MATH 115: FUNCTIONS OF A REAL VARIABLE

1. Introduction. No rational square equals 2. Induction.

The goal of this course is to rigorously study the key ideas in calculus: limits, sequences, continuity, the derivative, and the integral. On one hand, some of the things that we will uncover in the course might appear to be fairly routine. For example, by the end, you will be able give a full proof that the derivative of the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is $2x$, and you will have a rigorous notion of what the word "derivative" means to go alongside the ideas that you learned in your initial exposure to calculus.

On the other hand, there are some subtle issues that require a large amount of development. A key example lies in the study of the real numbers $\mathbb{R}$. In your initial exposure to calculus, you probably never thought to take the derivative of a function whose domain is $\mathbb{Z}$ (the integers) or $\mathbb{Q}$ (the rationals). You only worked with functions on $\mathbb{R}$. Why is this? Can one prove that working with functions on $\mathbb{R}$ is the "right setting" for calculus? What *is* a real number, anyway? You might have an intuitive definition in mind, but we will need a very comprehensive definition in order to develop calculus properly. In order to do this, we need to develop the theory of sequences and limits. Only after we are equipped with the comprehensive properties of $\mathbb{R}$ and the theory of sequences and limits, we will be able to handily develop the notions of continuity, differentiation, and integration.

What does $\mathbb{R}$ have that number systems like $\mathbb{Q}$ and $\mathbb{Z}$ don't have? To help motivate this, consider the problem of describing the number $x \geq 0$ such that $x^2 = 2$. Hopefully, it is clear that x is not an integer, but perhaps x might be rational.

Theorem 1.1. *There is no rational number x such that $x^2 = 2$.*

Proof. We begin by supposing that there does in fact exist a rational number x such that $x^2 = 2$. We will show that this supposition leads to a conclusion that we know is false, which means that our initial hypothesis (there is a rational number x such that $x^2 = 2$) is false. This is a classical example of **proof by contradiction**.

Suppose (to the contrary) that x is a rational number such that $x^2 = 2$. We may write x as a quotient of integers a/b with $a, b \neq 0$ and a and b having no common factor. Then $a^2 = 2b^2$, which implies that a^2 is even. Since the square of an odd integer is odd, it follows that a is even. We may now write a as $2c$, where c is a nonzero integer. Now, $2b^2 = a^2 = (2c)^2 = 4c^2$, or $b^2 = 2c^2$. Hence b^2 is even, and for the same reason as before, b is even. Thus a and b are nonzero integers which share 2 as a factor. This contradicts our hypothesis that a and b have no common factor, and therefore the hypothesis that x is rational must be false. $\square$

This proof indicates something about the layout of the course. We reduce the proof of a statement about the rationals to the proof of a statement about the integers. In this course, we will build the rationals from the integers, and then we will build the real numbers from the rationals. Once we have rigorously developed desirable properties about the reals, we will be able to properly address the usual topics in calculus.

Let us take the time to discuss one property of the **natural numbers** $\mathbb{N} = \{1, 2, 3, \ldots\}$. We make special mention of $\mathbb{N}$ because it is the setting for mathematical induction. This is a tool which is ubiquitous throughout mathematics and is incredibly powerful.

Principle of mathematical induction. *Let $P_1, P_2, P_3, \ldots$ be a list of statements or propositions (which may or may not be true) indexed by the natural numbers. Suppose that*

I1: *P_1 is true, and*

I2: *whenever P_n is true, P_{n+1} is also true.*

Then all of the statements $P_1, P_2, P_3, \ldots$ are true.

Here is an example of how useful mathematical induction can be.

Example 1.2. For every natural number n, we have that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.

Proof. For each natural number n, the statement we want to prove is P_n: "$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$". We proceed by induction on n. First we will prove that the *base case* P_1 is true. The statement P_1 reads $1 = \frac{1\cdot(1+1)}{2}$, which is true. Now, suppose that P_n is true; in other words, suppose that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$. Using this hypothesis, we want to prove the statement P_{n+1}. To achieve this, we add $n + 1$ to both sides to obtain

$$1 + 2 + \cdots + n + (n + 1) = \frac{n(n + 1)}{2} + n + 1 = \frac{n(n + 1) + 2(n + 1)}{2} = \frac{(n + 1)((n + 1) + 1)}{2}.$$

Thus P_{n+1} is true if P_n is true. By the principle of mathematical induction, P_n holds for all natural numbers n. □

Note that we did not prove P_n directly for any n except for $n = 1$. We just proved P_1, and we proved that if P_1 is true, so is P_2 (thus P_2 is true), and we proved that if P_2 is true, so is P_3 (thus P_3 is true), and we proved that if P_3 is true, so is P_4 (thus P_4 is true), etc.

Example 1.3. Fix $x > 0$. For every natural number n, we have that $(1+x)^{n+1} > 1+(n+1)x$.

Proof. We proceed by induction on n. We first prove the base case $n = 1$. We have $(1+x)^{1+1} = (1 + x)^2 = 1 + 2x + x^2$. If $x > 0$, then $1 + 2x + x^2 > 1 + 2x = 1 + (1 + 1)x$, as desired. Now, suppose that the inequality $(1 + x)^{n+1} > 1 + (n + 1)x$ has already been proven. We will show that $(1 + x)^{(n+1)+1} > 1 + ((n + 1) + 1)x$ is true (that is, $(1 + x)^{n+2} > 1 + (n + 2)x$ is true). Note that $(1 + x)^{n+2} = (1 + x)^{n+1}(1 + x)$.

By the inductive hypothesis, we have that $(1+x)^{n+2} > (1+(n+1)x)(1+x)$. This expands to $1 + (n + 2)x + (n + 1)x^2$. Since $x^2 > 0$ and $n + 1 > 0$, we have $(n + 1)x^2 > 0$. Hence $1 + (n + 2)x + (n + 1)x^2 > 1 + (n + 2)x$, as desired. It follows that if we fix $x > 0$, then for every natural number $\mathbb{N}$, we have that $(1 + x)^{n+1} > 1 + (n + 1)x$. □

Here is a prototype for all future proofs using mathematical induction.

Proposition 1.4 (Induction template)**.** *A property P_n is true for all natural numbers n.*

Proof. We proceed by induction on n (it's good to specify the variable if there are several variables in the statement you want to prove). We first verify the *base case* $n = 1$; in other words, we prove that P_1 is true. [Insert the proof of P_1 here]. Now, suppose that P_n has already been proven. We will show that P_{n+1} is true. [Insert the proof of P_{n+1}, assuming P_n is true]. It follows that P_n is true for all natural numbers. □

2. The integers.

From this point on, $\mathbb{Q}$ and $\mathbb{R}$ do not exist until we construct them. To begin, we will assume some familiarity with the basic properties of the positive integers (the **natural numbers**) as well as the set of all **integers**. One can build these properties from the Peano axioms of arithmetic (listed in §1 of Ross), but we will not do this.

We will denote the set of integers $\{\ldots, -2, 1, 0, 1, 2, \ldots\}$ by $\mathbb{Z}$. I am going to assume that you are comfortable with the ideas of addition, subtraction, and multiplication within $\mathbb{N}$ and $\mathbb{Z}$. The basic properties of $\mathbb{Z}$ follow from the following axioms.

Axiom 2.1 (Addition axioms)**.** *A set of numbers S is said to satisfy the* **addition axioms** *if the following hold for all members x, y, and z of S.*
A0*: Closure: If x and y are in S, then their sum $x + y$ is an integer.*
A1*: Associativity: $(x + y) + z = x + (y + z)$.*
A2*: Commutativity: $x + y = y + x$.*
A3*: Identity: There is a number in 0 in S such that $0 + x = x$.*
A4*: Inverse: To every x in S there corresponds a number $(-x)$ in S such that $x + (-x) = 0$.*
A5*: Well-definedness: If x and y are in S and $x = x'$, then $x+y = x'+y$ and $(-x) = (-(x'))$.*

Notation 2.2. We write $x - y$ as shorthand for $x + (-y)$.

Axiom 2.3 (Multiplication axioms)**.** *A set of numbers S is said to satisfy the* **multiplication axioms** *if the following hold for all numbers x, y, and z of S.*
M0*: Closure: If x and y are in S, then their product xy (also written $x \cdot y$) is in S.*
M1*: Associativity: $(xy)z = x(yz)$.*
M2*: Commutativity: $xy = yx$.*
M3*: Identity: There exists number in S, denoted 1, such that $1 \neq 0$ and $1x = x$.*
M4*: Well-definedness: If x and y are in S and $x = x'$, then $xy = x'y$.*

Axiom 2.4 (Distributive axiom)**.** *A set of numbers S is said to satisfy the* **distribution axiom** *if the following holds for all members x, y, and z of S.*
DL*: $x(y + z) = xy + xz$.*

The set $\mathbb{Z}$ also is equipped with an *ordering* which arises from the usual notion of inequality $<$. Let's take some time to spell out the pertinent definitions.

Definition 2.5. Let S be a set of numbers. An **order** on S is a relation, denoted by $<$, with the following two properties:

(i) (Trichotomy) If x and y are numbers in S, then exactly one of these statements is true:

$$x < y, \qquad x = y, \qquad y < x.$$

(ii) (Transitivity) If x, y, and z are numbers of S such that $x < y$ and $y < z$, then $x < z$.

The statement "$x < y$" reads "x is less than y". The statement "$y > x$" is interchangeable with "$x < y$". We use the notation $x \leq y$ to indicate that $x < y$ or $x = y$.

Theorem 2.6. *(1) The set $\mathbb{Z}$ is an ordered set of numbers that satisfies the addition axioms, the multiplication axioms, and the distributive axiom.*
(2) The set $\mathbb{N}$ is an ordered set of numbers that satisfies the addition axioms except for **A3** *and* **A4***, the multiplication axioms, and the distributive axiom.*

The order on $\mathbb{Z}$ is given as follows: For any integers x and y, we define the statement $x < y$ to mean that $y - x$ is a positive integer.

Using Theorem 2.6, we can rigorously prove a wide variety of results that you may have seen before and didn't prove.

Lemma 2.7. *If S is a set of numbers satisfying the addition axioms, and x, y, and z are members of S, then:*

(i) (Cancellation): if $x + y = x + z$, then $y = z$.
(ii) (Uniqueness of zero): if $x + y = x$, then $y = 0$.
(iii) (Uniqueness of additive inverse): if $x + y = 0$, then $y = -x$.
(iv) (Cancellation of $-$): $-(-x) = x$.

Proof. Here is a proof for (i). The rest is an exercise.
Suppose that $x + y = x + z$. By **A4**, there exists $(-x)$ in S such that $x + (-x) = 0$. By **A5**, we have that $(x + y) + (-x) = (x + z) + (-x)$. By two applications of **A2**, we have that $(-x) + (x + y) = (-x) + (x + z)$. By two applications of **A1**, we have that $((-x) + x) + y = ((-x) + x) + z$. By two applications of **A2**, we have that $(x + (-x)) + y = (x + (-x)) + z$. By two applications of **A4**, we have that $0 + y = 0 + z$. By two applications of **A3**, we conclude that $y = z$, as desired. □

Proposition 2.8. *If x, y, z are integers, then:*

(i) $0 \cdot x = 0$.
(ii) $(-x) \cdot y = -(x \cdot y)$
(iii) $(-x) \cdot (-y) = x \cdot y$.
(iv) If $x \cdot y = 0$, then $x = 0$ or $y = 0$ (or both).
(v) If $z \neq 0$ and $x \cdot z = y \cdot z$, then $x = y$.

Proof. (i) By **A3**, **A4**, and **DL**, $0 + 0 \cdot x = 0 \cdot x = (0 + 0) \cdot x = 0 \cdot x + 0 \cdot x$. Thus $0 + 0 \cdot x = 0 \cdot x + 0 \cdot x$. Then the cancellation proved in Lemma 2.7(i) now tells us that $x \cdot 0 = 0$.

(ii) Observe that $(-x)y + xy = ((-x) + x)y$ by **DL**. This equals $0y$ by **A4**, which equals 0 by Part (a). The result now follows by Lemma 2.7(iii). The other equality is proved similarly (but you should work it out!).

(iii) Exercise.

(iv) We will prove that if $x \neq 0$ and $y \neq 0$, then $xy \neq 0$. (Convince yourself that this is enough! This is the *contrapositive* of a statement we want to prove.)

First, suppose that $x > 0$ and $y > 0$. Then x and y are natural numbers, which are closed under multiplication. Thus $x \cdot y$ is a natural number, and zero is not.

Second, suppose that $x < 0$ and $y < 0$. Then $x = -m$ and $y = -n$ for certain positive integers m and n. Thus $x \cdot y = (-m) \cdot (-n)$, which equals $m \cdot n$ by Part (iii). By our initial argument, $m \cdot n \neq 0$. Thus $x \cdot y \neq 0$, as desired.

Finally, suppose that one of $x > 0$ and $y < 0$. (You can switch these up if you'd like; the proof will be the same.) Then x is a natural number, and $y = -n$ for some natural number n. Now, $x \cdot y = x \cdot (-n) = -(x \cdot n)$ by (ii). By our initial argument $x \cdot n \neq 0$. Thus $-(x \cdot n) \neq 0$.

(v) Exercise.

□

3. Equivalence relations

We now proceed to the construction of the rationals from the integers. But first, what should the rationals look like? For instance, integers should be rationals. Rationals should capture our usual notion of division. The addition, multiplication, and distribution axioms should hold. The rationals should be ordered.

Here is a trickier task: We want to properly ensure that $1/2 = 2/4 = 4/8 = (-1)/(-2) = (-2)/(-4) = (-4)/(-8) = \cdots$. We have infinitely many ways to write $1/2$. How do we rigorously say that they are all equal? How do we know that working with one "version" of $1/2$ is exactly the same as working with another "version" of $1/2$? This is a practical problem: If $1/2 = 2/4$, then how do we properly guarantee that $1/2 + 6/7 = 2/4 + 6/7$? How do we properly guarantee that $-(1/2) = -(2/4)$? How do we properly guarantee that $(1/2) \cdot (6/7) = (2/4) \cdot (6/7)$? Thus we need a very robust notion of "equality".

We will introduce *equivalence relations* and *equivalence classes* in order to address this issue. Our ideas will be robust enough to accommodate the reals when we come to them. We begin with some convenient shorthand notation (largely to help my hand at the board!).

Notation 3.1. Let A be any set (whose members may be numbers or other objects). We write $x \in A$ to indicate that "x is in A" or "x is a member of A" or "x is an element of A". If x is not in A, then we write $x \notin A$.

Notation 3.2. The statement "$x := y$ is shorthand for "x is *defined* to equal y."

Notation 3.3. Given a set X, we let $X \times X$ denote the ordered pairs (x_1, x_2), where x_1 and x_2 are members of X. (Think back to linear algebra; $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ is the set of ordered pairs of real numbers.) Order matters here!

Definition 3.4. A **binary relation** on a set X is a subset of $X \times X$. If $\sim$ is a binary relation on X and $x, y \in X$, we say that $x \sim y$ ("x is related to y") if $(x, y) \in \sim$.

Example 3.5. Let X be the set of all people in Indonesia; then $X \times X$ is the set of all ordered pairs of people in Indonesia. The set $\sim = \{(x, y) \colon x, y \in X \text{ and } x, y \text{ are on the same island}\}$ is a binary relation.

