Definition 5.1. Let \( x, y \in \mathbb{Q} \). We define the absolute value of \( x \), denoted \( |x| \) by

\[
|x| := \begin{cases} 
  x & \text{if } x > 0, \\
  0 & \text{if } x = 0, \\
  -x & \text{if } x < 0.
\end{cases}
\]

By splitting into four cases (i) \( x, y \geq 0 \), (ii) \( x, -y \geq 0 \), (iii) \(-x, y \geq 0 \), and (iv) \( x, y \leq 0 \), we can see that

\[
|xy| = |x| \cdot |y|
\]

for all \( x, y \in \mathbb{Q} \). In particular, \( |x| \geq 0 \) for all \( x \in \mathbb{Q} \).

Here is probably the most important and heavily used inequality in analysis.

**Theorem 5.2** (The triangle inequality). If \( x, y \in \mathbb{Q} \), then \( |x + y| \leq |x| + |y| \).

In order to prove the triangle inequality, we require two intermediate results.

**Lemma 5.3.** If \( x \in \mathbb{Q} \), then \(-|x| \leq x \leq |x| \).

*Proof.* Homework. \( \square \)

**Lemma 5.4.** If \( x, y \in \mathbb{Q} \), then \( |x| \leq |y| \) if and only if \(-|y| \leq x \leq |y| \).

*Proof.* Homework. Note that you need to prove two separate results here. You need to prove that (i) if \( |x| \leq |y| \), then \(-|y| \leq x \leq |y| \) AND (ii) if \(-|y| \leq x \leq |y| \), then \( |x| \leq |y| \). \( \square \)

*Proof of Theorem 5.2.* Let \( x, y \in \mathbb{Q} \). We see from the definition of absolute value that \( ||x| + |y|| = |x| + |y| \). Thus the statement we want to prove is the same as \( |x + y| \leq ||x| + |y|| \). By Lemma 5.4, this holds if and only if \(-||x| + |y|| \leq x + y \leq ||x| + |y|| \). But since \( ||x| + |y|| = |x| + |y| \), the statement that we seek to prove is the same as proving \(-(|x| + |y|) \leq x + y \leq |x| + |y| \).

This follows from adding together the inequalities

\[
-|x| \leq x \leq |x|,
\]
\[
-|y| \leq y \leq |y|.
\]

from Lemma 5.3. \( \square \)

The absolute value will play an indispensable role in our construction of the reals. Recall that \( \mathbb{Q} \) does not contain all of the numbers you know; it does not contain \( \sqrt{2} \), for example. (See §2 of Ross for a detailed discussion on how to produce an abundance of numbers which are not in \( \mathbb{Q} \).) So, even though every pair of distinct rationals \( x \) and \( y \) has many rationals in between them (like \((x + y)/2\)), \( \mathbb{Q} \) contains many “holes”. We have a good intuition for what a real number should look like and how it should behave. However, we do not have definitions to match our intuition (yet!).

Let’s start with \( \sqrt{2} \). We will not cover decimal expansions in detail, but you already know that \( 1.4 = 14/10 \), \( 1.41 = 141/100 \), \( 1.414 = 1414/1000 \), etc. While we cannot produce any
rational number whose square is 2, we can observe (experimentally, if you will):

\[
\begin{align*}
|1 - 2| &= 1 \\
|1.4^2 - 2| &= 0.04 \\
|1.41^2 - 2| &= 0.0119 \\
|1.414^2 - 2| &= 0.000604 \\
|1.4142^2 - 2| &= 0.00003836 \\
|1.41421^2 - 2| &= 0.0000100759.
\end{align*}
\]

The sequence of numbers 1, 1.4, 1.41, etc. looks like it converges to 2, in the sense that the absolute values \(|1 - 2|, |1.4 - 2|, |1.41 - 2|, \ldots\) get closer and closer to zero. If we proceed in this fashion using a suitable succession of rationals (whose absolute value when subtracted from 2 keeps getting closer and closer to zero), then it looks like we have a shot at actually defining \(\sqrt{2}\). Of course we must make precise what we mean by sequence and converge because we only can work with the rationals!

**Definition 5.5.** A sequence of rational numbers is a function \(a : \mathbb{Z} \to \mathbb{Q}\) whose domain contains set of the form \(\{n \in \mathbb{Z} : n \geq 1\}\) (though starting at 0 or another fixed integer also works). It is customary to use \((a_1, a_2, a_3, \ldots)\), \((a_n)_{n=1}^\infty\), or \((a_n)_{n \in \mathbb{N}}\) to denote a sequence rather than the function itself. Sometimes we will write \((a_n)\) when the domain is understood or when the results under discussion do not depend on the specific starting point.

We now restrict ourselves to \(a_n \in \mathbb{Q}\); we will later extend this to \(a_n \in \mathbb{R}\).

**Example 5.6.** Consider the function \(a : \mathbb{Z} \to \mathbb{Q}\) given by \(a(n) = 1/n\). This gives us the sequence \((1, 1/2, 1/3, 1/4, \ldots)\), also written \((1/n)_n=1^\infty\) or \((1/n)_{n \in \mathbb{N}}\). Since \(1/0\) is not a rational number, we might also write \((1/n)\).

**Example 5.7.** Consider the function \(a : \mathbb{Z} \to \mathbb{Q}\) given by

\[
a(n) = \begin{cases} 
1 & \text{if } n \text{ is a multiple of } 3, \\
0 & \text{otherwise}.
\end{cases}
\]

This gives us the sequence \((0, 0, 1, 0, 0, 1, 0, 0, 1, \ldots)\). It’s a little tricky to write this as \((a_n)\).

Key point: The set of values attained by a sequence is different from the sequence itself! In the last example, the sequence is \((0, 0, 1, 0, 0, 1, 0, 0, 1, \ldots)\), but the set of values attained by the sequence is \(\{0, 1\}\).

**Example 5.8.** Let \(n \geq 1\) be an integer, and let \(x \geq 0\) be a natural number. We use the shorthand \(x^n\) to denote that we add \(x\) to itself \(n\) times. Consider the function \(a(n) : \mathbb{Z} \to \mathbb{Q}\) given by \(a(n) = (1 + 1/n)^n\). This gives us the sequence \((2, (3/2)^2, (4/3)^3, (5/4)^4, \ldots)\). The decimal approximation looks like

\[
(2, 2.25, 2.3704, 2.4414, 2.4883, 2.5216, 2.5465, 2.5658, \ldots).
\]

To which number might we get really close to if we continue?

**Example 5.9.** Sequences can be defined recursively. For example, let \(a_1 = 0, a_2 = 1, a_3 = a_2 + a_1, a_4 = a_3 + a_2, a_5 = a_4 + a_3,\) and so on. We can write this as \(a_4 = 0, a_2 = 1,\) and \(a_{n+2} = a_{n+1} + a_n\) for all \(n \geq 3\). This gives us the Fibonacci sequence \((0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots)\).
6. Cauchy sequences

In preparation for constructing $\mathbb{R}$, we need to develop the theory of sequences. In our example $(a_n) = (1, 1.4, 1.41, 1.414, 1.4142, 1.41421, \ldots)$, it looks like all that we need is for the terms in the sequence to get closer and closer to $\sqrt{2}$ (in the sense that absolute value $|a_n^2 - 2|$ gets closer and closer to zero). Then we could define $\sqrt{2}$ to be the limit of the sequence. However, this has a subtle problem. The number $\sqrt{2}$ is also the limit of many other rational sequences, such as $(b_n) = (1.4, 1.414, 1.41421, 1.4142135, 1.414213562, \ldots)$, even though $a_n \neq b_n$ for all $n$. How then can it be seen that $\sqrt{2}$ is a unique element of the real numbers? Does the choice of sequence whose limit is $\sqrt{2}$ affect the way we do computations with $\sqrt{2}$?