Definition 3.6. A binary relation $\sim$ on a set X is an **equivalence relation** if:

(1) (Reflexivity) For all $x \in X$, $x \sim x$.
(2) (Symmetry) For all $x, y \in X$ such that $x \sim y$, we have $y \sim x$.
(3) (Transitivity) For all $x, y, z \in X$ such that $x \sim y$ and $y \sim z$, we have $x \sim z$.

Example 3.7. Referring to our previous example, suppose that x, y, z are people in Indonesia. We have that $x \sim y$ precisely when x is on the same island of Indonesia as y. If x is on a given island, then x is on the same island as x. Thus (x, x) is in $\sim$, so $\sim$ is reflexive.

If x and y are on the same island, then (x, y) is in $\sim$. But since y is on the same island as x, (y, x) is in $\sim$ as well. Thus $\sim$ is symmetric.

Suppose x and y are on the same island, and suppose that x and z are on the same island. Then (x, y) and (y, z) are in $\sim$. But since x shares the island with y and y shares the island with z, (x, z) is in $\sim$ as well. Thus $\sim$ is transitive, hence $\sim$ is an equivalence relation. Thus being on the same island of Indonesia is an equivalence on the people in Indonesia.

Definition 3.8. Let $\sim$ be an equivalence relation on a set X. We define the **equivalence class** of $x \in X$ (relative to the equivalence relation $\sim$) to be the set $[x] := \{y \in X \colon y \sim x\}$.

Example 3.9. Keeping with our example, for any person x on the island of Sumatra, the set {people on Sumatra} is the equivalence class $[x]$ of $x \in X$, the set of all people in Indonesia.

Lemma 3.10. *Let X be the set of all ordered pairs of integers with the second entry being non-zero. In set-builder notation, $X = \{(a,b)\colon a,b \in \mathbb{Z} \text{ and } b \neq 0\}$. The relation $\sim$ on X defined so that $(a,b) \sim (c,d)$ if $ad = cb$ is an equivalence relation. If $a,b \in \mathbb{Z}$ and $b \neq 0$, the equivalence class of (a,b) (relative to $\sim$) is*

$$\{(c,d)\colon c,d \in \mathbb{Z},\ d \neq 0,\ (a,b) \sim (c,d)\} = \{(c,d)\colon c,d \in \mathbb{Z},\ d \neq 0,\ ad = cb\}.$$

Proof. Let $a,b,c,d,r,s \in \mathbb{Z}$, and assume that b,d,r are all nonzero. First, since $ab = ba$, we have that $(a,b) \sim (a,b)$, so $\sim$ is reflexive. Second, if $(a,b) \sim (c,d)$, then $ad = cb$. But then $cb = ad$ by commutativity, so $(c,d) \sim (a,b)$. Finally, suppose $(a,b) \sim (c,d)$ and $(c,d) \sim (r,s)$. Then $ad = cb$ and $cs = rd$. We do not have "division" yet, but since b,d,s are non-zero, we have that $ads = bcs$ (multiply $ad = bc$ by s on both sides) and $bcs = rbd$ (multiply $cs = rd$ by b on both sides). Thus $ads = rbd$. Since d is nonzero, it follows from Proposition 2.8 that $as = rb$. Thus $(a,b) \sim (r,s)$, and $\sim$ is an equivalence relation. □

Example 3.11. The binary relation on $\mathbb{Z}$ given by $a \sim b$ if $a \leq b$ is not an equivalence relation. We have that $4 \leq 7$, but 7 is not less than or equal to 4. Thus $\sim$ is not reflexive. Similar problems exist for the binary relations given by $\geq$, $<$, and $>$.

Definition 3.12. Let X be a set. A **partition** of X is a grouping of the members of X into non-empty subsets in such a way that each element of X is in exactly one of the subsets.

Example 3.13. Let X be the set of all people in Indonesia. Everyone in Indonesia must be on one of its islands (the water line forms the border of the state for our discussion). One cannot be on two of these islands at the same time. Thus if we group the people in Indonesia by which island you are on, we establish a partition of the people in Indonesia.

Equivalence classes have the following useful and important property:

Lemma 3.14. *Suppose that $\sim$ is an equivalence relation on a set X. Then the set of equivalence classes in X (relative to $\sim$) form a partition of X. In other words, each $x \in X$ must lie in exactly one of the equivalence classes in X (relative to $\sim$).*

Proof. Homework. □

Another way to state the same thing is to say that the union of all the equivalence classes in X (relative to $\sim$) is X, and the equivalence classes are pairwise disjoint. Once we have this property, we are prepared to define the *quotient space* $X/\sim$ as follows.

Definition 3.15. Let $\sim$ be an equivalence relation on a set X. The **quotient space** $X/\sim$ is the set $\widetilde{X} := \{[x]\colon x \in X\}$, the set of all equivalence classes of X (relative to $\sim$).

Quotients...this sounds like a great direction. (Think about where the notation $\mathbb{Q}$ comes from.) Note that in Lemma 3.10, we have that $(1,2) \sim (2,4) \sim (-1,-2) \sim (-2,-4) \sim \ldots$

4. The rationals

Definition 4.1 (The rationals)**.** Let $X = \{(a, b)\colon a, b \in \mathbb{Z},\ b \neq 0\}$. Let $\sim$ be the equivalence relation from Lemma 3.10. A **rational number** is an equivalence class in $X/\sim$. If $(a, b) \in X$, then we write $a//b$ (instead of $[(a, b)]$) for the equivalence class of (a, b). We denote the set of rational numbers by $\mathbb{Q}$. If $a//b$ and $c//d$ are rational numbers, we define their sum

$$(a//b) + (c//d) := (ad + bc)//(b \cdot d),$$

their product

$$(a//b) \cdot (c//d) := (a \cdot c)//(b \cdot d),$$

and the negation $-(a//b) := (-a)//b$.

We have defined the rationals, but we have no idea how the arithmetic of $\mathbb{Q}$ works yet. We have to *prove* everything! The notation $a//b$ is meant to be *suggestive* of the fact that we'll eventually get to the usual a/b, but we're not there yet! But we can take advantage of our pre-existing knowledge about the arithmetic of fractions to guide us. This guided our definitions for adding/multiplying/negating rationals.

We already proved that if b and d are non-zero, then bd is also non-zero. Thus the sum and product of two rational numbers remains a rational number. But we encounter the subtle problem of *well-definedness*. Remember, $a//b = [(a, b)]$, which is an entire **equivalence class** of pairs of integers. Moreover, we can see that $[(1, 2)] = [(2, 4)]$. Does this necessarily mean that $1//2 + 6//7 = 2//4 + 6//7$? Does this necessarily mean that $(1//2) * (6//7) = (2//4) * (6//7)$? Does this necessarily mean that $-(1//2) = -(2//4)$? In particular, since there are infinitely many ways to represent a rational number,

(***) does the way we represent a rational number affect the arithmetic of $\mathbb{Q}$?

Lemma 4.2. *The sum, product, and negation operations on rational numbers are well-defined. That is, if $a//b$ and $c//d$ are rational numbers and $a'//b' = a//b$ (under the equivalence relation in Lemma 3.10), then $a//b + c//d = a'//b' + c//d$, and similarly for products and negation. (So the answer to the question (***) is NO.)*

Proof. I'll work out the proof for sums; I leave products and negation as homework.

Let $a//b$, $a'//b'$, and $c//d$ be rationals (but they are still equivalence classes!); thus a, b, a', b', c, d are integers, and b, b', d are non-zero. Suppose that $a//b = a'//b'$, in which case $ab' = a'b$. We shall show that

$$a//b + c//d = a'//b' + c//d.$$

By definition, the left-hand side is $(ad + bc)//bd$ and the right-hand side is $(a'd + b'c)//b'd$, so upon unravelling the definition of the equivalence relation, we have to show that $(ad+bc)b'd = (a'd+b'c)bd$. This expands to $ab'd^2+bb'cd = a'bd^2+bb'cd$. By additive cancellation for integers, it remains to prove that $ab'd^2 = a'bd^2$. Since $a'b = ab'$, we have the desired conclusion. □

Observe that the rational numbers $a//1$ behave just like the integers a:

$$(a//1) + (b//1) = (a + b)//1, \quad (a//1) \cdot (b//1) = (a \cdot b)//1, \quad -(a//1) = (-a)//1.$$

Also, $a//1 \sim b//1$ exactly when $a = b$. Because of these, we will identify each integer a with $a//1$; the above observation guarantees that the arithmetic of the integers is consistent with the arithmetic of the rationals, and we can think of the integers as *being embedded* (sitting inside) the rationals. We identify 0 with $0//1$ and 1 with $1//1$.

Observe that a rational number $a//b$ is equal to $0 = 0//1$ if and only if $a \cdot 1 = b \cdot 0$, i.e., if the numerator a is equal to 0. Thus if a and b are non-zero then so is $a//b$.

Definition 4.3. If $x = a//b$ is a non-zero rational number, (so $a, b \neq 0$), then the **reciprocal** x^{-1} of x is the rational number $x^{-1} := b//a$. (Check that the reciprocal is well-defined: $a//b = c//d$, then they have the same reciprocal.)

We now finally get to prove the usual arithmetic laws for $\mathbb{Q}$.

Proposition 4.4. *The set of numbers $\mathbb{Q}$ satisfies the addition axioms (Axiom 2.1), the multiplication axioms (Axiom 2.3), and distributive axiom (Axiom 2.4). Additionally, if x is a non-zero rational number, then $x \cdot x^{-1} = x^{-1} \cdot x = 1$.*

Proof. To give you an idea of what is needed, we will prove the longest part, namely that addition is associative. The rest will be left as homework.

Let x, y, z be rational numbers. We write $x = a//b$, $y = c//d$, and $z = e//f$ for certain integers a, c, e and certain nonzero integers b, d, f. Now, we compute

$$\begin{aligned}(x+y)+z = ((a//b)+(c//d))+e//f &= (ad+bc)//(bd)+e//f \\ &= (adf+bcf+bde)//bdf \\ &= a//b+(cf+de)//(df) \\ &= (a//b)+((c//d)+e//f) = x+(y+z).\end{aligned}$$

Thus we see that $(x + y) + z$ and $x + (y + z)$ are equal. □

Definition 4.5. If x, y are rational numbers and $y \neq 0$, we define the **quotient** of x and y by the formula $x/y := x \cdot y^{-1}$.

Example 4.6. $(3//4)/(5//6) = (3//4) \cdot (6//5) = 18//20 = 9//10$.

Using the definition of the quotient and viewing the integer a as the rational $a//1$, we find that $a//b = (a//1) \cdot (b//1)^{-1}$ corresponds naturally with the usual notion of fraction a/b that you are familiar with. Thus we can (finally!) discard the // notation and use the usual a/b notation instead. Similarly, we use the shorthand $x - y$ to denote $x + (-y)$.

Definition 4.7. A rational number a/b is defined to be **positive** when $x = a/b$ for some positive integers a and b and **negative** if $x = (-a)/b$ for some positive integers a and b.

Note that since $ab = (-a)(-b)$ (HW), we have that $(a, b) \sim (-a, -b)$. One now can appeal to Lemma 3.14 to see that $a//b = (-a)//(-b)$. In our new shorthand, this reads as $a/b = (-a)/(-b)$. It now follows from our definition of positivity that a/b is positive if and only if a and b are both negative. Thus our definition of positivity is comprehensive, and the same can be shown for our definition of negativity.

Definition 4.8. Let a/b and c/d be rational numbers. We say that $x > y$ precisely when $x - y$ is a positive rational. We say that $x < y$ precisely when $x - y$ is a negative rational number. We say that $x \geq y$ when either $x > y$ or $x = y$, and we similarly define $x \leq y$.

Proposition 4.9. *The relation $<$ in Definition 4.8 on $\mathbb{Q}$ makes $\mathbb{Q}$ an ordered set (recall Definition 2.5).*

Proof. Exercise. □

5. Absolute value and sequences

A key concept in analysis is that of the *absolute value*, which measures closeness to zero.

Definition 5.1. Let $x, y \in \mathbb{Q}$. We define the **absolute value** of x, denoted $|x|$ by

$$|x| := \begin{cases} x & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -x & \text{if } x < 0. \end{cases}$$

By splitting into four cases (i) $x, y \geq 0$, (ii) $x, -y \geq 0$, (iii) $-x, y \geq 0$, and (iv) $x, y \leq 0$, we can see that

$$|xy| = |x| \cdot |y|$$

for all $x, y \in \mathbb{Q}$. In particular, $|x| \geq 0$ for all $x \in \mathbb{Q}$.

Here is probably the most important and heavily used inequality in analysis.

Theorem 5.2 (The triangle inequality)**.** *If $x, y \in \mathbb{Q}$, then $|x + y| \leq |x| + |y|$.*

In order to prove the triangle inequality, we require two intermediate results.

Lemma 5.3. *If $x \in \mathbb{Q}$, then $-|x| \leq x \leq |x|$.*

Proof. Homework. □

Lemma 5.4. *If $x, y \in \mathbb{Q}$, then $|x| \leq |y|$ if and only if $-|y| \leq x \leq |y|$.*

Proof. Homework. Note that you need to prove <u>two</u> separate results here. You need to prove that (i) if $|x| \leq |y|$, then $-|y| \leq x \leq |y|$ **AND** (ii) if $-|y| \leq x \leq |y|$, then $|x| \leq |y|$. □

Proof of Theorem 5.2. Let $x, y \in \mathbb{Q}$. We see from the definition of absolute value that $||x| + |y|| = |x| + |y|$. Thus the statement we want to prove is the same as $|x + y| \leq ||x| + |y||$. By Lemma 5.4, this holds if and only if $-||x|+|y|| \leq x+y \leq ||x|+|y||$. But since $||x|+|y|| = |x|+|y|$, the statement that we seek to prove is the same as proving $-(|x|+|y|) \leq x+y \leq |x|+|y|$. This follows from adding together the inequalities

$$\begin{aligned} -|x| &\leq x \leq |x|, \\ -|y| &\leq y \leq |y|. \end{aligned}$$

from Lemma 5.3. □

The absolute value will play an indispensable role in our construction of the reals. Recall that $\mathbb{Q}$ does not contain all of the numbers you know; it does not contain $\sqrt{2}$, for example. (See §2 of Ross for a detailed discussion on how to produce an abundance of numbers which are not in $\mathbb{Q}$.) So, even though every pair of distinct rationals x and y has many rationals in between them (like $(x+y)/2$), $\mathbb{Q}$ contains many "holes". We have a good intuition for what a real number should look like and how it should behave. However, we do *not* have definitions to match our intuition (yet!).

Let's start with $\sqrt{2}$. We will not cover decimal expansions in detail, but you already know that $1.4 = 14/10$, $1.41 = 141/100$, $1.414 = 1414/1000$, etc. While we cannot produce any rational number whose square is 2, we can observe (experimentally, if you will):

$$\begin{aligned}|1-2| &= 1\\ |1.4^2-2| &= 0.04\\ |1.41^2-2| &= 0.0119\\ |1.414^2-2| &= 0.000604\\ |1.4142^2-2| &= 0.00003836\\ |1.41421^2-2| &= 0.0000100759.\end{aligned}$$

The *sequence* of numbers 1, 1.4, 1.41, etc. looks like it *converges* to 2, in the sense that the absolute values $|1-2|$, $|1.4-2|$, $|1.414-2|$, etc. get closer and closer to zero. If we proceed in this fashion using a suitable succession of rationals (whose absolute value when subtracted from 2 keeps getting closer and closer to zero), then it looks like we have a shot at actually defining $\sqrt{2}$. Of course we must make precise what we mean by *sequence* and *converge* because we only can work with the rationals!