But suppose we could resolve this issue. What should $(a_n)$ converge to? We are incapable of making a prediction because we don’t have an explicit description of $a_n$; also, what would we do if the value to which $(a_n)$ converges is not rational? The following definition allows us to imagine sequences $(a_n)$ with the $a_n$’s “getting close to something”, but the idea of “getting close” will be defined entirely in terms of the $a_n$’s, which is exactly what we need!

Definition 6.1. A Cauchy sequence of rational numbers is a sequence $(a_1, a_2, a_3, \ldots)$ of rational numbers such that for every rational $\varepsilon > 0$, there exists a positive integer $N_\varepsilon$ (which is allowed to depend on $\varepsilon$) such that

$$|a_m - a_n| < \varepsilon \quad \text{whenever } m, n \geq N_\varepsilon.$$

(The restriction that $\varepsilon$ needs to be a rational number is there purely because we don’t know what a real number is yet. Later we will consider Cauchy sequences of real numbers and we will think of $\varepsilon$ as being any positive real number. You should not think about this distinction too much, as it will not be important in the long run.)

Let us delve into Definition 6.1. It says that we must first pick a error, which we call $\varepsilon$. You can pick your error to be as small as you like (say, $\varepsilon = 1/100$ or $\varepsilon = 1/10^{100}$), as long as it is positive. So pick the error that you want, and we will move on to the next part.

Now that you have picked your error, there is a threshold (a positive integer $N_\varepsilon$ which is allowed to depend on the error $\varepsilon$ that you chose) with a very special property that we want to have: Eventually (i.e., after $m, n$ get past the threshold $N_\varepsilon$), the terms $a_m$ and $a_n$ get REALLY close together (i.e., $|a_m - a_n|$ is less than the error $\varepsilon$ that you chose at the beginning). This needs to hold for ALL $m, n \geq N$ (e.g., not just for $n$ and $m = n + 1$). The definition now says that a sequence is Cauchy if this special property holds for any choice of $\varepsilon > 0$. So you should be able to make your error at the beginning as small as you could possibly imagine (and then even smaller).

This is a lot to ask! Think about it: The definition is saying that if $n \geq N_\varepsilon$, then $|a_n - a_{n+1}|$, $|a_n - a_{n+2}|$, $|a_n - a_{n+20}|$, $|a_n - a_{n+10000}|$, etc. are all smaller than $\varepsilon$. That’s a very special kind of sequence. Moreover, you can take $\varepsilon$ to be arbitrarily small!

Proposition 6.2. The sequence $(1/n)_{n \in \mathbb{N}} = (1, 1/2, 1/3, 1/4, 1/5, \ldots)$ is a Cauchy sequence.

Sometimes, for these proofs, it’s good to “work backwards”. Let’s first recall the definition: Pick an error $\varepsilon > 0$. We would like to come up with a threshold $N$ (a positive integer) such that $|1/m - 1/n| < \varepsilon$ whenever $m, n \geq N$. Since $m, n \geq N$, we can use the triangle inequality to obtain $|1/m - 1/n| \leq 1/m + 1/n \leq 2/N$. Therefore, it is enough to come up with a positive integer $N$ for which $2/N < \varepsilon$, or alternatively $N > 2/\varepsilon$. Does such a positive integer exist?

Proposition 6.3 (Archimedean property). If $x \in \mathbb{Q}$, there exists $n \in \mathbb{N}$ such that $n > x$. 
Proof. Since \( x \in \mathbb{Q} \), we can write \( x = p/q \) for some integers \( p, q \) with \( q \geq 1 \). If \( p \leq 0 \), then \( 1 > x \), as desired. So we may now assume that \( p \geq 1 \). But then \( p/q \leq p < p+1 \), so \( p+1 > x \), as desired. (Notice how convenient the trichotomy is!) \( \square \)

This supplies the missing piece in Proposition 6.2, so we can now write a formal proof.

Proof of Proposition 6.2. Let \( \varepsilon > 0 \). Let \( N \) be an integer such that \( N > 2/\varepsilon \). Then, if \( m, n \geq N \), we have \(|1/m - 1/n| \leq 1/N + 1/N < \varepsilon/2 + \varepsilon/2 = \varepsilon \). Thus \((1/n)_{n \in \mathbb{N}} \) is Cauchy. \( \square \)

But not all sequences are Cauchy sequences.

Proposition 6.4. For each \( n \in \mathbb{N} \), let \( H_n = 1 + 1/2 + 1/3 + \cdots + 1/n \). The sequence \((H_n)_{n \in \mathbb{N}}\) is not a Cauchy sequence.

To show that a sequence \((a_n)\) is not Cauchy, \((a_n)\) must satisfy the negation of the definition of a Cauchy sequence. Thus we need to produce a specific \( \varepsilon > 0 \) such that for all thresholds \( N \in \mathbb{N} \), we have \(|a_m - a_n| \geq \varepsilon\) for some \( m, n \geq N \). So no matter how big you make your threshold \( N \), you will always be able to find a pair \( m, n \) for which the distance between \( a_m \) and \( a_n \) is always greater than \( \varepsilon \).

Proof of Proposition 6.4. Let \( \varepsilon = 1/2 \) (others could work). For each \( n \in \mathbb{N} \), consider

\[
|H_{2n} - H_n| = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n}.
\]

There are \( n \) terms in the sum above, and each one is at least as big as \( 1/(2n) \). Thus

\[
|H_{2n} - H_n| = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n} \geq \frac{1}{2n} + \cdots + \frac{1}{2n} = \frac{1}{2}.
\]

Therefore, for all \( N \in \mathbb{N} \), we can always find a pair of integers \( m \) and \( n \) (namely \( n \) and \( 2n \)) such that \(|a_m - a_n| \geq 1/2 \). Thus \((H_n)\) is not Cauchy. \( \square \)

Proving whether a sequence is Cauchy by brute force (like the last two propositions) can be very tough. We will develop tools to help us work with Cauchy sequences more efficiently. This will require a notion of boundedness.

Definition 6.5. A sequence \((a_n)\) (of rational numbers) is bounded if there exists a rational number \( M \geq 0 \) such that \(|a_n| \leq M\) for all \( n \geq 1 \). We then say that \( \langle a_n \rangle \) is bounded by \( M \).

Not all sequences are bounded. Consider the sequence \((n)_{n \in \mathbb{N}}\).

Proposition 6.6. Every Cauchy sequence of rational numbers is bounded.