Definition 5.5. A **sequence** of rational numbers is a function $a : \mathbb{Z} \to \mathbb{Q}$ whose domain contains set of the form $\{n \in \mathbb{Z}\colon n \geq 1\}$ (though starting at 0 or another fixed integer also works). It is customary to use $(a_1, a_2, a_3, \ldots)$, $(a_n)_{n=1}^{\infty}$, or $(a_n)_{n\in\mathbb{N}}$ to denote a sequence rather than the function itself. Sometimes we will write (a_n) when the domain is understood or when the results under discussion do not depend on the specific starting point.

We now restrict ourselves to $a_n \in \mathbb{Q}$; we will later extend this to $a_n \in \mathbb{R}$.

Example 5.6. Consider the function $a : \mathbb{Z} \to \mathbb{Q}$ given by $a(n) = 1/n$. This gives us the sequence $(1, 1/2, 1/3, 1/4, \ldots)$, also written $(1/n)_{n=1}^{\infty}$ or $(1/n)_{n\in\mathbb{N}}$. Since $1/0$ is not a rational number, we might also write $(1/n)$.

Example 5.7. Consider the function $a : \mathbb{Z} \to \mathbb{Q}$ given by

$$a(n) = \begin{cases} 1 & \text{if } n \text{ is a multiple of 3,} \\ 0 & \text{otherwise.} \end{cases}$$

This gives us the sequence $(0, 0, 1, 0, 0, 1, 0, 0, 1, \ldots)$. It's a little tricky to write this as (a_n).

Key point: The set of values attained by a sequence is **different** from the sequence itself! In the last example, the sequence is $(0, 0, 1, 0, 0, 1, 0, 0, 1, \ldots)$, but the set of values attained by the sequence is $\{0, 1\}$.

Example 5.8. Let $n \geq 1$ be an integer, and let $x \geq 0$ be a natural number. We use the shorthand x^n to denote that we multiply x by itself n times. Consider the function $a(n)$: $\mathbb{Z} \to \mathbb{Q}$ given by $a(n) = (1 + 1/n)^n$. This gives us the sequence $(2, (3/2)^2, (4/3)^3, (5/4)^4, \ldots)$. The decimal approximation looks like

$$(2, 2.25\ldots, 2.3704\ldots, 2.4414\ldots, 2.4883\ldots, 2.5216\ldots, 2.5465\ldots, 2.5658\ldots, \ldots).$$

To which number might we get really close to if we continue?

Example 5.9. Sequences can be defined *recursively*. For example, let $a_1 = 0$, $a_2 = 1$, and let $a_3 = a_2 + a_1$, $a_4 = a_3 + a_2$, $a_5 = a_4 + a_3$, and so on. We can write this as $a_1 = 0$, $a_2 = 1$, and $a_{n+2} = a_{n+1} + a_n$ for all $n \geq 3$. This gives us the *Fibonacci sequence* $(0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots)$.

6. Cauchy sequences

In preparation for constructing $\mathbb{R}$, we need to develop the theory of sequences. In our example $(a_n) = (1, 1.4, 1.41, 1.414, 1.4142, 1.41421, \ldots)$, it looks like all that we need is for the terms in the sequence to get closer and closer to $\sqrt{2}$ (in the sense that absolute value $|a_n^2 - 2|$ gets closer and closer to zero). Then we could *define* $\sqrt{2}$ to be the limit of the sequence. However, this has a subtle problem. The number $\sqrt{2}$ is also the limit of many other rational sequences, such as $(b_n) = (1.4, 1.414, 1.41421, 1.4142135, 1.414213562, \ldots)$, even though $a_n \neq b_n$ for all n. How then can it be seen that $\sqrt{2}$ is a *unique* element of the real numbers? Does the choice of sequence whose limit is $\sqrt{2}$ affect the way we do computations with $\sqrt{2}$?

But suppose we could resolve this issue. What should (a_n) converge to? We are incapable of making a prediction because we don't have an explicit description of a_n; also, what would we do if the value to which (a_n) converges is not rational? The following definition allows us to imagine sequences (a_n) with the a_n's "getting close to something", but the idea of "getting close" will be defined entirely in terms of the a_n's, which is exactly what we need!

Definition 6.1. A **Cauchy sequence** of rational numbers is a sequence $(a_1, a_2, a_3, \ldots)$ of rational numbers such that for every *rational* $\varepsilon > 0$, there exists a positive integer N_ε (which is allowed to depend on ε) such that

$$|a_m - a_n| < \varepsilon \qquad \text{whenever } m, n \geq N_\varepsilon.$$

(The restriction that ε needs to be a rational number is there purely because we don't know what a real number is yet. Later we will consider Cauchy sequences of real numbers and we will think of ε as being any positive real number. You should not think about this distinction too much, as it will not be important in the long run.)

Let us delve into Definition 6.1. It says that we must first pick a *error*, which we call ε. You can pick your error to be as small as you like (say, $\varepsilon = 1/100$ or $\varepsilon = 1/10^{100}$), as long as it is positive. So pick the error that you want, and we will move on to the next part.

Now that you have picked your error, there is a *threshold* (a positive integer N_ε which is allowed to depend on the error ε that you chose) with a very special property that we want to have: Eventually (i.e., after m, n get past the threshold N_ε), the terms a_m and a_n get REALLY close together (i.e., $|a_m - a_n|$ is less than the error ε that you chose at the beginning). This needs to hold for **ALL** $m, n \geq N$ (e.g., not just for n and $m = n + 1$). The definition now says that a sequence is *Cauchy* if this special property holds for *any* choice of $\varepsilon > 0$. So you should be able to make your error at the beginning as small as you could possibly imagine (and then even smaller).

This is a lot to ask! Think about it: The definition is saying that if $n \geq N_\varepsilon$, then $|a_n - a_{n+1}|$, $|a_n - a_{n+2}|$, $|a_n - a_{n+20}|$, $|a_n - a_{n+10000}|$, etc. are all smaller than ε. That's a very special kind of sequence. Moreover, you can take ε to be arbitrarily small!

Proposition 6.2. *The sequence* $(1/n)_{n \in \mathbb{N}} = (1, 1/2, 1/3, 1/4, 1/5, \ldots)$ *is a Cauchy sequence.*

Sometimes, for these proofs, it's good to "work backwards". Let's first recall the definition: Pick an error $\varepsilon > 0$. We would like to come up with a threshold N (a positive integer) such that $|1/m - 1/n| < \varepsilon$ whenever $m, n \geq N$. Since $m, n \geq N$, we can use the triangle inequality to obtain $|1/m - 1/n| \leq 1/m + 1/n \leq 2/N$. Therefore, it is enough to come up with a positive integer N for which $2/N < \varepsilon$, or alternatively $N > 2/\varepsilon$. Does such a positive integer exist?

Proposition 6.3 (Archimedean property)**.** *If* $x \in \mathbb{Q}$*, there exists* $n \in \mathbb{N}$ *such that* $n > x$*.*

Proof. Since $x \in \mathbb{Q}$, we can write $x = p/q$ for some integers p, q with $q \geq 1$. If $p \leq 0$, then $1 > x$, as desired. So we may now assume that $p \geq 1$. But then $p/q \leq p < p+1$, so $p+1 > x$, as desired. (Notice how convenient the trichotomy is!) □

This supplies the missing piece in Proposition 6.2, so we can now write a formal proof.

Proof of Proposition 6.2. Let $\varepsilon > 0$. Let N be an integer such that $N > 2/\varepsilon$. Then, if $m, n \geq N$, we have $|1/m - 1/n| \leq 1/N + 1/N < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Thus $(1/n)_{n\in\mathbb{N}}$ is Cauchy. □

But not all sequences are Cauchy sequences.

Proposition 6.4. *For each $n \in \mathbb{N}$, let $H_n = 1 + 1/2 + 1/3 + \cdots + 1/n$. The sequence $(H_n)_{n\in\mathbb{N}}$ is not a Cauchy sequence.*

To show that a sequence (a_n) is not Cauchy, (a_n) must satisfy the negation of the definition of a Cauchy sequence. Thus we need to produce a specific $\varepsilon > 0$ such that for all thresholds $N \in \mathbb{N}$, we have $|a_m - a_n| \geq \varepsilon$ for *some* $m, n \geq N$. So no matter how big you make your threshold N, you will always be able to find a pair m, n for which the distance between a_m and a_n is always greater than ε.

Proof of Proposition 6.4. Let $\varepsilon = 1/2$ (others could work). For each $n \in \mathbb{N}$, consider

$$|H_{2n} - H_n| = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n}.$$

There are n terms in the sum above, and each one is at least as big as $\frac{1}{2n}$. Thus

$$|H_{2n} - H_n| = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n} \geq \frac{1}{2n} + \cdots + \frac{1}{2n} = \frac{1}{2}.$$

Therefore, for all $N \in \mathbb{N}$, we can always find a pair of integers m and n (namely n and $2n$) such that $|a_m - a_n| \geq 1/2$. Thus (H_n) is not Cauchy. □

Proving whether a sequence is Cauchy by brute force (like the last two propositions) can be very tough. We will develop tools to help us work with Cauchy sequences more efficiently. This will require a notion of *boundedness*.

Definition 6.5. A sequence (a_n) (of rational numbers) is **bounded** if there exists a rational number $M \geq 0$ such that $|a_n| \leq M$ for all $n \geq 1$. We then say that "(a_n) is bounded by M".

Not all sequences are bounded. Consider the sequence $(n)_{n\in\mathbb{N}}$.

Proposition 6.6. *Every Cauchy sequence of rational numbers is bounded.*

Proof. Suppose (a_n) is a Cauchy sequence. Taking $\varepsilon = 1$, this means that there exists an integer $N \geq 1$ such that $|a_m - a_n| < 1$ for all integers $m, n \geq N$. Consider the set of absolute values $\{|a_1|, |a_2|, |a_3|, \ldots, |a_N|\}$. Let M_0 denote the largest of these absolute values (this is fine because this is a finite set of numbers). Now, let $M = M_0 + 1$.

We will show that each a_n in the sequence with $n > N$ satisfies $|a_n| \leq M$. To see this, we use the "adding zero trick". In particular, we observe that $a_n = a_n - a_N + a_N$. Now, an application of the triangle inequality yields $|a_n| = |a_n - a_N + a_N| \leq |a_n - a_N| + |a_N|$. Since $n > N$, we have from before that $|a_n - a_N| < 1$, that $|a_N| \leq M_0$, and $M = M_0 + 1$. Thus

$$|a_n| \leq |a_n - a_N| + |a_N| \leq 1 + M_0 = M.$$

Since $M_0 < M$, we find that $|a_n| \leq M$ for all $n \in \mathbb{N}$, as desired. □

The "adding zero trick" is a very important tool!!! Remember this one.

7. Arithmetic of Cauchy sequences

We now show that we can "add" and "multiply" Cauchy sequences.

Proposition 7.1. *If (a_n) and (b_n) are Cauchy sequences of rational numbers, then so are $(a_n + b_n)$, $(a_n b_n)$, and $(-a_n)$.*

Proof. Pick an error $\varepsilon > 0$. Since (a_n) and (b_n) are Cauchy, there exist positive integers $N_a(\varepsilon)$ and $N_b(\varepsilon)$ such that for all $m, n \geq N_a(\varepsilon)$ and $m, n \geq N_b(\varepsilon)$, we have

$$|a_m - a_n| < \frac{\varepsilon}{2} \qquad \text{and} \qquad |b_m - b_n| < \frac{\varepsilon}{2}.$$

(You'll see shortly why we chose $\varepsilon/2$; convince yourself that this is OK.) Let $N_\varepsilon = \max\{N_a(\varepsilon), N_b(\varepsilon)\}$, and suppose that $m, n \geq N$. Then

$$|(a_m + b_m) - (a_n + b_n)| = |(a_m - a_n) + (b_m + b_n)| \leq |a_m - a_n| + |b_m - b_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus $(a_n + b_n)$ is a Cauchy sequence.

For products, we will use Proposition 6.6. This tells us that there exists numbers $M_a \geq 0$ and $M_b \geq 0$ such that $|a_n| \leq M_a$ and $|b_n| \leq M_b$ for all $n \in \mathbb{N}$. Set $M = \max\{M_a, M_b\}$. Since (a_n) and (b_n) are Cauchy, there exist natural numbers $N_a(\varepsilon, M)$ and $N_b(\varepsilon, M)$ (they can, and probably will, depend on M) such that if $m, n \geq N_a(\varepsilon, M)$ and $m, n \geq N_b(\varepsilon, M)$, then

$$|a_m - a_n| < \frac{\varepsilon}{2M} \qquad \text{and} \qquad |b_m - b_n| < \frac{\varepsilon}{2M}.$$

Let $N(\varepsilon, M) = \max\{N_a(\varepsilon, M), N_b(\varepsilon, M)\}$. Then if $m, n \geq N(\varepsilon, M)$, we have, using the "adding zero trick" and the triangle inequality,

$$\begin{aligned} |a_m b_m - a_n b_n| &= |a_m b_m - a_m b_n + a_m b_n - a_n b_n| \\ &\leq |a_m b_m - a_m b_n| + |a_m b_n - a_n b_n| \\ &= |a_m| \cdot |b_m - b_n| + |b_n| \cdot |a_m - a_n| < M\frac{\varepsilon}{2M} + M\frac{\varepsilon}{2M} = \varepsilon. \end{aligned}$$

Thus $(a_n b_n)$ is Cauchy. We leave the proof that $(-a_n)$ is Cauchy as an exercise. □

It is starting to look like Cauchy sequences enjoy a structure similar to the rationals. Is there an additive identity? Yes, namely $(0, 0, 0, 0, \ldots)$. Is there a multiplicative identity? Yes, namely $(1, 1, 1, 1, \ldots)$.

What about division? Tempting as it is to say that (a_n/b_n) is Cauchy if (a_n) and (b_n) are, we have a problem: What if *any* of the numbers b_n equal zero? That would ruin the division for the whole sequence. So division wouldn't work even if we "divide by" $(1, 0, 1, 1, 1, 1, 1, \ldots)$.

A more interesting example comes from comparing the three Cauchy sequences

$$(a_n) = (1/n), \qquad (b_n) = (2/n), \qquad (c_n) = (3/n).$$

Each "looks like" their terms are approaching 0, but $a_n \neq b_n \neq c_n$ for all n. So the sequences are definitely not the same. But is there a notion of "equality" for (a_n), (b_n), and (c_n) that registers with our observation that the terms a_n, b_n, and c_n each approach 0?

Definition 7.2. We define a binary relation $\sim$ on the set of Cauchy sequences of rational numbers (a_n) as follows. We define $(a_n) \sim (b_n)$ (that is, (a_n) is "equivalent to" or "related to" (b_n)) to mean that for every rational $\varepsilon > 0$, there exists a positive integer N_ε such that if $n \geq N$, then

$$|a_n - b_n| < \varepsilon.$$

Lemma 7.3. *The binary relation in Definition 7.2 is an equivalence relation.*

Proof. Homework. □

This notion of equivalence allows us to come to a good definition for the "division" of two Cauchy sequences. This requires some further development.

Definition 7.4. A Cauchy sequence of rational numbers (b_n) is said to be **bounded away from zero** if there exists a rational number $\ell > 0$ such that $|b_n| \geq \ell$ for every $n \in \mathbb{N}$.

Proposition 7.5. *Suppose that (a_n) is a Cauchy sequence of rational numbers that is not equivalent to the zero sequence $(0, 0, 0, \ldots)$. Then there exists a Cauchy sequence of rational numbers (b_n) such that*

(1) (b_n) is bounded away from zero (so there exists $\ell > 0$ such that $|b_n| \geq \ell$ for all n),
(2) $(a_n) \sim (b_n)$.

Proof. Suppose that (a_n) is not equivalent to the zero sequence. We need to negate the definition of $\sim$: *There exists some* fixed rational $\varepsilon_0 > 0$ (we might not know the exact number) such that *for each* $N \geq 1$, *there exists* a number $n \geq N$ for which

$$|a_n - 0| = |a_n| \geq \varepsilon_0.$$

Now, (a_n) is a Cauchy sequence, so there exists some (fixed) N_0 (we might not know the exact number) for which

$$|a_m - a_n| < \frac{\varepsilon_0}{2} \qquad \text{whenever } m, n \geq N_0.$$

And for this fixed N_0, there is a fixed $n_0 \geq N_0$ for which

$$|a_{n_0} - 0| = |a_{n_0}| \geq \varepsilon_0. \tag{7.1}$$

<u>Claim</u>: $|a_n| \geq \varepsilon_0/2$ for every $n \geq N_0$.
<u>Proof of the claim</u>: Suppose to the contrary that $|a_{n_1}| < \varepsilon_0/2$ for some $n_1 \geq N_0$. Then, by the triangle inequality and the "adding zero trick",

$$|a_{n_0}| = |a_{n_0} - a_{n_1} + a_{n_1}| \leq |a_{n_0} - a_{n_1}| + |a_{n_1}| < \frac{\varepsilon_0}{2} + \frac{\varepsilon_0}{2} = \varepsilon_0.$$

But this contradicts (7.1), so we must have $|a_n| \geq \varepsilon_0/2$ for every $n \geq N_0$. This proves the claim.

This almost finishes the proof. We have shown that a_n eventually satisfies $|a_n| \geq \ell$, where $\ell = \varepsilon_0/2$. Now, we simply define

$$b_n = \begin{cases} \ell & \text{if } 1 \leq n < N_0, \\ a_n & \text{if } n > N_0. \end{cases}$$

Since (a_n) is Cauchy, (b_n) is Cauchy as well. Clearly, (b_n) is bounded away from zero. Now, $(a_n) \sim (b_n)$ because $|a_n - b_n| = 0$ for all $n \geq N_0$. This finishes the proof. □

8. Arithmetic of Cauchy sequences, II. The definition of a real number.

We can now give conditions under which (a_n) being Cauchy implies that (a_n^{-1}) is Cauchy.

Proposition 8.1. *Suppose that (a_n) is a Cauchy sequence of rational numbers which is bounded away from zero. Then the sequence (a_n^{-1}) is also a Cauchy sequence.*

Proof. Since (a_n) is bounded away from zero, there exists some fixed rational number $\ell > 0$ such that $|a_n| \geq \ell$ for all n. Thus for any positive integers m, n, we have that

$$|a_m^{-1} - a_n^{-1}| = \left|\frac{a_m - a_n}{a_m a_n}\right| \leq \frac{|a_m - a_n|}{\ell^2} = \frac{1}{\ell^2} \cdot |a_m - a_n|.$$

Now, since (a_n) is Cauchy, we have that for any rational error $\varepsilon > 0$, there exists a positive integer N_ε (our threshold) such that $|a_m - a_n| < \ell^2\varepsilon$ whenever $m, n \geq N$. For that same ε and N_ε, we also have $|a_m^{-1} - a_n^{-1}| \leq \frac{1}{\ell^2} \cdot |a_m - a_n| < \frac{1}{\ell^2} \cdot \ell^2\varepsilon = \varepsilon$ whenever $m, n \geq N$. □

Now, we can define the "division" of two Cauchy sequences of rational numbers (a_n) and (b_n) to be $(a_n b_n^{-1})$, provided that (b_n) is bounded away from zero. Additionally, Proposition 7.5 shows that any sequence (b_n) which is not equivalent to $(0, 0, 0, \ldots)$ is equivalent to a sequence (c_n) which *is* bounded away from zero.

But we encounter a problem. Suppose that (a_n), (b_n), (c_n) are Cauchy sequences of rational numbers. Suppose also (b_n) and (c_n) are bounded away from zero and that $(b_n) \sim (c_n)$. Is it necessarily the case that $(a_n b_n^{-1}) \sim (a_n c_n^{-1})$? In other words, is this notion of "division" well-defined, or does the output from the "division" process depend on the choice of element you pick from the equivalence class of (b_n)?

Lemma 8.2. *"Division" of Cauchy sequences is well-defined. That is, if (a_n), (b_n), (c_n) are Cauchy sequences of rational numbers, $(b_n) \sim (c_n)$, and both (b_n) and (c_n) are bounded away from zero, then $(a_n b_n^{-1}) \sim (a_n c_n^{-1})$.*

Proof. Exercise. □

We can now define the real numbers! Before we do, let's reflect on what happened when we defined the rationals. We looked at equivalence classes of ordered pairs of integers. We came up with reasonable definitions of adding, multiplying, and negating. Then, we showed that these operations are *well-defined* (e.g., addition doesn't change if you use 1/2 instead of 2/4, etc.) using properties of the *integers*. Then we proved all of the pertinent properties of the rationals, which reduced to studying ordered pairs of *integers*. We now proceed along a similar path for the reals, now appealing to the properties of the *rationals*.

Definition 8.3. Let X be the set of all Cauchy sequences of rational numbers, and let $\sim$ be the equivalence relation given by Definition 7.2. A **real number** is an equivalence class in $X/\sim$. We denote the set of all real numbers by $\mathbb{R}$. Recall the notation that $[(a_n)]$ is the set of Cauchy sequences of rationals which are equivalent to (a_n) under $\sim$. Let $x = [(a_n)]$ and $y = [(b_n)]$ be real numbers. We define their sum $x + y := [(a_n + b_n)]$, their product $x \cdot y := [(a_n \cdot b_n)]$, and negation $-x := [(-a_n)]$. If (b_n) is bounded away from zero, then we define the reciprocal $y^{-1} := [(b_n^{-1})]$. We say that $x = 0$ if $[(a_n)] = [(0, 0, 0, 0, \ldots)]$.

This does not look at all like the definition of a number! How might we actually get something that resembles a number out of this mess? What we would **really** like to do is be able to say that a real number is a limit of a convergent Cauchy sequence of reals, and that two real numbers are equal if the pertinent Cauchy sequences are equivalent under the $\sim$ from Definition 7.2. Here is the strategy to approach this:

- Get a sense of the arithmetic of $\mathbb{R}$ (like we did with $\mathbb{Q}$)
- (Finally!) Define notions of *convergence* and *limit*
- Prove that all Cauchy sequences of rational numbers converge to a real number
- Prove that equivalent Cauchy sequences converge to the same real number

With these ideas in place, we can associate to any equivalence class of Cauchy sequences to their collective limit, which is precisely the sort of idea that guided us before.

It is not clear that rationals are real numbers yet (rational numbers are not equivalence classes of sequences of rational numbers), but there is a sense in which we can *embed* $\mathbb{Q}$ into $\mathbb{R}$. For each $x \in \mathbb{Q}$, the sequence $(x, x, x, \ldots)$ is Cauchy (why?), so the equivalence class $[(x, x, x, \ldots)]$ is a real number. In this way, we can think of $\mathbb{Q}$ as sitting inside of $\mathbb{R}$.

Lemma 8.4. *If $x \neq 0$, then there exists a Cauchy sequence (b_n) of rationals which is bounded away from zero such that $x^{-1} = [(b_n^{-1})]$. In particular, if $x \neq 0$, then x^{-1} exists.*

Proof. If $x \neq 0$, then we can write x as an equivalence class $[(a_n)]$ such that $(a_n) \not\sim (0, 0, 0, \ldots)$. By Proposition 7.5, we can find a Cauchy sequence of rationals (b_n) which is bounded away from zero such that $(a_n) \sim (b_n)$. By Lemma 3.14, it follows that $[(a_n)] = [(b_n)]$. Since (b_n) is bounded away from zero, Proposition 8.1 tells us that (b_n^{-1}) is a Cauchy sequence, and so Definition 8.3 tells us that $[(b_n^{-1})] = [(b_n)]^{-1} = x^{-1}$. □

As with the rationals, we first check the well-definedness of adding, multiplying, negating, and reciprocating Cauchy sequences of rationals.

Lemma 8.5. *Addition, multiplication, negation, and reciprocation of the reals are well-defined. That is, suppose that $x = [(a_n)]$, $y = [(b_n)]$, and $z = [(c_n)]$ are real numbers and $x = y$. (Here, (a_n), (b_n), and (c_n) are Cauchy sequences of rationals.) Then*

(1) $x + z = y + z$,
(2) $x \cdot z = y \cdot z$,
(3) $-x = -y$,
(4) If $x \neq 0$, then $x^{-1} = y^{-1}$.

Proof. Part 4 is the hardest, so we will prove that here. The rest are left as homework. Part 4 relies on the well-definedness of multiplication in Part 2.

Assume that $x = y$, in which case $[(a_n)] = [(b_n)]$. Since $x \neq 0$ (hence $y \neq 0$), Lemma 8.4 tells us that we can assume that (a_n) and (b_n) are bounded away from zero. Thus the reciprocal sequences (a_n^{-1}) and (b_n^{-1}) are Cauchy sequences, and so $x^{-1} = [(a_n^{-1})]$ and $y^{-1} = [(b_n^{-1})]$ by definition. Consider the product P given by $P := (x^{-1} \cdot x) \cdot y^{-1}$. By the definition of multiplication, we have that

$$P = \Big([(a_n^{-1})] \cdot [(a_n)]\Big) \cdot [(b_n^{-1})] = [\big(a_n^{-1} \cdot a_n\big) \cdot b_n^{-1})] = [(b_n^{-1})] = [(b_n)]^{-1} = y^{-1}.$$

On the other hand, it follows from the above calculation, associativity of multiplication in $\mathbb{Q}$, and the definition of multiplication in $\mathbb{R}$ that

$$P = [(\big(a_n^{-1} \cdot a_n\big) \cdot b_n^{-1})] = [(a_n^{-1} \cdot \big(a_n \cdot b_n^{-1}\big))] = [(a_n^{-1})] \cdot [(a_n \cdot b_n^{-1})].$$

By part (2), the fact that $x = y$ implies $x \cdot y^{-1} = y \cdot y^{-1}$. Thus $[(a_n \cdot b_n^{-1})] = [(b_n \cdot b_n^{-1})]$, so

$$P = [(a_n^{-1})]\cdot[(a_n\cdot b_n^{-1})] = [(a_n^{-1})]\cdot[(b_n\cdot b_n^{-1})] = [(a_n^{-1}\cdot\big(b_n\cdot b_n^{-1}\big))] = [(a_n^{-1}\cdot 1)] = [(a_n^{-1})] = [(a_n)]^{-1} = x^{-1}.$$

It now follows that $y^{-1} = P = x^{-1}$, as desired. □

We now define division of the reals via $x/y := x \cdot y^{-1}$.

9. Arithmetic of the reals

Proposition 9.1. *The set of real numbers* $\mathbb{R}$ *satisfies the addition axioms (Axiom 2.1), the multiplication axioms (Axiom 2.3), and the distributive axiom (Axiom 2.3). Additional, if* x *is a non-zero real number, then* $x^{-1} \cdot x = x^{-1} \cdot x = [(1, 1, 1, \dots)]$ *(which we now think of as* 1*).*

Proof. We have done most of the hard work already. This is left as an exercise. □

We now move on to show that the reals are ordered.

Definition 9.2. A nonzero real number $x = [(a_n)]$ is **positive** if there is at least one Cauchy sequence $(b_n) \in [(a_n)]$ such that

- (b_n) is bounded away from zero, and
- b_n is positive for each $n \in \mathbb{N}$.

Similarly, $x = [(a_n)]$ is **negative** if there is at least one Cauchy sequence $(c_n) \in [(a_n)]$ such that

- (c_n) is bounded away from zero, and
- c_n is negative for each $n \in \mathbb{N}$.

Lemma 9.3. *Every real number is exactly one of positive, negative, or zero.*

Proof. Let $x \in \mathbb{R}$, and suppose that $x \neq 0$. We need to prove that $x < 0$ or $x > 0$. Since $x \neq 0$, Proposition 7.5 implies that there exists a Cauchy sequence (a_n) bounded away from zero such that $x = [(a_n)]$. Thus there is some fixed rational number $\ell > 0$ such that $|a_n| \geq \ell$ for all n.

Claim: There is a positive integer N such that a_n has the same sign (i.e., a_n is always positive or always negative) for all $n \geq N$.
Proof of the claim: Let $\varepsilon = \ell$. There exists a natural number N such that

$$|a_m - a_n| < \ell \qquad \text{whenever } m, n \geq N. \tag{9.1}$$

Suppose to the contrary that a_m and a_n have different signs. Then, since $|a_n| \geq \ell$ for all n,

$$|a_m - a_n| = ||a_m| + |a_n|| = |a_m| + |a_n| \geq \ell + \ell = 2\ell > \ell.$$

But this contradicts (9.1). Thus a_m and a_n have the same sign; that is, both are positive or both are negative. This concludes the proof of the claim.

So all of the terms a_n with $n \geq N$ have the same sign. In order to ensure that every term a_n with $n \geq 1$ has the same sign, we can change the beginning of the sequence (since changing finitely many terms at the beginning of a sequence yields an equivalent sequence). □

Definition 9.4. If $x, y \in \mathbb{R}$, we say that $x < y$ if $y - x$ is a positive real number.

Exercise. *Prove that* $x > 0$ *if and only if* x *is positive, and* $x < 0$ *if and only if* x *is negative.*

We now define the absolute value in the way we did for the rationals:

$$|x| := \begin{cases} x & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -x & \text{if } x < 0. \end{cases}$$

This definition *extends* the one for the rationals in the sense that if we think of $r \in \mathbb{Q}$ as $x = [(r, r, r, \dots)]$, then $|x| = [(|r|, |r|, |r|, \dots)]$. In fact, we have more generally,

Lemma 9.5. *If $x = [(a_n)]$ is a real number, then $|x| = [(|a_n|)]$.*

Proof. Exercise. □

Proposition 9.6. *The set $\mathbb{R}$ is an ordered set under $<$. Moreover, we have the triangle inequality for $\mathbb{R}$: If $x, y \in \mathbb{R}$, then $|x + y| \leq |x| + |y|$.*

Proof. Exercise. □

The following is another important property which shows how $\mathbb{Q}$ sits in $\mathbb{R}$.