Proof. Suppose \((a_n)\) is a Cauchy sequence. Taking \( \varepsilon = 1 \), this means that there exists an integer \( N \geq 1 \) such that \(|a_m - a_n| < 1\) for all integers \( m, n \geq N \). Consider the set of absolute values \(|a_1|, |a_2|, |a_3|, \ldots, |a_N|\). Let \( M_0 \) denote the largest of these absolute values (this is fine because this is a finite set of numbers). Now, let \( M = M_0 + 1 \).

We will show that each \( a_n \) in the sequence with \( n > N \) satisfies \(|a_n| \leq M \). To see this, we use the “adding zero trick”. In particular, we observe that \( a_n = a_n - a_N + a_N \). Now, an application of the triangle inequality yields \(|a_n| = |a_n - a_N + a_N| \leq |a_n - a_N| + |a_N|\). Since \( n > N \), we have from before that \(|a_n - a_N| < 1\), that \(|a_n| \leq M_0\), and \( M_0 = M + 1 \). Thus

\[
|a_n| \leq |a_n - a_N| + |a_N| \leq 1 + M_0 = M.
\]

Since \( M_0 < M \), we find that \(|a_n| \leq M\) for all \( n \in \mathbb{N} \), as desired. \( \square \)

The “adding zero trick” is a very important tool!!! Remember this one.
7. Arithmetic of Cauchy sequences

We now show that we can “add” and “multiply” Cauchy sequences.

**Proposition 7.1.** If \((a_n)\) and \((b_n)\) are Cauchy sequences of rational numbers, then so are \((a_n + b_n), (a_n b_n), \) and \((-a_n)\).

**Proof.** Pick an error \(\varepsilon > 0\). Since \((a_n)\) and \((b_n)\) are Cauchy, there exist positive integers \(N_a(\varepsilon)\) and \(N_b(\varepsilon)\) such that for all \(m, n \geq N(\varepsilon)\) and \(m, n \geq N(\varepsilon)\), we have

\[
|a_m - a_n| < \frac{\varepsilon}{2} \quad \text{and} \quad |b_m - b_n| < \frac{\varepsilon}{2}.
\]

(You’ll see shortly why we chose \(\varepsilon/2\); convince yourself that this is OK.) Let \(N(\varepsilon) = \max\{N_a(\varepsilon), N_b(\varepsilon)\}\), and suppose that \(m, n \geq N\). Then

\[
|(a_m + b_m) - (a_n + b_n)| = |(a_m - a_n) + (b_m - b_n)| \leq |a_m - a_n| + |b_m - b_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

Thus \((a_n + b_n)\) is a Cauchy sequence.

For products, we will use Proposition 6.6. This tells us that there exist numbers \(M_a \geq 0\) and \(M_b \geq 0\) such that \(|a_n| \leq M_a\) and \(|b_n| \leq M_b\) for all \(n \in \mathbb{N}\). Set \(M = \max\{M_a, M_b\}\). Since \((a_n)\) and \((b_n)\) are Cauchy, there exist natural numbers \(N_a(\varepsilon, M)\) and \(N_b(\varepsilon, M)\) (they can, and probably will, depend on \(M\)) such that if \(m, n \geq N(\varepsilon, M)\) and \(m, n \geq N(\varepsilon, M)\), then

\[
|a_m - a_n| < \frac{\varepsilon}{2M} \quad \text{and} \quad |b_m - b_n| < \frac{\varepsilon}{2M}.
\]

Let \(N(\varepsilon, M) = \max\{N_a(\varepsilon, M), N_b(\varepsilon, M)\}\). Then if \(m, n \geq N(\varepsilon, M)\), we have, using the “adding zero trick” and the triangle inequality,

\[
|a_m b_m - a_n b_n| = |a_m b_m - a_m b_n + a_m b_n - a_n b_n| \\
\leq |a_m b_m - a_m b_n| + |a_m b_n - a_n b_n| \\
= |a_m| \cdot |b_m - b_n| + |b_n| \cdot |a_m - a_n| < M \frac{\varepsilon}{2M} + M \frac{\varepsilon}{2M} = \varepsilon.
\]

Thus \((a_n b_n)\) is Cauchy. We leave the proof that \((-a_n)\) is Cauchy as an exercise. \(\square\)

It is starting to look like Cauchy sequences enjoy a structure similar to the rationals. Is there an additive identity? Yes, namely \((0, 0, 0, 0, \ldots)\). Is there a multiplicative identity? Yes, namely \((1, 1, 1, 1, 1, 1, \ldots)\).

What about division? Tempting as it is to say that \((a_n/b_n)\) is Cauchy if \((a_n)\) and \((b_n)\) are, we have a problem: What if any of the numbers \(b_n\) equal zero? That would ruin the division for the whole sequence. So division wouldn’t work even if we “divide by” \((1, 0, 1, 1, 1, 1, 1, \ldots)\).

A more interesting example comes from comparing the three Cauchy sequences

\[
(a_n) = (1/n), \quad (b_n) = (2/n), \quad (c_n) = (3/n).
\]

Each “looks like” their terms are approaching 0, but \(a_n \neq b_n \neq c_n\) for all \(n\). So the sequences are definitely not the same. But is there a notion of “equality” for \((a_n), (b_n), \) and \((c_n)\) that registers with our observation that the terms \(a_n, b_n, \) and \(c_n\) each approach 0?

**Definition 7.2.** We define a binary relation \(\sim\) on the set of Cauchy sequences of rational numbers \((a_n)\) as follows. We define \((a_n) \sim (b_n)\) (that is, \((a_n)\) is “equivalent to” or “related to” \((b_n)\)) to mean that for every rational \(\varepsilon > 0\), there exists a positive integer \(N(\varepsilon)\) such that if \(n \geq N\), then

\[
|a_n - b_n| < \varepsilon.
\]
Lemma 7.3. The binary relation in Definition 7.2 is an equivalence relation.

Proof. Homework. □

This notion of equivalence allows us to come to a good definition for the “division” of two Cauchy sequences. This requires some further development.

Definition 7.4. A Cauchy sequence of rational numbers \( (b_n) \) is said to be bounded away from zero if there exists a rational number \( \ell > 0 \) such that \( |b_n| \geq \ell \) for every \( n \in \mathbb{N} \).

Proposition 7.5. Suppose that \( (a_n) \) is a Cauchy sequence of rational numbers that is not equivalent to the zero sequence \( (0, 0, 0, \ldots) \). Then there exists a Cauchy sequence of rational numbers \( (b_n) \) such that

1. \( (b_n) \) is bounded away from zero (so there exists \( \ell > 0 \) such that \( |b_n| \geq \ell \) for all \( n \)),
2. \( (a_n) \sim (b_n) \).

Proof. Suppose that \( (a_n) \) is not equivalent to the zero sequence. We need to negate the definition of \( \sim : \) There exists some fixed rational \( \varepsilon_0 > 0 \) (we might not know the exact number) such that for each \( N \geq 1 \), there exists a number \( n \geq N \) for which

\[ |a_n - 0| = |a_n| \geq \varepsilon_0. \]

Now, \( (a_n) \) is a Cauchy sequence, so there exists some (fixed) \( N_0 \) (we might not know the exact number) for which

\[ |a_m - a_n| < \frac{\varepsilon_0}{2} \quad \text{whenever } m, n \geq N_0. \]

And for this fixed \( N_0 \), there is a fixed \( n_0 \geq N_0 \) for which

\[ |a_{n_0} - 0| = |a_{n_0}| \geq \varepsilon_0. \]

Claim: \( |a_n| \geq \varepsilon_0/2 \) for every \( n \geq N_0 \).