Proposition 9.7. *The set $\mathbb{Q}$ is* **dense** *in $\mathbb{R}$. In other words, for every pair of real numbers x and y with $x < y$, there exists a rational number q such that $x < q < y$.*

Proof. Exercise. □

To summarize, $\mathbb{R}$ is a set of numbers which satisfies the addition, multiplication, and distribution axioms, has a notion of division (we call such sets **fields**) which is ordered (with respect to $<$) and which contains $\mathbb{Q}$ as a dense subset. We have really made progress! But remember, $\mathbb{Q}$ has "holes" in it (like $\sqrt{2}$), and we want to show that $\mathbb{R}$ actually fills the holes. (For example, we need to show that $\sqrt{2} \in \mathbb{R}$.) But so far all we've done is constructed something that may not even be better than $\mathbb{Q}$.

To show that $\mathbb{R}$ fills these holes, we need some definitions. The first might look familiar.

Definition 9.8. Let (a_n) be a sequence of real numbers. We say that (a_n) is a **Cauchy sequence** if for every error $\varepsilon > 0$ there exists a natural number N_ε (our threshold) for which

$$|a_m - a_n| < \varepsilon \qquad \text{whenever } m, n \geq N_\varepsilon.$$

We no longer need to let $\varepsilon > 0$ be rational, because we have finally defined the reals! That being said, convince yourself that it suffices to take ε to be rational.

And finally, the definition of convergence and limit!

Definition 9.9. Let (a_n) be a sequence of real numbers. We say that (a_n) **converges** to a **limit** $L \in \mathbb{R}$ if for every error $\varepsilon > 0$, there exists a threshold N_ε for which

$$|a_n - L| < \varepsilon \qquad \text{for all } n \geq N_\varepsilon.$$

If (a_n) converges to L, then we write

$$\lim a_n = L \qquad \text{or} \qquad \lim_{n\to\infty} a_n = L.$$

If there does not exist any $L \in \mathbb{R}$ such that (a_n) converges to L, we say that (a_n) **diverges**.

It should come as no surprise that these notions are related.

Proposition 9.10. *Let (a_n) be a sequence of real numbers. If (a_n) converges to a limit $L \in \mathbb{R}$, then (a_n) is Cauchy.*

Proof. Let $\varepsilon > 0$. Since (a_n) converges to L, there is a number N such that

$$|a_n - L| < \frac{\varepsilon}{2} \qquad \text{for all } n \geq N.$$

Using the "adding zero trick" and the triangle inequality, if $m, n \geq N$, then

$$|a_m - a_n| = |a_m - L - (a_n - L)| \leq |a_n - L| + |a_m - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus (a_n) is Cauchy. □

Intermission: Arithmetic of $\mathbb{Q}$ and $\mathbb{R}$

Let X equal either $\mathbb{Q}$ or $\mathbb{R}$. At this point, we have proven lots of results about X and its arithmetic. Let's take stock of what we can prove directly from Axiom 2.1, Axiom 2.3, Axiom 2.4, and the notion of a reciprocal. Some of these may be repeats from previous results in the notes, graded problems, or additional problems. We will assume these results henceforth, but **you are responsible for all of the proofs**.

Exercise. If $x, y, z \in X$, then the following are true.

(1) If $x + y = x + z$, then $y = z$.
(2) If $x + y = x$, then $y = 0$.
(3) If $x + y = 0$, then $y = -x$.
(4) $-(-x) = x$.

Exercise. If $x, y, z \in X$, then the following are true.

(1) If $x \neq 0$ and $xy = xz$, then $y = z$.
(2) If $x \neq 0$ and $xy = x$, then $y = 1$.
(3) If $x \neq 0$ and $xy = 1$, then $y = x^{-1}$.
(4) If $x \neq 0$, then $(x^{-1})^{-1} = x$.

Exercise. If $x, y \in X$, then the following are true.

(1) $0x = 0$.
(2) If $x \neq 0$ and $y \neq 0$, then $xy \neq 0$.
(3) $(-x)y = -(xy) = x(-y)$.
(4) $(-x)(-y) = xy$.

Definition. Let $x \in X$. To raise x to the power 0, we define $x^0 := 1$. In particular, we define $0^0 := 1$. Now, suppose inductively that x^n has been defined for some nonnegative integer n. We then define $x^{n+1} := x^n \cdot x$.

Exercise. Let $x, y \in X$, and let $m, n \geq 0$ be integers. Prove the following statements:

(1) We have $x^n x^m = x^{n+m}$.
(2) We have $(x^n)^m = x^{nm}$.
(3) We have $(xy)^n = x^n y^n$.

(Side note: Similar proofs give comparable properties for negative exponents once we define $x^{-n} := (x^n)^{-1}$.)

It is part of your homework to establish that $\mathbb{R}$ is ordered; you already proved this for $\mathbb{Q}$. It is part of your additional problems to prove that if $x \in \mathbb{R}$ is represented as an equivalence class of Cauchy sequences of rationals, say $x = [(a_n)]$, then $|x| = [(|a_n|)]$. Once you finish those, then **you will be responsible for the following exercises**.

Exercise. If $x, y, z \in X$, then the following are true.

(1) If $x > 0$, then $-x < 0$, and vice versa.
(2) If $x > 0$ and $y < z$, then $xy < xz$.
(3) If $x < 0$ and $y < z$, then $xy > xz$.

(4) If $x \neq 0$, then $x^2 > 0$. In particular, $1 > 0$.
(5) If $0 < x < y$, then $0 < y^{-1} < x^{-1}$.

Exercise. Let $x, y \in X$, and let $m, n \geq 0$ be integers. Prove the following statements.
(1) Suppose $n \geq 1$. Then we have $x^n = 0$ if and only if $x = 0$.
(2) If $x \geq y \geq 0$, then $x^n \geq y^n \geq 0$.
(3) If $x > y \geq 0$ and $n > 0$, then $x^n > y^n \geq 0$.
(4) We have $|x^n| = |x|^n$.

(Side note: Similar proofs give comparable properties for negative exponents once we define $x^{-n} := (x^n)^{-1}$.)

10. Convergence and limits. The completeness of $\mathbb{R}$.

Proposition 9.10 is not so shocking. But what if the converse statement were true: What if Cauchy sequences of real numbers were the *only* sequences that converged?

Definition 10.1. A set of numbers X is said to be **complete** if all Cauchy sequences of numbers in X converge to a number inside of X.

Theorem 10.2. *The set of real numbers $\mathbb{R}$ is complete.*

This is vindication! We wanted earlier to define the reals to be limits of convergent sequences rational numbers, but we could not because the limit might not be rational, and we had not yet defined the reals. Cauchy sequences may have seemed like a peculiar mechanism by which we might construct the reals. Now, our efforts have paid off: Cauchy sequences of real numbers are in fact the ONLY convergent sequences of real numbers! So we were right all along; we just could not articulate why.

Proposition 10.3. *Let (a_n) be a Cauchy sequence of rationals. Let x be the real number equal to $[(a_n)]$. Then $\lim a_n = x$.*

Proof. Let $\varepsilon > 0$ be rational. Since (a_k) is Cauchy, there exists $N > 0$ such that

$$|a_k - a_m| < \varepsilon/2 \qquad \text{whenever } k, m \geq N. \tag{10.1}$$

We want to prove that for the same N, we also have

$$|a_n - x| < \varepsilon \qquad \text{whenever } n \geq N.$$

To begin, **fix** $n \geq N$. Since a_n is rational, we can write a_n as a real number in the form $a_n = [(a_n, a_n, a_n, \ldots)]$. Now, by Lemma 9.5, we want to show that

$$\begin{aligned} |a_n - x| &= |[(a_n, a_n, a_n, \ldots)] - [(a_1, a_2, a_3, \ldots)]| \\ &= |[(a_n - a_1, a_n - a_2, a_n - a_3, \ldots)]| \\ &= [(|a_n - a_1|, |a_n - a_2|, |a_n - a_3|, \ldots)] \\ &= [(|a_n - a_k|)_{k\in\mathbb{N}}] \end{aligned}$$

is less than ε. Since $\varepsilon \in \mathbb{Q}$, we may write ε as the equivalence class of the Cauchy sequence $(\varepsilon, \varepsilon, \varepsilon, \ldots)$. From our definitions, we now seek to prove that the real number

$$[(\varepsilon - |a_n - a_k|)_{k\in\mathbb{N}}]$$

is positive. This follows from (10.1) — if $k \geq N$, then

$$\varepsilon - |a_n - a_k| \geq \varepsilon - \varepsilon/2 = \varepsilon/2.$$

Thus the sequence (b_n) given by

$$b_n = \begin{cases} \varepsilon/2 & \text{if } k \leq N, \\ \varepsilon - |a_n - a_k| & \text{if } k > N \end{cases}$$

is a sequence, all of whose terms are positive, which is bounded away from zero — $|b_n| \geq \ell := \varepsilon/2$ for all $n \in \mathbb{N}$ — and satisfies $(b_n) \in [(\varepsilon - |a_n - a_k|)_{k\in\mathbb{N}}]$. Thus the real number $[(\varepsilon - |a_n - a_k|)_{k\in\mathbb{N}}]$ is positive, as desired.

We now show that we may take $\varepsilon > 0$ to be real. By Proposition 9.7, we know that there exists a rational ε' such that $0 < \varepsilon' < \varepsilon$. Our work above shows that there exists an $N' > 0$ (depending on ε') such that if $n \geq N'$, then $|a_n - x| < \varepsilon'$. But since $0 < \varepsilon' < \varepsilon$, we have that $|a_n - x| < \varepsilon$, provided that $n \geq N'$. This was precisely the conclusion we sought. ☐

Proof of Theorem 10.2. Let (a_n) be a Cauchy sequence of real numbers. Let $\varepsilon > 0$ be given. For each n, it follows by Proposition 9.7 that there is a rational number q_n such that

$$|a_n - q_n| < 1/(3n). \tag{10.2}$$

Claim: The sequence (q_n) is Cauchy. Proof of the claim: Observe that

$$|q_m - q_n| = |q_m - a_m + a_m - a_n + a_n - q_n| \leq |q_m - a_m| + |a_m - a_n| + |a_n - q_n|.$$

Since (a_n) is Cauchy, we have the existence of some $N \in \mathbb{N}$ (depending on ε) such that if $m, n \geq N$, then $|a_m - a_n| < \varepsilon/3$ whenever $m, n \geq N$. Now, let $N' \in \mathbb{N}$ satisfy $N' \geq \max\{N, 1/\varepsilon\}$. If $m, n \geq N'$, then we can use (10.2) to conclude

$$|q_m - a_m| + |a_m - a_n| + |a_n - q_n| \leq \frac{1}{3m} + \frac{\varepsilon}{3} + \frac{1}{3n} \leq \frac{1}{3N'} + \frac{\varepsilon}{3} + \frac{1}{3N'} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

This proves the claim.

Let $x = [(q_n)]$. We will show that $\lim a_n = x$. Let $\varepsilon > 0$ be given. By Proposition 10.3, (q_n) converges to x, so there exists some threshold $N_\varepsilon \in \mathbb{N}$ such that $|q_n - x| < \varepsilon/2$ for all $n \geq N$. It now follows from the claim that

$$|a_n - x| = |a_n - q_n + q_n - x| \leq |a_n - q_n| + |q_n - x| < \varepsilon/3 + \varepsilon/2 < \varepsilon$$

for all $n \geq N$, as desired. □

Thus we have proved that the set of convergent sequences of reals equals the set of Cauchy sequences of reals. Our proofs of properties for Cauchy sequences of rationals largely carry over for Cauchy sequences of reals. For instance:

Proposition 10.4. *Convergent sequences of real numbers are bounded.*

Proof. The proof is identical to that of Proposition 6.6. □

Proposition 10.5. *Suppose that (a_n) and (b_n) are convergent sequences of reals. Then:*

(1) $\lim(a_n + b_n) = \lim a_n + \lim b_n$.
(2) $\lim(a_n b_n) = (\lim a_n) \cdot (\lim b_n)$.
(3) If $c \in \mathbb{R}$, then $\lim ca_n = c \lim a_n$.
(4) If $b_n \neq 0$ for all n and if $\lim b_n \neq 0$, *then* $\lim(a_n/b_n) = (\lim a_n)/(\lim b_n)$.

Proof. We'll do Part 2; the other proofs are similar. Let $\lim a_n = A \in \mathbb{R}$ and $\lim b_n = B \in \mathbb{R}$. Since convergent sequences of reals are bounded, there exists a real number $M \geq 1$ such that $|b_n| \leq M$ for all $n \in \mathbb{N}$. Moreover, for all $\varepsilon > 0$, there exist $N_a, N_b \in \mathbb{N}$ such that

$$|a_n - A| < \frac{\varepsilon}{2M}, \qquad |b_n - B| < \frac{\varepsilon}{2\max\{|A|, 1\}}$$

when $n \geq N := \max\{N_a, N_b\}$. Now, observe by the triangle inequality that

$$\begin{aligned} |a_n b_n - AB| = |a_n b_n - Ab_n + Ab_n - AB| &= |b_n(a_n - A) + A(b_n - B)| \\ &\leq |b_n| \cdot |a_n - A| + |A| \cdot |b_n - B| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Hence $\lim a_n b_n = AB = (\lim a_n)(\lim b_n)$, as desired. □

11. The least upper bound property

Proposition 11.1. *Suppose* (a_n) *converges. If* $a_n \leq M$ *for all* $n \geq 1$*, then* $\lim a_n \leq M$*. If* $a_n \geq m$ *for all* $n \geq 1$*, then* $\lim a_n \geq m$*.*

Proof. Homework. □

We now get to the heart of why completeness is so important to calculus.

Definition 11.2. Let E be a subset $\mathbb{R}$. Let $M, m \in \mathbb{R}$. We say that M is an **upper bound** for E if $x \leq M$ for all $x \in E$. We say that m is a **lower bound** for E if $x \geq m$ for all $x \in E$.

Example 11.3. Let $E = \{x \in \mathbb{R}\colon 0 \leq x \leq 1\}$. The numbers 1, 4, and 999/1000 are upper bounds for E; the numbers $-1/2$, 0, and $-1/1000$ are lower bounds for E.

Definition 11.4. Let E be a subset $\mathbb{R}$. Let $M, m \in \mathbb{R}$. We say that M is a **least upper bound** for E if (1) M is an upper bound for E, and (2) if M' is another upper bound for E, then $M \leq M'$. We say that m is a **greatest lower bound** for E if (1) m is an lower bound for E, and (2) if m' is another lower bound for E, then $m \geq m'$.

Lemma 11.5. *If* M *and* M' *are two least upper bounds for* E*, then* $M = M'$*, and similarly for greatest lower bounds.*

Proof. By Definition 11.4, we have that $M \geq M'$ and $M' \geq M$. Thus $M = M'$ (why?). □

Notation 11.6. Let E be a subset of $\mathbb{R}$. If E has a least upper bound, it is unique by Lemma 11.5, so we write *the* least upper bound of E as $\sup E$ (the **supremum** of E). If E has a greatest lower bound, it is unique by Lemma 11.5, so we write *the* greatest lower bound bound of E as $\inf E$ (the **infimum** of E).

Theorem 11.7. *Any nonempty subset* E *of* $\mathbb{R}$ *satisfies the* **least upper bound property***: If* E *has an upper bound, then* $\sup E$ *exists and is in* $\mathbb{R}$*.*

Proof. Since E is nonempty has an upper bound, there exists $a_1 \in E$ and $b_1 \in \mathbb{R}$ with $a_1 \leq b_1$. We will define two sequences (a_i) and (b_i) inductively such that $a_n \leq b_n$ for all $n \in \mathbb{N}$. For each n, suppose we have constructed a_n and b_n such that $a_n \leq b_n$. Let $K_n = (a_n + b_n)/2$. Note that $a_n \leq K_n \leq b_n$. If K_n is an upper bound of E, let $a_{n+1} = a_n$ and $b_{n+1} = K$; then $|b_{n+1} - a_{n+1}| = (b_n - a_n)/2$. If K_n is not an upper bound for E, then there exists $x_n \in E$ such that $x_n > K_n$; then we let $a_{n+1} = x_n$ and $b_{n+1} = b_n$, in which case $|b_{n+1} - a_{n+1}| = b_n - x < b_n - K = (b_n - a_n)/2$.

To summarize, we begin with $a_1 \in E$ and b_1 an upper bound for E. Then $a_1 \leq b_1$. Now, for each $n \geq 1$,

$$a_{n+1} = \begin{cases} a_n & \text{if } \frac{a_n+b_n}{2} \text{ is an upper bound for } E, \\ x_n & \text{if } \frac{a_n+b_n}{2} \text{ is not an upper bound for } E \text{ and } x_n \in E \text{ satisfies } x_n > \frac{a_n+b_n}{2} \end{cases}$$

and

$$b_{n+1} = \begin{cases} \frac{a_n+b_n}{2} & \text{if } \frac{a_n+b_n}{2} \text{ is an upper bound for } E, \\ b_n & \text{if } \frac{a_n+b_n}{2} \text{ is not an upper bound for } E. \end{cases}$$

In either case,

(1) $|b_{n+1} - a_{n+1}| \leq (b_n - a_n)/2$,
(2) $a_{n+1} \geq a_n$ for all $n \in \mathbb{N}$,
(3) $b_{n+1} \leq b_n$ for all $n \in \mathbb{N}$, and

(4) $a_n \in E$ for all $n \in \mathbb{N}$ and b_n is an upper bound for E for all $n \in \mathbb{N}$.

Claim 1: For each $n \in \mathbb{N}$,

$$|b_{n+1} - a_{n+1}| = b_{n+1} - a_{n+1} \leq (b_1 - a_1)/2^n. \tag{11.1}$$

Proof of Claim 1: Exercise. (Hint: use induction.)

Claim 2: (a_n) and (b_n) are Cauchy.

Proof of Claim 2: Note that $a_{n+1} \geq a_n$ and $b_n \geq a_{n+1}$. By (11.1),

$$|a_{n+1} - a_n| = a_{n+1} - a_n \leq b_n - a_n \leq \frac{b_1 - a_1}{2^{n-1}} = \frac{2(b_1 - a_1)}{2^n}.$$

If $a_1 = b_1$, then it follows from the above calculation that $a_n = a_1 = b_1$ for all n, in which case (a_n) is Cauchy. If $a_1 < b_1$, then

$$|a_{n+1} - a_n| \leq 2(b_1 - a_1) \cdot \frac{1}{2^n},$$

which implies (by the last problem on HW2) that

$$\left(\frac{a_n}{2(b_1 - a_1)}\right)_{n\in\mathbb{N}}$$

is Cauchy. Thus (a_n) is Cauchy. A similar calculation (left to you!) shows that (b_n) is Cauchy. **QED Claim 2**.

Note that since (a_n) and (b_n) are Cauchy, they converge to real numbers!

Claim 3: If $\lim a_n = A \in \mathbb{R}$ and $\lim b_n = B \in \mathbb{R}$, then $A = B$.

Proof of Claim 3: Exercise. (Hint: Use (11.1) to prove that $(a_n) \sim (b_n)$.)

Claim 4: Let $L = \lim a_n = \lim b_n$. Then L is the least upper bound for E.

Proof of Claim 4: Let $x \in E$ be arbitrary. Then $x \leq b_n$ (since each b_n is an upper bound for E), so $x \leq L$ by Proposition 11.1. Since $x \in E$ was arbitrary, we conclude that L is an upper bound for E. On the other hand, if L' is another least upper bound for E, then $L' \geq a_n$ for all n since each $a_n \in E$. Thus $L' \geq L$ by Proposition 11.1 again. Thus L is the least upper bound for E. **QED Claim 4**. □

Let us now consider some frequently-encountered sets of numbers.

Example 11.8. Let $a, b \in \mathbb{R}$ satisfy $a < b$. Define $[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$, $(a, b) := \{x \in \mathbb{R} : a < x < b\}$, $[a, b) := \{x \in \mathbb{R} : a \leq x < b\}$, and $(a, b] := \{x \in \mathbb{R} : a < x \leq b\}$. These sets of numbers have the same supremum (namely b) and the same infimum (namely a).

Example 11.9. Let $a \in \mathbb{R}$. Define $[a, \infty) := \{x \in \mathbb{R} : x \geq a\}$, $(a, \infty) := \{x \in \mathbb{R} : x > a\}$, $(-\infty, a] := \{x \in \mathbb{R} : x \leq a\}$, and $(-\infty, a) := \{x \in \mathbb{R} : x < a\}$. The first two sets have no supremum (they have no upper bound) and the same infimum (namely a). The last two sets have the same supremum (namely a) and no infimum (they have no lower bound).

Example 11.10. Let (a_n) be a sequence of reals, and let $A = \{a_n : n \in \mathbb{N}\}$. We define $\sup a_n = \sup(A)$ (when it exists) and $\inf a_n = \inf(A)$ (when it exists). For instance, if $a_n = 1/n^2$, then $\sup a_n = 1$ and $\inf a_n = 0$. On the other hand, if $b_n = n^{(-1)^n}$, then $\sup b_n$ does not exist, while $\inf b_n = 0$.

12. Useful consequences of the least upper bound property for $\mathbb{R}$

We use Theorem 11.7 to prove that there exists a positive real number x such that $x^2 = 2$.

Theorem 12.1. *For every real $x > 0$ and every $n \in \mathbb{N}$, there is exactly one positive $y \in \mathbb{R}$ such that $y^n = x$. We sometimes write y as $x^{1/n}$ or $\sqrt[n]{x}$.*

Proof. Let $E = \{t \in \mathbb{R}\colon t > 0,\ t^n < x\}$. If $t = x/(1+x)$, then $0 \leq t < 1$ and $t^n \leq t < x$; thus E is nonempty. By Theorem 11.7, $\sup E$ exists and is a real number. Let $y = \sup E$. We will establish that the statements $y^n < x$ and $y^n > x$ lead to contradictions. By the trichotomy for $\mathbb{R}$, we conclude that $y = x^n$. Lemma 11.5 ensures that y is unique.

Let $0 < a < b$. One can prove by induction that $b^n - a^n = (b-a)\sum_{j=0}^{n-1} a^j b^{n-1-j}$. By HW2, this yields the inequality

$$b^n - a^n < (b-a)nb^{n-1}. \tag{12.1}$$

Suppose to the contrary that $y^n < x$. Then there exists $h \in (0,1)$ such that

$$h < \frac{x - y^n}{n(y+1)^{n-1}} \tag{12.2}$$

(why?). Putting $a = y$ and $b = h + y$ in (12.1), we deduce from (12.2) that

$$(y+h)^n - y^n < hn(y+h)^{n-1} < hn(y+1)^{n-1} < x - y^n.$$

Thus $(y+h)^n < x$, hence $y + h \in E$. But since $y + h > y$, this contradicts the fact that y is the *least* upper bound for E. Thus the statement $y^n < x$ is false. A similar construction also leads us to see that the statement $y^n > x$ is false. □

Corollary 12.2. *If $a, b \in \mathbb{R}$ are positive and $n \in \mathbb{N}$, then $(ab)^{1/n} = a^{1/n}b^{1/n}$.*

Proof. We have $ab = (a^{1/n})^n \cdot (b^{1/n})^n = (a^{1/n}b^{1/n})^n$. Thus $(ab)^{1/n} = a^{1/n}b^{1/n}$ by the uniqueness assertion from Theorem 12.1. □

Corollary 12.3. *The irrational reals are dense in $\mathbb{R}$. In other words, if $x, y \in \mathbb{R}$ and $x < y$, there exists a real number $r \in \mathbb{R}$ such that $r \notin \mathbb{Q}$ and $x < r < y$.*

Proof. Homework. □

Proposition 12.4 (Proposition 9.7 of Ross)**.** *(1) If $p > 0$ and $p \in \mathbb{Q}$, then $\lim 1/n^p = 0$.*
(2) If $|a| < 1$, then $\lim a^n = 0$. If $a = 1$, $\lim a^n = 1$. Otherwise, (a^n) does not converge.
(3) $\lim n^{1/n} = 1$.
(4) If $a > 0$, then $\lim a^{1/n} = 1$.

Proof. We prove the first one. Let $\varepsilon > 0$ and $N \in \mathbb{N}$ satisfy $N > (1/\varepsilon)^{1/p}$. If $n \geq N$, then $n^p > 1/\varepsilon$. Since $1/n^p > 0$, it follows that $|1/n^p - 0| < \varepsilon$, as desired. □

We can now address decimal expansions of reals. Let $x > 0$ be real. Let n_0 be the largest integer such that $n_0 \leq x$. We now proceed recursively; having chosen $n_0, n_1, \ldots, n_{k-1}$, let n_k be the largest integer such that $n_0 + n_1/10 + \cdots + n_k/10^k \leq x$. Let

$$E = \Big\{ \sum_{j=0}^{k} \frac{n_j}{10^j} \colon k = 0, 1, 2, \ldots \Big\}. \tag{12.3}$$

Then $x = \sup E$. The decimal expansion of x is then

$$n_0.n_1n_2n_3\cdots. \tag{12.4}$$

Conversely, for any infinite decimal (12.4), the set of numbers (12.3) is bounded above, and (12.4) is the decimal expansion of $\sup E$.

Definition 12.5. We say that $\lim a_n = \infty$ if for all $M > 0$ there exists $N > 0$ such that $a_n > M$ whenever $n \geq N$. We say that $\lim a_n = -\infty$ if for all $M > 0$ there exists $N > 0$ such that $a_n < -M$ whenever $n \geq N$.

Theorem 12.6 (Theorem 9.9 in Ross)**.** *Let* (a_n) *and* (b_n) *be sequences such that* $\lim a_n = \infty$ *and* $\lim b_n > 0$ *(*$\lim b_n$ *can be finite or* ∞*). Then* $\lim a_n b_n = \infty$*.*

Proof. Since $\lim b_n > 0$, we can find a real $\ell \in (0, \lim b_n)$. Regardless of whether $\lim b_n$ is finite, it follows by Proposition 11.1 that there exists $N_1 > 1$ such that $b_n > \ell$ for all $n \geq N_1$.

Let $M > 0$. Since $\lim a_n = \infty$, there exists $N_2 > 0$ such that $a_n > M/\ell$. Now, if $n \geq \max\{N_1, N_2\}$, then $a_n b_n > \frac{M}{\ell} \cdot \ell = M$. □

Theorem 12.7 (Theorem 9.10 in Ross)**.** *Let* (a_n) *be a sequence of positive reals. We have* $\lim a_n = \infty$ *if and only if* $\lim a_n^{-1} = 0$*.*

Proof. ($\Rightarrow$): Suppose that $\lim a_n = \infty$. Then for each $M > 0$, there exists an $N > 0$ such that $a_n > M$ for all $n \geq N$. Now, let $\varepsilon > 0$, and choose $M = 1/\varepsilon$. If $n \geq N$ (and N here now depends on ε since we chose M in terms of ε), then $0 < a_n^{-1} < 1/M = \varepsilon$ since $a_n > 0$ for all n. Thus for all $\varepsilon > 0$, there exists $N > 0$ such that $|a_n^{-1}| < 1/M = \varepsilon$, as desired.

($\Leftarrow$): Suppose that $\lim a_n^{-1} = 0$. Then for each $\varepsilon > 0$, there exists $N > 0$ such that $|a_n^{-1}| < \varepsilon$ once $n \geq N$. But since a_n is positive, we have $0 < a_n < \varepsilon$. Thus $0 < \varepsilon^{-1} < a_n$. Now, choose $M > 0$ and let $\varepsilon = 1/M$. Then if $n \geq N$, we have that $a_n > 1/\varepsilon = M$ once $n \geq N$. In other words, $\lim a_n = \infty$, as desired. □

Definition 12.8. A sequence (a_n) of reals is **monotonically increasing** (resp. monotonically decreasing) if $a_n \leq a_{n+1}$ (resp. $a_n \geq a_{n+1}$) for all n. Lecture shorthand: $a_n \uparrow$ and $a_n \downarrow$.

Theorem 12.9. *If* (a_n) *is monotonic, then* (a_n) *converges if and only if* (a_n) *is bounded.*

Proof. We already saw that convergent sequences are bounded. Now, suppose that $a_n \leq a_{n+1}$ for all n (the decreasing case is analogous and left as an exercise). Let E be the set of values attained by a_n. If (a_n) is bounded, let $a = \sup a_n$. Then $a_n \leq a$ for all n. For all $\varepsilon > 0$, there exists $N > 0$ such that $a - \varepsilon < a_N \leq a$ (otherwise, $a - \varepsilon$ would be an upper bound for E). But since $a_n \uparrow$, we have $a - \varepsilon < a_n \leq a$ for all $n \geq N$. Thus $|a_n - a| < \varepsilon$ for $n \geq N$. □

13. Subsequences, the Bolzano-Weierstrass theorem, and lim sup/lim inf

Example 13.1. Let (a_n) be defined by $a_n = (-1)^n$. It should be clear that this does not converge, but if we look at $(a_2, a_4, a_6, \ldots, a_{2n}, \ldots) = (1, 1, 1, \ldots)$, this clearly converges (to 1). If we look at $(a_1, a_3, a_5, \ldots, a_{2n+1}, \ldots) = (-1, -1, -1, \ldots)$, this clearly converges to (to -1). These are examples of *subsequences.*

Definition 13.2. Given a sequence $(a_n)_{n\in\mathbb{N}}$, consider a sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers such that $n_1 < n_2 < n_3 < \cdots$. Then the sequence $(a_{n_k})_{k\in\mathbb{N}}$ is called a **subsequence** of $(a_n)_{n\in\mathbb{N}}$. If $(a_{n_k})_{k\in\mathbb{N}}$ converges, its limit is called a **subsequential limit** of $(a_n)_{n\in\mathbb{N}}$.

Example 13.3. Let $a_n = n^2$, so $(a_n)_{n\in\mathbb{N}} = (1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, \ldots)$. If we take $n_k = 2k + 3$, then $(n_k)_{k\in\mathbb{N}} = (5, 7, 9, 11, \ldots)$. Then

$$(a_{n_k})_{k\in\mathbb{N}} = (a_{2k+3})_{k\in\mathbb{N}} = (25, 49, 81, 121, \ldots).$$

There is precisely one subsequential limit, namely ∞. See §11 of Ross for more examples.