Proof of the claim: Suppose to the contrary that \( |a_{n_1}| < \varepsilon_0/2 \) for some \( n_1 \geq N_0 \). Then, by the triangle inequality and the “adding zero trick”,

\[ |a_{n_0}| = |a_{n_0} - a_{n_1} + a_{n_1}| \leq |a_{n_0} - a_{n_1}| + |a_{n_1}| < \frac{\varepsilon_0}{2} + \frac{\varepsilon_0}{2} = \varepsilon_0. \]

But this contradicts (7.1), so we must have \( |a_n| \geq \varepsilon_0/2 \) for every \( n \geq N_0 \). This proves the claim.

This almost finishes the proof. We have shown that \( a_n \) eventually satisfies \( |a_n| \geq \ell \), where \( \ell = \varepsilon_0/2 \). Now, we simply define

\[ b_n = \begin{cases} \ell & \text{if } 1 \leq n < N_0, \\ a_n & \text{if } n > N_0. \end{cases} \]

Since \( (a_n) \) is Cauchy, \( (b_n) \) is Cauchy as well. Clearly, \( (b_n) \) is bounded away from zero. Now, \( (a_n) \sim (b_n) \) because \( |a_n - b_n| = 0 \) for all \( n \geq N_0 \). This finishes the proof. □
8. Arithmetic of Cauchy sequences, II. The definition of a real number.

We can now give some conditions under which \((a_n)\) being Cauchy implies that \((a_n^{-1})\) is Cauchy.

**Proposition 8.1.** Suppose that \((a_n)\) is a Cauchy sequence of rational numbers which is bounded away from zero. Then the sequence \((a_n^{-1})\) is also a Cauchy sequence.

**Proof.** Since \((a_n)\) is bounded away from zero, there exists some fixed rational number \(\ell > 0\) such that \(|a_n| \geq \ell\) for all \(n\). Thus for any positive integers \(m, n\), we have that

\[
|a_m^{-1} - a_n^{-1}| = \left| \frac{a_m - a_n}{a_m a_n} \right| \leq \frac{|a_m - a_n|}{\ell^2} = \frac{1}{\ell^2} \cdot |a_m - a_n|.
\]

Now, since \((a_n)\) is Cauchy, we have that for any rational error \(\varepsilon > 0\), there exists a positive integer \(N_\varepsilon\) (our threshold) such that

\[
|a_m - a_n| < \ell^2 \varepsilon \quad \text{whenever } m, n \geq N.
\]

For that same \(\varepsilon\) and \(N_\varepsilon\), we also have

\[
|a_m^{-1} - a_n^{-1}| \leq \frac{1}{\ell^2} \cdot |a_m - a_n| < \frac{1}{\ell^2} \cdot \ell^2 \varepsilon = \varepsilon \quad \text{whenever } m, n \geq N.
\]

Thus \((a_n^{-1})\) is Cauchy, as desired. \(\square\)

Now, we can define the “division” of two Cauchy sequences of rational numbers \((a_n)\) and \((b_n)\) to be \((a_n b_n^{-1})\), provided that \((b_n)\) is bounded away from zero. Additionally, Proposition 7.5 shows that any sequence \((b_n)\) which is not equivalent to \((0, 0, 0, \ldots)\) is equivalent to a sequence \((c_n)\) which is bounded away from zero.

But we encounter a problem. Suppose that \((a_n), (b_n), (c_n)\) are Cauchy sequences of rational numbers. Suppose also \((b_n)\) and \((c_n)\) are bounded away from zero and that \((b_n) \sim (c_n)\). Is it necessarily the case that \((a_n b_n^{-1}) \sim (a_n c_n^{-1})\)? In other words, is this notion of “division” well-defined, or does the output from the “division” process depend on the choice of element you pick from the equivalence class of \((b_n)\)?

**Lemma 8.2.** “Division” of Cauchy sequences is well-defined. That is, if \((a_n), (b_n), (c_n)\) are Cauchy sequences of rational numbers, \((b_n) \sim (c_n)\), and both \((b_n)\) and \((c_n)\) are bounded away from zero, then \((a_n b_n^{-1}) \sim (a_n c_n^{-1})\).

**Proof.** Exercise. \(\square\)

We can now define the real numbers! Before we do, let’s reflect on what happened when we defined the rationals. We looked at equivalence classes of ordered pairs of integers. We came up with reasonable definitions of adding, multiplying, and negating. Then, we showed that these operations are well-defined (e.g., addition doesn’t change if you use 1/2 instead of 2/4, etc.) using properties of the integers. Then we proved all of the pertinent properties of the rationals, which reduced to studying ordered pairs of integers. We now proceed along a similar path for the reals, now appealing to the properties of the rationals.

**Definition 8.3.** Let \(X\) be the set of all Cauchy sequences of rational numbers, and let \(\sim\) be the equivalence relation given by Definition 7.2. A **real number** is an equivalence class in \(X/\sim\). We denote the set of all real numbers by \(\mathbb{R}\). Recall the notation that \([(a_n)]\) is the set of Cauchy sequences of rationals which are equivalent to \((a_n)\) under \(\sim\). Let \(x = [(a_n)]\) and \(y = [(b_n)]\) be real numbers. We define their sum \(x + y := [(a_n + b_n)]\), their product
By the definition of multiplication, we have that sequences \((a_n \cdot b_n)\), and negation \(-x := \([-a_n]\). If \((b_n)\) is bounded away from zero, then we define the reciprocal \(y^{-1} := \([b_n^{-1}]\). We say that \(x = 0\) if \([a_n] = [(0, 0, 0, \ldots)]\).

This does not look at all like the definition of a number! How might we actually get something that resembles a number out of this mess? What we would really like to do is be able to say that a real number is a limit of a convergent Cauchy sequence of reals, and that two real numbers are equal if the pertinent Cauchy sequences are equivalent under the ~ from Definition 7.2. Here is the strategy to approach this:

- Get a sense of the arithmetic of \(\mathbb{R}\) (like we did with \(\mathbb{Q}\))
- (Finally!) Define notions of convergence and limit
- Prove that all Cauchy sequences of rational numbers converge to a real number
- Prove that equivalent Cauchy sequences converge to the same real number

With these ideas in place, we can associate to any equivalence class of Cauchy sequences to their collective limit, which is precisely the sort of idea that guided us before.

It is not clear that rationals are real numbers yet (rational numbers are not equivalence classes of sequences of rational numbers), but there is a sense in which we can embed \(\mathbb{Q}\) into \(\mathbb{R}\). For each \(x \in \mathbb{Q}\), the sequence \((x, x, x, \ldots)\) is Cauchy (why?), so the equivalence class \[][] is a real number. In this way, we can think of \(\mathbb{Q}\) as sitting inside of \(\mathbb{R}\).

**Lemma 8.4.** If \(x \neq 0\), then there exists a Cauchy sequence \((b_n)\) of rationals which is bounded away from zero such that \(x^{-1} = \([b_n^{-1}]\) . In particular, if \(x \neq 0\), then \(x^{-1}\) exists.

**Proof.** If \(x \neq 0\), then we can write \(x\) as an equivalence class \([a_n]\) such that \((a_n) \not\sim (0, 0, 0, \ldots)\). By Proposition 7.5, we can find a Cauchy sequence of rationals \((b_n)\) which is bounded away from zero such that \((a_n) \sim (b_n)\). By Lemma 3.14, it follows that \([a_n] = [b_n]\) . Since \((b_n)\) is bounded away from zero, Proposition 8.1 tells us that \((b_n^{-1})\) is a Cauchy sequence, and so Definition 8.3 tells us that \([b_n^{-1}] = [(b_n)^{-1}] = x^{-1}\).

As with the rationals, we first check the well-definedness of adding, multiplying, negating, and reciprocating Cauchy sequences of rationals.

**Lemma 8.5.** Addition, multiplication, negation, and reciprocation of the reals are well-defined. That is, suppose that \(x = [(a_n)]\), \(y = [(b_n)]\), and \(z = [(c_n)]\) are real numbers and \(x = y\). (Here, \((a_n)\), \((b_n)\), and \((c_n)\) are Cauchy sequences of rationals.) Then

1. \(x + z = y + z\),
2. \(x \cdot z = y \cdot z\),
3. \(-x = -y\),
4. If \(x \neq 0\), then \(x^{-1} = y^{-1}\).

**Proof.** Part 4 is the hardest, so we will prove that here. The rest are left as homework. Part 4 relies on the well-definedness of multiplication in Part 2.

Assume that \(x = y\), in which case \([a_n] = [b_n]\). Since \(x \neq 0\) (hence \(y \neq 0\)), Lemma 8.4 tells us that we can assume that \((a_n)\) and \((b_n)\) are bounded away from zero. Thus the reciprocal sequences \((a_n^{-1})\) and \((b_n^{-1})\) are Cauchy sequences, and so \(x^{-1} = \([a_n^{-1}]\) and \(y^{-1} = \([b_n^{-1}]\) by definition. Consider the product \(P\) given by

\[P := (x^{-1} \cdot x) \cdot y^{-1}\]

By the definition of multiplication, we have that

\[P = \left(\([a_n^{-1}]\) \cdot \([a_n]\)\right) \cdot \([b_n^{-1}]\) = \([a_n^{-1} \cdot a_n \cdot b_n^{-1}]\) = \([b_n^{-1}]\) = \([b_n]\)^{-1} = y^{-1} \cdot \]
On the other hand, it follows from the above calculation, associativity of multiplication in \( \mathbb{Q} \), and the definition of multiplication in \( \mathbb{R} \) that

\[ P = \left[ \left( \left( a_n^{-1} \cdot a_n \right) \cdot b_n^{-1} \right) \right] = \left[ \left( a_n^{-1} \cdot \left( a_n \cdot b_n^{-1} \right) \right) \right] = \left( a_n^{-1} \right) \cdot \left( a_n \cdot b_n^{-1} \right) \].

By part (2), the fact that \( x = y \) implies \( x \cdot y^{-1} = y \cdot y^{-1} \). Thus \( \left( a_n \cdot b_n^{-1} \right) \) = \( \left( b_n \cdot b_n^{-1} \right) \), so

\[ P = \left[ \left( a_n^{-1} \right) \cdot \left( a_n \cdot b_n^{-1} \right) \right] = \left( a_n^{-1} \right) \cdot \left( b_n \cdot b_n^{-1} \right) = \left( a_n^{-1} \cdot b_n^{-1} \right) = \left( a_n^{-1} \right) = \left( a_n \right)^{-1} = x^{-1}. \]

It now follows that \( y^{-1} = P = x^{-1} \), as desired. \( \square \)

We now define division of the reals via \( x/y := x \cdot y^{-1} \).
9. Arithmetic of the Reals

**Proposition 9.1.** The set of real numbers \( \mathbb{R} \) satisfies the addition axioms (Axiom 2.1), the multiplication axioms (Axiom 2.3), and the distributive axiom (Axiom 2.3). Additionally, if \( x \) is a non-zero real number, then \( x^{-1} \cdot x = x^{-1} : x = [(1,1,1,\ldots)] \) (which we now think of as 1).

**Proof.** We have done most of the hard work already. This is left as an exercise. \( \square \)

We now move on to show that the reals are ordered.

**Definition 9.2.** A nonzero real number \( x = [(a_n)] \) is **positive** if there is at least one Cauchy sequence \( (b_n) \in [(a_n)] \) such that

1. \( (b_n) \) is bounded away from zero, and
2. \( b_n \) is positive for each \( n \in \mathbb{N} \).

Similarly, \( x = [(a_n)] \) is **negative** if there is at least one Cauchy sequence \( (c_n) \in [(a_n)] \) such that

1. \( (c_n) \) is bounded away from zero, and
2. \( c_n \) is negative for each \( n \in \mathbb{N} \).

**Lemma 9.3.** Every real number is exactly one of positive, negative, or zero.

**Proof.** Let \( x \in \mathbb{R} \), and suppose that \( x \neq 0 \). We need to prove that \( x < 0 \) or \( x > 0 \). Since \( x \neq 0 \), Proposition 7.5 implies that there exists a Cauchy sequence \( (a_n) \) bounded away from zero such that \( x = [(a_n)] \). Thus there is some fixed rational number \( \ell > 0 \) such that \(|a_n| \geq \ell \) for all \( n \).

Claim: There is a positive integer \( N \) such that \( a_n \) has the same sign (i.e., \( a_n \) is always positive or always negative) for all \( n \geq N \).

**Proof of the claim:** Let \( \varepsilon = \ell \). There exists a natural number \( N \) such that

\[
|a_m - a_n| < \ell \quad \text{whenever } m, n \geq N.
\]

Suppose to the contrary that \( a_m \) and \( a_n \) have different signs. Then, since \(|a_n| \geq \ell \) for all \( n \),

\[
|a_m - a_n| = ||a_m| + |a_n|| = |a_m| + |a_n| \geq \ell + \ell = 2\ell > \ell.
\]

But this contradicts (9.1). Thus \( a_m \) and \( a_n \) have the same sign; that is, both are positive or both are negative. This concludes the proof of the claim.

So all of the terms \( a_n \) with \( n \geq N \) have the same sign. In order to ensure that every term \( a_n \) with \( n \geq 1 \) has the same sign, we can change the beginning of the sequence (since changing finitely many terms at the beginning of a sequence yields an equivalent sequence). \( \square \)

**Definition 9.4.** If \( x, y \in \mathbb{R} \), we say that \( x < y \) if \( y - x \) is a positive real number.

**Exercise.** Prove that \( x > 0 \) if and only if \( x \) is positive, and \( x < 0 \) if and only if \( x \) is negative.