Proposition 13.4. *If (a_n) converges to a limit $L \in \mathbb{R}$, then every subsequence of (a_n) also converges to L. If $\lim a_n = \pm\infty$, then every subsequence of (a_n) also converges to $\pm\infty$.*

Proof. Let (a_n) be a sequence, and let $(a_{n_k})_{k\in\mathbb{N}}$ be a subsequence. Clearly $n_1 \geq 1$; suppose that $n_k \geq k$ for some k. Since $n_{k+1} > n_k$, we have that $n_{k+1} \geq n_k + 1 \geq k + 1$. Thus by induction, we have $n_k \geq k$ for all k.

Suppose $\lim_{n\to\infty} a_n = L$ for some $L \in \mathbb{R}$. Then for all $\varepsilon > 0$, there exists some $N > 0$ such that if $n \geq N$, then $|a_n - L| < \varepsilon$. If $k \geq N$, then $n_k \geq k \geq N$, so $|a_{n_k} - L| < \varepsilon$ as well. Thus $\lim_{k\to\infty} a_{n_k} = L$, as desired. (The infinite limit cases are left to the reader.) □

Proposition 13.5. *Every sequence has a monotonic subsequence.*

Proof. Let (a_n) be a sequence. Some terminology: We say that n is a **peak** if $a_m < a_n$ for all $m > n$. Suppose first that (a_n) has infinitely many peaks at $n_1 < n_2 < n_3 < \cdots < n_j < \cdots$. The subsequence $(a_{n_j})_{j\in\mathbb{N}}$ corresponding to these peaks is monotonically decreasing.

Second, suppose there are finitely many peaks. Let n_1 be greater than the last peak (or let $a_{n_1} = a_1$ if there are no peaks). Then

$$\text{given } N \geq n_1, \text{ there exists } m > N \text{ such that } a_m \geq a_N. \tag{13.1}$$

We apply (13.1) with $N = n_1$, selecting $n_2 > n_1$ such that $s_{n_2} \geq s_{n_1}$. Suppose that $n_1, n_2, \ldots, n_{k-1}$ have been selected so that

$$n_1 < n_2 < \cdots < n_{k-1} \qquad \text{and} \qquad a_{n_1} \leq a_{n_2} \leq \cdots \leq a_{n_{k-1}}. \tag{13.2}$$

Applying (13.1) with $N = n_{k-1}$, we select $n_k > n_{k-1}$ such that $s_{n_k} \geq s_{n_{k-1}}$. Then (13.2) holds with k in place of $k - 1$ This inductively produces a monotonically increasing sequence. □

Theorem 13.6 (Bolzano-Weierstrass)**.** *If (a_n) is a bounded sequence of reals, then (a_n) has a convergent subsequence.*

Proof. By Proposition 13.5, (a_n) has a monotonic subsequence; this subsequence must be bounded since (a_n) is. By Theorem 12.9, this monotonic subsequence must converge. □

Example 13.7. Let $a_1 = 1/2$, $a_2 = -1/2$, $a_3 = -1$, $a_4 = -1/2$, $a_5 = 1/2$, $a_6 = 1$, and $a_{n+6} = a_n$ for all $n \geq 1$. Clearly, (a_n) does not converge. It is also clear that (a_n) is bounded, and we can find plenty of subsequences that converge. For instance, let $(n_k) = (6k + 1)_{k\in\mathbb{N}}$, then $(a_{n_k}) = (a_{6k+1})_{k\in\mathbb{N}} = (1/2, 1/2, 1/2, 1/2, \ldots)$.

If (a_n) is not bounded, we can still give a description of when (a_n) has a convergent subsequence, even if the limit happens to be $\pm\infty$. Recall the definition of a limit: If $\lim a_n = L$, then for all $\varepsilon > 0$, we can find some $N > 0$ such that if $n \geq N$, then $|a_n - L| < \varepsilon$. Thus the set $\{n \in \mathbb{N} \colon |a_n - L| < \varepsilon\}$ contains all but finitely many $n \in \mathbb{N}$; in particular, the set is infinite.

Proposition 13.8. *Let (a_n) be a sequence.*

(1) Let $t \in \mathbb{R}$. There is a subsequence of (a_n) converging to t if and only if the set $\{n \in \mathbb{N} \colon |a_n - t| < \varepsilon\}$ is infinite for every $\varepsilon > 0$.
(2) If (a_n) is unbounded from above, then it has a subsequence with limit ∞.
(3) If (a_n) is unbounded from below, then it has a subsequence with limit $-\infty$.

In each case, the subsequence can be taken to be monotonic.

Proof. See Ross (Theorem 11.2). □

Definition 13.9. Let (a_n) be a sequence. If (a_n) is bounded, then define

$$\limsup a_n = \lim_{N\to\infty} \sup\{a_n \colon n > N\}, \qquad \liminf a_n = \lim_{N\to\infty} \inf\{a_n \colon n > N\}.$$

If (a_n) is not bounded above, then $\limsup a_n = \infty$ (since $\{a_n \colon n > N\}$ has no upper bound). If (a_n) is not bounded below, then $\liminf a_n = -\infty$ (since $\{a_n \colon n > N\}$ has no lower bound).

Note: $\limsup a_n$ may not equal $\sup\{a_n \colon n \in \mathbb{N}\}$, but we always have $\limsup a_n \leq \sup\{a_n \colon n \in \mathbb{N}\}$. Think of all of the values that **infinitely many** of the a_n can get close to; the supremum of these values is $\limsup a_n$. Similar remarks hold for $\liminf a_n$.

Theorem 13.10. *Let (a_n) be a sequence.*

(1) If $\lim a_n$ is defined (as a real number, ∞, or $-\infty$), then $\liminf a_n = \lim a_n = \limsup a_n$.
(2) If $\liminf a_n = \limsup a_n$, then $\lim a_n$ is defined and $\liminf a_n = \lim a_n = \limsup a_n$.

Proof. See Ross (Theorem 10.7). The proof is straightforward. □

More generally, we have the following result.

Theorem 13.11. *Let (a_n) be a sequence. There exists a monotonic subsequence whose limit is $\limsup a_n$, and there exists a monotonic subsequence whose limit is $\liminf a_n$.*

Proof. If (a_n) is unbounded from above or below, we may simply use Proposition 13.8 (parts 2 and 3). It remains to consider when (a_n) is bounded from above or below. We will prove the case when (a_n) is bounded from above; the other case is similar (you should check it).

Suppose that (a_n) is bounded from above. Then $t = \limsup a_n \in \mathbb{R}$. Let $\varepsilon > 0$. By the definition of $\limsup$, there exists an $N > 0$ such that

$$|\sup\{a_n \colon n > N\} - t| < \varepsilon.$$

But since $\sup\{a_n \colon n > N\} \geq t$, we refine this say that

$$\sup\{a_n \colon n > N\} < t + \varepsilon.$$

Thus $a_n < t + \varepsilon$ for all $n > N$ (since $a_n \leq \sup\{a_n \colon n > N\}$).

Claim: The set $A = \{n \in \mathbb{N} \colon |a_n - t| < \varepsilon\}$ is infinite, so the theorem follows from Proposition 13.8.

Proof of Claim: Suppose to the contrary that the set is finite; that is, there exists $N' \geq N$ such that if $n \geq N'$, then $a_n \notin A$. Since $a_n < t+\varepsilon$ for all $n \geq N$, then for $n \geq N'$, we find that $a_n \leq t - \varepsilon$ (otherwise $a_n \in A$ with $n \geq N'$, a contradiction). But then $\limsup a_n \leq t - \varepsilon < t$, a contradiction. Thus A is infinite. □

13.1. **Alternate proof of Bolzano-Weierstrass.** This proof will use nested intervals, similar to the presentation by Or on April 29 in that it will use nested intervals. First, we introduce some notation.

Notation 13.12. Let X and Y be sets. We write $X \subseteq Y$ if $x \in X$ implies that $x \in Y$. One may equivalently write $Y \supseteq X$.

Suppose (a_n) is a bounded sequence of reals. Then there exist constants $s, S \in \mathbb{R}$ such that $s \leq a_n \leq S$ for all $n \in \mathbb{N}$. If $s = S$, then (a_n) is a constant sequence, and the desired result follows immediately. So we may suppose that $s < S$.

Let $I_1 = [s, S]$. At least one of the intervals $[s, \frac{s+S}{2}]$ and $[\frac{s+S}{2}, S]$ has infinitely many terms (a_n); pick one such interval and call if $I_2 \subseteq I_1$. Notice that I_2. Similarly split I_2 in half. One of the halves has infinitely many terms in the sequence; call it I_3. Notice that $I_3 \subseteq I_2$. Proceed inductively constructing intervals $I_1, I_2, I_3, \ldots, I_k, \ldots$ such that

(1) $I_{k+1} \subseteq I_k$ for all $k \in \mathbb{N}$, and
(2) I_k contains infinitely many terms in (a_n).

Let $b_k = \inf I_k$ and $B_k = \sup I_k$. Note that our halving process implies that (b_k) is monotonically increasing, (B_k) is monotonically decreasing, $b_i \leq B_j$ for all $i, j \in \mathbb{N}$, and

$$|B_k - b_k| \leq \frac{S-s}{2^{k-1}}$$

for all $k \in \mathbb{N}$. Proceeding in a manner similar to our proof of the least upper bound property, we see that (b_k) and (B_k) converge to the same real limit L.

Since (b_k) monotonically increases to L and (B_k) monotonically decreases to L, it follows that $b_k \leq L \leq B_k$, hence $L \in I_k$, for every $k \in \mathbb{N}$. Thus for each $k \in \mathbb{N}$, our halving process and the fact that infinitely many terms in (a_n) lie in each I_k implies that

$$\left\{n \in \mathbb{N} \colon a_n \in I_k,\ |a_n - L| < \frac{S-s}{2^{k-1}}\right\}$$

is infinite for each $k \geq 3$. Since $k \leq 2^{k-1}$ for $k \geq 3$ (could prove by induction), we have that

$$\left\{n \in \mathbb{N} \colon a_n \in I_k,\ |a_n - L| < \frac{S-s}{k}\right\}$$

is infinite. Now, choose $\varepsilon > 0$. If we take $k \in \mathbb{N}$ so that $k > \max\{3, (S-s)/\varepsilon\}$, then

$$\{n \in \mathbb{N} \colon |a_n - L| < \varepsilon\}$$

is infinite. Thus by Proposition 13.8, there is a subsequence of (a_n) converging to L.

14. Series

We now transition to a discussion of series. We begin with some notation:

Notation 14.1. We write $a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k$

Definition 14.2. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence. An **infinite series** is an object of the form

$$\sum_{k=1}^{\infty} a_k. \tag{14.1}$$

To give meaning to (14.1), consider the sequence $(s_n)_{n\in\mathbb{N}}$ given by

$$s_n = \sum_{k=1}^{n} a_k.$$

We call s_n the **n-th partial sum**. We say that (14.1) **converges** if $\lim s_n$ exists and is a real number. Otherwise, we say that (14.1) **diverges** Sometimes, we can be more precise about how (14.1) diverges by saying that (14.1) **diverges to ∞** if $\lim s_n = \infty$ and **diverges to $-\infty$** if $\lim s_n = -\infty$. If $\sum_{k=1}^{\infty} |a_k|$ converges, then we say that $\sum_{k=1}^{\infty} a_k$ **converges absolutely**.

While we have written our series as starting with $k = 1$, we can modify our definitions so that we start at $k = m$ for any m, in which case $s_n = a_m + a_{m+1} + \cdots + a_n$. Often, it will not matter where the index begins, so we write $\sum a_k$ for short.

Because $\mathbb{R}$ is complete, (s_n) converges if and only if (s_n) is Cauchy. By applying the definition of what it means for (s_n) to be a Cauchy sequence, we obtain the following result.

Proposition 14.3 (Cauchy criterion)**.** *A series $\sum a_n$ converges if and only if for each $\varepsilon > 0$ there exists $N > 0$ such that*

$$\Big|\sum_{k=m}^{n} a_k\Big| < \varepsilon \qquad \text{whenever } n \geq m \geq N.$$

Corollary 14.4 (Divergence test)**.** *If $\sum a_n$ converges, then $\lim a_n = 0$.*

Proof. Applying the Cauchy criterion with $m = n$, we find that for all $\varepsilon > 0$, there exists $N > 0$ such that $|a_n| < \varepsilon$ whenever $n \geq N$. □

We can do arithmetic for series using what we have already proved for limits. For example, if $\sum a_n$ and $\sum b_n$ both converge, then the corresponding sequences of partial sums (s_n) and (s'_n) converge. Thus $\lim s_n + \lim s'_n = \lim(s_n + s'_n)$; in other words,

$$\sum a_n + \sum b_n = \sum (a_n + b_n).$$

Similarly, if $c \in \mathbb{R}$, then

$$c \sum a_n = \sum c \cdot a_n.$$

Given $r \in \mathbb{R}$, a **geometric series** is a series of the form

$$\sum r^n.$$

Proposition 14.5. *If $|r| < 1$, then the geometric series $\sum_{k=0}^{\infty} r^k$ converges absolutely to $1/(1-r)$. If $|r| \geq 1$, then $\sum r^k$ diverges.*

Proof. If $r = 1$, then the series clearly diverges. Otherwise, we prove that

$$s_n = \sum_{k=0}^{n} r^k = \frac{1 - r^n}{1 - r}.$$

By the limit laws in Propositions 10.5 and 12.4, the series converges if and only if $\lim r^n$ is finite, which is true if and only if $|r| < 1$. In that case, $\lim r^n = 0$, proving the claimed limit. Finally, if $|r| < 1$, then $\sum |r^n| = \sum |r|^n$ converges, so $\sum r^n$ converges absolutely. □

Proposition 14.6 (Comparison test)**.** *Let (a_n) be a sequence, and let $a_n \geq 0$ for all n.*

(1) If $\sum a_n$ converges and $|b_n| \leq a_n$ for all n, then $\sum b_n$ converges.
(2) If $\sum a_n = \infty$ and $b_n \geq a_n$ for all n, then $\sum b_n$ diverges.

Proof. (1) Suppose $\sum a_n$ converges. By the Cauchy criterion and the nonnegativity of a_n, for all $\varepsilon > 0$ there exists $N > 0$ such that

$$\sum_{k=m}^{n} a_k = \Big| \sum_{k=m}^{n} a_k \Big| < \varepsilon$$

for all $n \geq m \geq N$. Thus for $n \geq m \geq N$, the triangle inequality gives

$$\Big| \sum_{k=m}^{n} b_k \Big| \leq \sum_{k=m}^{n} |b_k| \leq \sum_{k=m}^{n} a_k < \varepsilon.$$

Thus $\sum b_n$ satisfies the Cauchy criterion and converges.

(2) Let (s_n) (resp. (t_n)) be the sequence of partial sums for $\sum a_n$ (resp. $\sum b_n$). Then $t_n \geq s_n$ for all n. Since $\lim s_n = \infty$, we have $\lim t_n = \infty$ too.

□

Since $a_n \leq |a_n|$, the next corollary is immediate from the comparison test.

Corollary 14.7. *Absolutely convergent series converge.*

Let us consider another class of series that arises frequently.