We now define the absolute value in the way we did for the rationals:

\[
|x| := \begin{cases} 
  x & \text{if } x > 0, \\
  0 & \text{if } x = 0, \\
  -x & \text{if } x < 0.
\end{cases}
\]

This definition extends the one for the rationals in the sense that if we think of \( r \in \mathbb{Q} \) as \( x = [(r,r,r,\ldots)] \), then \( |x| = [(|r|,|r|,|r|,\ldots)] \). In fact, we have more generally,
Lemma 9.5. If \( x = [(a_n)] \) is a real number, then \(|x| = [|(a_n)|]|\).

Proof. Exercise.

\[ \] Proposition 9.6. The set \( \mathbb{R} \) is an ordered set under \(<\). Moreover, we have the triangle inequality for \( \mathbb{R} \): If \( x, y \in \mathbb{R} \), then \(|x + y| \leq |x| + |y|\).

Proof. Exercise.

The following is another important property which shows how \( \mathbb{Q} \) sits in \( \mathbb{R} \).

Proposition 9.7. The set \( \mathbb{Q} \) is dense in \( \mathbb{R} \). In other words, for every pair of real numbers \( x \) and \( y \) with \( x < y \), there exists a rational number \( q \) such that \( x < q < y \).

Proof. Exercise.

To summarize, \( \mathbb{R} \) is a set of numbers which satisfies the addition, multiplication, and distribution axioms, has a notion of division (we call such sets fields which is ordered (with respect to \(<\)) and which contains \( \mathbb{Q} \) as a dense subset. We have really made progress! But remember, \( \mathbb{Q} \) has “holes” in it (like \( \sqrt{2} \)), and we want to show that \( \mathbb{R} \) actually fills the holes. (For example, we need to show that \( \sqrt{2} \in \mathbb{R} \).) But so far all we’ve done is constructed something that may not even be better than \( \mathbb{Q} \).

To show that \( \mathbb{R} \) fills these holes, we need some definitions. The first might look familiar.

Definition 9.8. Let \( (a_n) \) be a sequence of real numbers. We say that \( (a_n) \) is a Cauchy sequence if for every error \( \varepsilon > 0 \) there exists a natural number \( N_\varepsilon \) (our threshold) for which

\[ |a_m - a_n| < \varepsilon \quad \text{whenever } m, n \geq N_\varepsilon. \]

We no longer need to let \( \varepsilon > 0 \) be rational, because we have finally defined the reals! That being said, convince yourself that it suffices to take \( \varepsilon \) to be rational.

And finally, the definition of convergence and limit!

Definition 9.9. Let \( (a_n) \) be a sequence of real numbers. We say that \( (a_n) \) converges to a limit \( L \in \mathbb{R} \) if for every error \( \varepsilon > 0 \), there exists a threshold \( N_\varepsilon \) for which

\[ |a_n - L| < \varepsilon \quad \text{for all } n \geq N_\varepsilon. \]

If \( (a_n) \) converges to \( L \), then we write

\[ \lim_{n \to \infty} a_n = L \quad \text{or} \quad \lim_{n \to \infty} a_n = L. \]

If there does not exist any \( L \in \mathbb{R} \) such that \( (a_n) \) converges to \( L \), we say that \( (a_n) \) diverges.

It should come as no surprise that these notions are related.

Proposition 9.10. Let \( (a_n) \) be a sequence of real numbers. If \( (a_n) \) converges to a limit \( L \in \mathbb{R} \), then \( (a_n) \) is Cauchy.

Proof. Let \( \varepsilon > 0 \). Since \( (a_n) \) converges to \( L \), there is a number \( N \) such that

\[ |a_n - L| < \frac{\varepsilon}{2} \quad \text{for all } n \geq N. \]

Using the “adding zero trick” and the triangle inequality, if \( m, n \geq N \), then

\[ |a_m - a_n| = |a_m - L - (a_n - L)| \leq |a_n - L| + |a_m - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \]

Thus \( (a_n) \) is Cauchy. \( \square \)
Intermission: Arithmetic of $\mathbb{Q}$ and $\mathbb{R}$

Let $X$ equal either $\mathbb{Q}$ or $\mathbb{R}$. At this point, we have proven lots of results about $X$ and its arithmetic. Let’s take stock of what we can prove directly from Axiom 2.1, Axiom 2.3, Axiom 2.4, and the notion of a reciprocal. Some of these may be repeats from previous results in the notes, graded problems, or additional problems. We will assume these results henceforth, but you are responsible for all of the proofs.

Exercise. If $x, y, z \in X$, then the following are true.

1. If $x + y = x + z$, then $y = z$.
2. If $x + y = x$, then $y = 0$.
3. If $x + y = 0$, then $y = -x$.
4. $-(-x) = x$.

Exercise. If $x, y, z \in X$, then the following are true.

1. If $x \neq 0$ and $xy = xz$, then $y = z$.
2. If $x \neq 0$ and $xy = x$, then $y = 1$.
3. If $x \neq 0$ and $xy = 1$, then $y = x^{-1}$.
4. If $x \neq 0$, then $(x^{-1})^{-1} = x$.

Exercise. If $x, y \in X$, then the following are true.

1. $0x = 0$.
2. If $x \neq 0$ and $y \neq 0$, then $xy \neq 0$.
3. $(-x)y = -(xy) = x(-y)$.
4. $(-x)(-y) = xy$.

Definition. Let $x \in X$. To raise $x$ to the power 0, we define $x^0 := 1$. In particular, we define $0^0 := 1$. Now, suppose inductively that $x^n$ has been defined for some nonnegative integer $n$. We then define $x^{n+1} := x^n \cdot x$.

Exercise. Let $x, y \in X$, and let $m, n \geq 0$ be integers. Prove the following statements:

1. We have $x^nx^m = x^{n+m}$.
2. We have $(x^n)^m = x^{nm}$.
3. We have $(xy)^n = x^ny^n$.

(Side note: Similar proofs give comparable properties for negative exponents once we define $x^{-n} := (x^n)^{-1}$.)

It is part of your homework to establish that $\mathbb{R}$ is ordered; you already proved this for $\mathbb{Q}$. It is part of your additional problems to prove that if $x \in \mathbb{R}$ is represented as an equivalence class of Cauchy sequences of rationals, say $x = [(a_n)]$, then $|x| = [|a_n|]$. Once you finish those, then you will be responsible for the following exercises.

Exercise. If $x, y, z \in X$, then the following are true.

1. If $x > 0$, then $-x < 0$, and vice versa.
2. If $x > 0$ and $y < z$, then $xy < xz$.
3. If $x < 0$ and $y < z$, then $xy > xz$.
(4) If $x \neq 0$, then $x^2 > 0$. In particular, $1 > 0$.
(5) If $0 < x < y$, then $0 < y^{-1} < x^{-1}$.

**Exercise.** Let $x, y \in X$, and let $m, n \geq 0$ be integers. Prove the following statements.

1. Suppose $n \geq 1$. Then we have $x^n = 0$ if and only if $x = 0$.
2. If $x \geq y \geq 0$, then $x^n \geq y^n \geq 0$.
3. If $x > y \geq 0$ and $n > 0$, then $x^n > y^n \geq 0$.
4. We have $|x^n| = |x|^n$.

(Side note: Similar proofs give comparable properties for negative exponents once we define $x^{-n} := (x^n)^{-1}$.)
10. Convergence and limits. The completeness of \( \mathbb{R} \).

Proposition 9.10 is not so shocking. But what if the converse statement were true: What if Cauchy sequences of real numbers were the only sequences that converged?

**Definition 10.1.** A set of numbers \( X \) is said to be complete if all Cauchy sequences of numbers in \( X \) converge to a number inside of \( X \).

**Theorem 10.2.** The set of real numbers \( \mathbb{R} \) is complete.

This is vindication! We wanted earlier to define the reals to be limits of convergent sequences rational numbers, but we could not because the limit might not be rational, and we had not yet defined the reals. Cauchy sequences may have seemed like a peculiar mechanism by which we might construct the reals. Now, our efforts have paid off: Cauchy sequences of real numbers are in fact the ONLY convergent sequences of real numbers! So we were right all along; we just could not articulate why.

**Proposition 10.3.** Let \( (a_n) \) be a Cauchy sequence of rationals. Let \( x \) be the real number equal to \( [(a_n)] \). Then \( \lim a_n = x \).

*Proof.* Let \( \varepsilon > 0 \) be rational. Since \( (a_k) \) is Cauchy, there exists \( N > 0 \) such that

\[
|a_k - a_m| < \varepsilon/2 \quad \text{whenever } k, m \geq N.
\]

We want to prove that for the same \( N \), we also have

\[
|a_n - x| < \varepsilon \quad \text{whenever } n \geq N.
\]

To begin, fix \( N \geq N \). Since \( a_n \) is rational, we can write \( a_n \) as a real number in the form \( a_n = [(a_n, a_n, a_n, \ldots)] \). Now, by Lemma 9.5, we want to show that

\[
|a_n - x| = |[(a_n, a_n, a_n, \ldots)] - [(a_1, a_2, a_3, \ldots)]|
\]

\[
= |[(a_n - a_1, a_n - a_2, a_n - a_3, \ldots)]|
\]

\[
= |(a_n - a_1, a_n - a_2, a_n - a_3, \ldots)|
\]

\[
= \left\|(a_n - a_k)_{k \in \mathbb{N}} \right\|
\]

is less than \( \varepsilon \). Since \( \varepsilon \in \mathbb{Q} \), we may write \( \varepsilon \) as the equivalence class of the Cauchy sequence \((\varepsilon, \varepsilon, \varepsilon, \ldots)\). From our definitions, we now seek to prove that the real number

\[
[(\varepsilon - |a_n - a_k|)_{k \in \mathbb{N}}]
\]

is positive. This follows from (10.1) — if \( k \geq N \), then

\[
\varepsilon - |a_n - a_k| \geq \varepsilon - \varepsilon/2 = \varepsilon/2.
\]

Thus the sequence \( (b_n) \) given by

\[
b_n = \begin{cases} 
\varepsilon/2 & \text{if } k \leq N, \\
\varepsilon - |a_n - a_k| & \text{if } k > N
\end{cases}
\]

is a sequence, all of whose terms are positive, which is bounded away from zero — \( |b_n| \geq \ell := \varepsilon/2 \) for all \( n \in \mathbb{N} \) — and satisfies \( (b_n) \in [(\varepsilon - |a_n - a_k|)_{k \in \mathbb{N}}] \). Thus the real number \([\varepsilon - |a_n - a_k|]_{k \in \mathbb{N}}\) is positive, as desired.

We now show that we may take \( \varepsilon > 0 \) to be real. By Proposition 9.7, we know that there exists a rational \( \varepsilon' \) such that \( 0 < \varepsilon' < \varepsilon \). Our work above shows that there exists an \( N' > 0 \) (depending on \( \varepsilon' \)) such that if \( n \geq N' \), then \( |a_n - x| < \varepsilon' \). But since \( 0 < \varepsilon' < \varepsilon \), we have that \( |a_n - x| < \varepsilon \), provided that \( n \geq N' \). This was precisely the conclusion we sought. \( \Box \)
Proof of Theorem 10.2. Let \((a_n)\) be a Cauchy sequence of real numbers. Let \(\varepsilon > 0\) be given. For each \(n\), it follows by Proposition 9.7 that there is a rational number \(q_n\) such that
\[
|a_n - q_n| < 1/(3n).
\]
Claim: The sequence \((q_n)\) is Cauchy. Proof of the claim: Observe that
\[
|q_m - q_n| = |q_m - a_m + a_m - a_n + a_n - q_n| \leq |q_m - a_m| + |a_m - a_n| + |a_n - q_n|.
\]
Since \((a_n)\) is Cauchy, we have the existence of some \(N \in \mathbb{N}\) (depending on \(\varepsilon\)) such that if \(m, n \geq N\), then \(|a_m - a_n| < \varepsilon/3\) whenever \(m, n \geq N\). Now, let \(N' \in \mathbb{N}\) satisfy \(N' \geq \max\{N, 1/\varepsilon\}\). If \(m, n \geq N'\), then we can use (10.2) to conclude
\[
|q_m - a_m| + |a_m - a_n| + |a_n - q_n| \leq \frac{1}{3m} + \frac{\varepsilon}{3} + \frac{1}{3n} \leq \frac{1}{3N'} + \frac{\varepsilon}{3} + \frac{1}{3N'} \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\]
This proves the claim.

Let \(x = [(q_n)]\). We will show that \(\lim a_n = x\). Let \(\varepsilon > 0\) be given. By Proposition 10.3, \((q_n)\) converges to \(x\), so there exists some threshold \(N_\varepsilon \in \mathbb{N}\) such that \(|q_n - x| < \varepsilon/2\) for all \(n \geq N\). It now follows from the claim that
\[
|a_n - x| = |q_n - q_n + q_n - x| \leq |a_n - q_n| + |q_n - x| < \varepsilon/3 + \varepsilon/2 < \varepsilon
\]
for all \(n \geq N\), as desired. \(\square\)

Thus we have proved that the set of convergent sequences of reals equals the set of Cauchy sequences of reals. Our proofs of properties for Cauchy sequences of rationals largely carry over for Cauchy sequences of reals. For instance:

Proposition 10.4. Convergent sequences of real numbers are bounded.

Proof. The proof is identical to that of Proposition 6.6. \(\square\)

Proposition 10.5. Suppose that \((a_n)\) and \((b_n)\) are convergent sequences of reals. Then:

1. \(\lim(a_n + b_n) = \lim a_n + \lim b_n\).
2. \(\lim(a_n b_n) = (\lim a_n) \cdot (\lim b_n)\).
3. If \(c \in \mathbb{R}\), then \(\lim ca_n = c \lim a_n\).
4. If \(b_n \neq 0\) for all \(n\) and if \(\lim b_n \neq 0\), then \(\lim(a_n/b_n) = (\lim a_n)/(\lim b_n)\).

Proof. We’ll do Part 2; the other proofs are similar. Let \(\lim a_n = A \in \mathbb{R}\) and \(\lim b_n = B \in \mathbb{R}\). Since convergent sequences of reals are bounded, there exists a real number \(M \geq 1\) such that \(|b_n| \leq M\) for all \(n \in \mathbb{N}\). Moreover, for all \(\varepsilon > 0\), there exist \(N_a, N_b \in \mathbb{N}\) such that
\[
|a_n - A| < \frac{\varepsilon}{2M}, \quad |b_n - B| < \frac{\varepsilon}{2 \max\{|A|, 1\}}
\]
when \(n \geq N := \max\{N_a, N_b\}\). Now, observe by the triangle inequality that
\[
|a_n b_n - AB| = |a_n b_n - Ab_n + Ab_n - AB| = |b_n(a_n - A) + A(b_n - B)|
\]\[
\leq |b_n| \cdot |a_n - A| + |A| \cdot |b_n - B|
\]\[
< \varepsilon/2 + \varepsilon/2 = \varepsilon.
\]
Hence \(\lim a_n b_n = AB = (\lim a_n)(\lim b_n)\), as desired. \(\square\)
11. The least upper bound property

Proposition 11.1. Suppose \( (a_n) \) converges. If \( a_n \leq M \) for all \( n \geq 1 \), then \( \lim_{n \to \infty} a_n \leq M \). If \( a_n \geq m \) for all \( n \geq 1 \), then \( \lim_{n \to \infty} a_n \geq m \).

Proof. Homework. \qed

We now get to the heart of why completeness is so important to calculus.

Definition 11.2. Let \( E \) be a subset \( \mathbb{R} \). Let \( M, m \in \mathbb{R} \). We say that \( M \) is an upper bound for \( E \) if \( x \leq M \) for all \( x \in E \). We say that \( m \) is a lower bound for \( E \) if \( x \geq m \) for all \( x \in E \).

Example 11.3. Let \( E = \{ x \in \mathbb{R} : 0 \leq x \leq 1 \} \). The numbers 1, 4, and 999/1000 are upper bounds for \( E \); the numbers \(-1/2, 0, \) and \(-1/1000 \) are lower bounds for \( E \).

Definition 11.4. Let \( E \) be a subset \( \mathbb{R} \). Let \( M, m \in \mathbb{R} \). We say that \( M \) is a least upper bound for \( E \) if (1) \( M \) is an upper bound for \( E \), and (2) if \( M' \) is another upper bound for \( E \), then \( M \leq M' \). We say that \( m \) is a greatest lower bound for \( E \) if (1) \( m \) is a lower bound for \( E \), and (2) if \( m' \) is another lower bound for \( E \), then \( m \geq m' \).

Lemma 11.5. If \( M \) and \( M' \) are two least upper bounds for \( E \), then \( M = M' \), and similarly for greatest lower bounds.

Proof. By Definition 11.4, we have that \( M \geq M' \) and \( M' \geq M \). Thus \( M = M' \) (why?). \qed

Notation 11.6. Let \( E \) be a subset of \( \mathbb{R} \). If \( E \) has a least upper bound, it is unique by Lemma 11.5, so we write the least upper bound of \( E \) as \( \sup E \) (the supremum of \( E \)). If \( E \) has a greatest lower bound, it is unique by Lemma 11.5, so we write the greatest lower bound of \( E \) as \( \inf E \) (the infimum of \( E \)).

Theorem 11.7. Any nonempty subset \( E \) of \( \mathbb{R} \) satisfies the least upper bound property: If \( E \) has an upper bound, then \( \sup E \) exists and is in \( \mathbb{R} \).

Proof. Since \( E \) is nonempty has an upper bound, there exists \( a_1 \in E \) and \( b_1 \in \mathbb{R} \) with \( a_1 \leq b_1 \). We will define two sequences \( (a_n) \) and \( (b_n) \) inductively such that \( a_n \leq b_n \) for all \( n \in \mathbb{N} \). For each \( n \), suppose we have constructed \( a_n \) and \( b_n \) such that \( a_n \leq b_n \). Let \( K_n = (a_n + b_n)/2 \). Note that \( a_n \leq K_n \leq b_n \). If \( K_n \) is an upper bound of \( E \), let \( a_{n+1} = a_n \) and \( b_{n+1} = K_n \); then \( |b_{n+1} - a_{n+1}| = (b_n - a_n)/2 \). If \( K_n \) is not an upper bound for \( E \), then there exists \( x_n \in E \) such that \( x_n > K_n \); then we let \( a_{n+1} = x_n \) and \( b_{n+1} = b_n \), in which case \( |b_{n+1} - a_{n+1}| = b_n - x_n > b_n - K_n = (b_n - a_n)/2 \).

To summarize, we begin with \( a_1 \in E \) and \( b_1 \) an upper bound for \( E \). Then \( a_1 \leq b_1 \). Now, for each \( n \geq 1 \),

\[
a_{n+1} = \begin{cases} a_n & \text{if } \frac{a_n + b_n}{2} \text{ is an upper bound for } E, \\ x_n & \text{if } \frac{a_n + b_n}{2} \text{ is not an upper bound for } E \text{ and } x_n \in E \text{ satisfies } x_n > \frac{a_n + b_n}{2} \end{cases}
\]

and

\[
b_{n+1} = \begin{cases} \frac{a_n + b_n}{2} & \text{if } \frac{a_n + b_n}{2} \text{ is an upper bound for } E, \\ b_n & \text{if } \frac{a_n + b_n}{2} \text{ is not an upper bound for } E. \end{cases}
\]

In either case,

1. \(|b_{n+1} - a_{n+1}| \leq (b_n - a_n)/2\),
2. \(a_{n+1} \geq a_n\) for all \( n \in \mathbb{N}\),
3. \(b_{n+1} \leq b_n\) for all \( n \in \mathbb{N}\), and
Example 11.8. Let \( a, b \in \mathbb{R} \) satisfy \( a < b \). Define \([a, b] := \{ x \in \mathbb{R} : a \leq x \leq b \}\), \((a, b) := \{ x \in \mathbb{R} : a < x < b \}\), \([a, b) := \{ x \in \mathbb{R} : a \leq x < b \}\), and \((a, b] := \{ x \in \mathbb{R} : a < x \leq b \}\). These sets of numbers have the same supremum (namely \( b \)) and the same infimum (namely \( a \)).

Example 11.9. Let \( a \in \mathbb{R} \). Define \([a, \infty) := \{ x \in \mathbb{R} : x \geq a \}\), \((a, \infty) := \{ x \in \mathbb{R} : x > a \}\), \((\infty, a] := \{ x \in \mathbb{R} : x \leq a \}\), and \((\infty, a) := \{ x \in \mathbb{R} : x < a \}\). The first two sets have no supremum (they have no upper bound) and the same infimum (namely \( a \)). The last two sets have the same supremum (namely \( a \)) and no infimum (they have no lower bound).

Example 11.10. Let \((a_n)\) be a sequence of reals, and let \( A = \{a_n : n \in \mathbb{N} \} \). We define \( \sup a_n = \sup(A) \) (when it exists) and \( \inf a_n = \inf(A) \) (when it exists). For instance, if \( a_n = 1/n^2 \), then \( \sup a_n = 1 \) and \( \inf a_n = 0 \). On the other hand, if \( b_n = n^{-1/n} \), then \( \sup b_n \) does not exist, while \( \inf b_n = 0 \).