Definition 14.8. We call series of the form $\sum_{n=1}^{\infty} 1/n^p$ a p-series. (Technically, we can only take p to be rational at this time because we can raise to the m-th power and take n-th roots with m an integer and $n \geq 1$ an integer, but once we can take p to be real, the proofs will be exactly the same.)

Lemma 14.9. *If $p \leq 1$, then $\sum_{n=1}^{\infty} 1/n^p$ diverges.*

Proof. For $p = 1$, we proved in Proposition 6.4 that the sequence of partial sums is not Cauchy; it follows from the proof that the sequence of partial sums is not bounded from above. Thus the series diverges to infinity. Since $1/n < 1/n^p$ for all $p < 1$, the same conclusion follows from part 2 of the Comparison Test. □

15. Series and Tests

Proposition 15.1. *If $p > 1$, then $\sum_{n=1}^{\infty} 1/n^p$ converges.*

Proof. Let $p > 1$, and let $s_n = s_n(p)$ be the n-th partial sum. Clearly, the sequence (s_n) is clearly monotonically increasing. We will prove that it is also bounded. Thus the series converges by Theorem 12.9.

The sequence of partial sums is

$$s_n = s_n(p) = \sum_{k=1}^{n} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p}.$$

We can write s_{2n} as follows:

$$s_{2n} = 1+\Big(\frac{1}{2^p}+\frac{1}{4^p}+\cdots+\frac{1}{(2n)^p}\Big)+\Big(\frac{1}{3^p}+\frac{1}{5^p}+\cdots+\frac{1}{(2n-1)^p}\Big) = 1+\sum_{k=1}^{n}\frac{1}{(2k)^p}+\sum_{k=2}^{n}\frac{1}{(2k-1)^p}.$$

Since $p > 1$, we observe that $k \geq 1$, $(2k+1)^p > (2k)^p$. Thus

$$s_n < s_{2n} < 1 + 2\sum_{k=1}^{n}\frac{1}{(2k)^p} = 1 + 2\sum_{k=1}^{n}\frac{1}{2^p k^p} = 1 + \frac{2}{2^p}\sum_{k=1}^{n}\frac{1}{k^p} = 1 + \frac{2}{2^p}s_n.$$

Solving this inequality for s_n yields

$$s_n < \Big(1 - \frac{2}{2^p}\Big)^{-1}.$$

Since the right hand side of this last inequality is independent of n, we see that (s_n) is bounded, as desired. ☐

The final convergence test that we will discuss in the alternating series test. There are many other convergence tests, but we do not have time to cover all of them. (If infinite series intrigue you, I highly recommend you take a course in complex analysis or analytic number theory).

Proposition 15.2 (Alternating series test)**.** *Let (a_n) be a monotonically decreasing sequence of numbers with $a_n \geq 0$ for all n. If $\lim a_n = 0$, then the alternating series $\sum(-1)^{n+1}a_n$ converges. Moreover, if $\sum(-1)^{n+1}a_n$ converges to $L \in \mathbb{R}$ and s_n is the n-th partial sum, then $|L - s_n| \leq a_{n+1}$ for all n.*

Proof. Let (a_n) be a nonnegative sequence monotonically converging to zero, and let (s_n) be the sequence of partial sums of $((-1)^{n+1}a_n)$. Observe that

$$s_{2n} = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2n-1} - a_{2n}).$$

Because (a_n) monotonically decreases, $a_{2k-1} - a_{2k} \geq 0$ for all $k \in \mathbb{N}$. Thus

$$s_{2(n+1)} - s_{2n} = a_{2n+1} - a_{2n+2} \geq 0,$$

so (s_{2n}) monotonically increases. Using again the fact that $a_{2k-1} - a_{2k} \geq 0$ for all $k \in \mathbb{N}$, we can also write

$$s_{2n} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2n-2} - a_{2n-1}) - a_{2n} \leq a_1 - a_{2n} \leq a_1.$$

Thus (s_{2n}) is also bounded from above, and hence it converges to a real number L by Theorem 12.9. Hence for all $\varepsilon > 0$, there exists an integer $N > 0$ such if $2n \geq N$, then

$$|s_{2n} - L| < \varepsilon/2 \qquad \text{and} \qquad |a_{2n+1}| < \varepsilon/2.$$

(We have that $|a_{2n+1}| < \varepsilon$ for large enough n since $\lim a_{2n+1} = \lim a_n = 0$.) Now, if $2n \geq N$, then $2n+1 \geq N$ as well, and we have

$$|s_{2n+1} - L| = |(-1)^{2n+1} a_{2n+1} + (s_{2n} - L)| \leq |a_{2n+1}| + |s_{2n} - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, regardless of whether n is even or odd, if $n \geq N$, then $|s_n - L| < \varepsilon$. Thus (s_n) converges, hence $\sum (-1)^{n+1} a_n$ converges.

For the second part, we observe that if $n \in \mathbb{N}$, then

$$\begin{aligned} |s_{2n+1} - L| &= s_{2n+1} - L \leq s_{2n+1} - s_{2n+2} = a_{2n+2} \leq a_{(2n+1)+1}, \\ |s_{2n} - L| &= L - s_{2n} \leq s_{2n+1} - s_{2n} = a_{2n+1}. \end{aligned}$$

□

Example 15.3. Even though we proved that $\sum 1/n$ diverges to infinity, the series $\sum (-1)^{n+1}/n$ converges since $1/n$ monotonically decreases and $\lim 1/n = 0$.

Definition 15.4. A convergent series that does not converge absolutely is said to **converge conditionally**.

Example 15.5. The series $\sum (-1)^{n+1}/n$ converges conditionally because $|(-1)^{n+1}/n| = 1/n$ and $\sum 1/n$ diverges.

Here is an example of why conditionally convergent series are hard to work with. We just showed that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots$$

converges conditionally. But notice that since every $n \geq 1$ is either odd, 2 times an odd, or 4 times an odd, we could consider the sum

$$\Big(1 - \frac{1}{2} - \frac{1}{4}\Big) + \Big(\frac{1}{3} - \frac{1}{6} - \frac{1}{8}\Big) + \Big(\frac{1}{5} - \frac{1}{10} - \frac{1}{12}\Big) + \Big(\frac{1}{7} - \frac{1}{14} - \frac{1}{16}\Big) + \cdots$$

Every term in $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ appears exactly once (with the appropriate sign). But this re-arranged sum equals

$$\sum_{k=1}^{\infty} \Big(\frac{1}{2k-1} - \frac{1}{2(2k-1)} - \frac{1}{4k}\Big) = \sum_{k=1}^{\infty} \Big(\frac{1}{2(2k-1)} - \frac{1}{4k}\Big) = \frac{1}{2} \sum_{k=1}^{\infty} \Big(\frac{1}{2k-1} - \frac{1}{2k}\Big),$$

which equals

$$\frac{1}{2}\Big(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots\Big) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}.$$

This phenomenon where you can re-arrange the series in order to obtain a different value actually defines what it means to be conditionally convergent; this phenomenon cannot happen when a series converges absolutely.

Midterm Review

Topics.

(1) $\mathbb{N}$ and $\mathbb{Z}$ (Induction, basic properties)
(2) Definition of an ordered set
(3) Equivalence relations
 (a) Definition of a binary relation
 (b) Definition of equivalence relation
 (c) Definition of equivalence class
 (d) Proof that equivalence classes partition a set
 (e) Be able to identify whether a given binary relation on a given set is an equivalence relation, determine the equivalence classes, use the partition to prove properties of the set
(4) $\mathbb{Q}$
 (a) Equivalence class definition (including definitions of addition, multiplication, negation, quotient)
 (b) Be able to prove properties of $\mathbb{Q}$ using properties of $\mathbb{Z}$
 (c) Describe the manner in which we embed $\mathbb{Z}$ in to $\mathbb{Q}$ (think $a//1$)
 (d) Determine whether numbers like $\sqrt{3}$ or $2^{1/3}$ lie in $\mathbb{Q}$
(5) $\mathbb{R}$
 (a) Definition of a Cauchy sequence of rationals
 (b) Definition of what it means for $(a_n) \sim (b_n)$.
 (c) If (a_n) and (b_n), what can you say about $(a_n + b_n)$, $(-a_n)$, $(a_n b_n)$?
 (d) Definition of a Cauchy sequence bounded away from zero; proof that if $(a_n) \not\sim 0$, then there exists a sequence $(b_n) \sim (a_n)$ which is bounded away from zero and which satisfies the criterion that each b_n has the same (nonzero) sign; prove criterion for which (a_n) Cauchy implies (a_n^{-1}) Cauchy.
 (e) Equivalence class definition of $\mathbb{R}$ (including definitions of addition, multiplication, negation, quotient)
 (f) Be able to prove properties of $\mathbb{R}$ using properties of $\mathbb{Q}$
 (g) Definition of limit and convergence; prove that convergent sequences are Cauchy
 (h) Proof that $\mathbb{Q}$ is dense in $\mathbb{R}$; proof of archimedean property for $\mathbb{R}$.
 (i) (*) Proof that if (a_n) is a Cauchy sequence of rationals and $x = [(a_n)] \in \mathbb{R}$, then $x = \lim a_n$.
 (j) Use (*) to prove that $\mathbb{R}$ is complete (all Cauchy sequences converge)
 (k) Definition of $\sup E$ and $\inf E$
 (l) Full statement of the least upper bound property (or the greatest lower bound property); be able to apply it in various settings
(6) Limit properties
 (a) If (a_n) and (b_n) are convergent, what can be said about $\lim(a_n + b_n)$, $\lim(a_n/b_n)$, $\lim(a_n b_n)$, $\lim c \cdot a_n$? How do these relate to the results proven about Cauchy sequences?
 (b) Be able to prove that an explicitly given sequence tends to a given limit (and find N in terms of ε)
 (c) Definitions of what it means for $\lim a_n = \infty$ or $\lim a_n = -\infty$.
(7) Sequences
 (a) Definition of monotonically increasing/decreasing sequences

   (b) Proof that if (a_n) is bounded, then (a_n) converges if and only if (a_n) is bounded
   (c) Definition of subsequence
   (d) Statement of Bolzano-Weierstrass theorem and its proof
   (e) Definition of lim sup and lim inf
(8) Series
   (a) Definition of a series
   (b) Partial sums approach to series; define what it means for a series to converge in terms of partial sums
   (c) Fully state, prove, and apply the Cauchy criterion
   (d) Definition of absolute convergence
   (e) Fully state and prove the comparison test
   (f) Fully state and prove the alternating series test

Milestones.

(1) Equivalence class definitions of rationals and reals
(2) Sequences of reals converge if and only if they are Cauchy (completeness of $\mathbb{R}$)
(3) Least upper bound property of $\mathbb{R}$
(4) Bolzano–Weierstrass
(5) Cauchy criterion for convergence of infinite series

Exam Logistics.

(1) Wednesday, May 8, IN CLASS (50 minutes)
(2) No notes, no books, no collaboration, no collusion, no internet, etc.
(3) Exam content: Everything from notes (and corresponding sections in Ross), graded HW, and additional HW up through and including Lecture 15

16. Continuity

We now put our study of sequences to use in order to study functions; this is typically what people think of when the word "calculus" is used. First, what exactly do we mean when we say "function"? Recall that for non-empty sets X and Y, $X \times Y$ is the set of ordered pairs (x, y) with $x \in X$ and $y \in Y$.

Definition 16.1. Let X and Y be non-empty sets. A **function** is a subset G of $X \times Y$ with the property that for every $x \in X$, there is exactly one $y \in Y$ such that $(x, y) \in G$. We think of this as the existence of a rule f for assigning $x \in X$ to $y \in Y$; we write this as $f : X \to Y$ with the property that if $x = y$, then $f(x) = f(y)$ (this is called **well-definedness**). We call X the **domain** of f (sometimes written $\mathrm{dom}(f)$) and Y the **codomain** of f.

Example 16.2. Let $X = [-1, 1]$ and $Y = \mathbb{R}$. Consider the function $f : X \to Y$ given by $f(x) = \sqrt{1 - x^2}$. But $f(x)$ only returns values in the set $[0, 1]$. Thus we could have been more efficient and written $f : X \to [0, 1]$. Sometimes it is important for us to be as exact as possible when stating the codomain; other times, such precision is not completely necessary.

Our attention is centered on functions $f : X \to \mathbb{R}$, where X is a subset of $\mathbb{R}$. We call such functions **real-valued functions**.

Notation 16.3. Let U and V be sets. We write $U \subseteq V$ ("U is a subset of V") to mean that each $x \in U$ also satisfies $x \in V$. We write $U \cup V = \{x \colon x \in U \text{ or } x \in V\}$ and $U \cap V = \{x \colon x \in U \text{ and } x \in V\}$.

Definition 16.4 (Sequential definition of continuity)**.** Let $X \subseteq \mathbb{R}$, let $f : X \to \mathbb{R}$ be a function, and let $a \in X$. We say that f is **continuous at** a if, for every sequence (a_n) such that (1) $a_n \in X$ for all n, and (2) $\lim a_n = a$, we have $\lim f(a_n) = f(\lim a_n) = f(a)$.

If you are not familiar with this definition, then maybe you are with:

Definition 16.5 ($\varepsilon - \delta$ definition of continuity)**.** Let $X \subseteq \mathbb{R}$, let $f : X \to \mathbb{R}$ be a function, and let $a \in X$. We say that f is **continuous at** a if, for all $\varepsilon > 0$, there exists $\delta > 0$ (depending on ε) such that $|f(x) - f(a)| < \varepsilon$ whenever $x \in X$ and $|x - a| < \delta$.

It would be bad if these two definitions were not equivalent (the notion of continuity would not be well-defined!). Let us check that the two definitions are indeed equivalent.

Theorem 16.6. *Definition 16.4 and Definition 16.5 are equivalent.*

Proof. First, suppose that $f : X \to \mathbb{R}$ satisfies Definition 16.5 at the point $a \in X$. Let $\varepsilon > 0$ be given. Then there exists a $\delta > 0$ such that

$$|f(x) - f(a)| < \varepsilon \qquad \text{whenever } x \in X \text{ and } |x - a| < \delta.$$

Let (a_n) be a sequence such $a_n \in X$ for all n and $\lim a_n = a$. By the definition of the limit, there exists an integer $N > 0$ such that if $n \geq N$, then $|a_n - a| < \delta$. Thus for $n \geq N$, we have $|f(a_n) - f(a)| < \varepsilon$. Thus $(f(a_n))$ converges to $(f(a))$, as required by Definition 16.4.

Second, suppose that $f : X \to \mathbb{R}$ does not satisfy Definition 16.5 at the point $a \in X$. Then there exists an $\varepsilon > 0$ such that for every $\delta > 0$, there exists a point $x \in X$ such that

$$|x - a| < \delta \qquad \text{but } |f(x) - f(a)| \geq \varepsilon. \tag{16.1}$$

Since this holds for *every* $\delta > 0$, it holds for $\delta = 1/n$ for every $n \in \mathbb{N}$. Thus for each $n \in \mathbb{N}$, there exists a point $a_n \in X$ such that

$$|a_n - a| < 1/n \qquad \text{but } |f(a_n) - f(a)| \geq \varepsilon.$$

The sequence (a_n) clearly converges to a, but we have established that $(f(a_n))$ does not converge to $f(a)$. Thus $f : X \to \mathbb{R}$ does not satisfy Definition 16.5 at a. □

Definition 16.7. Let $A \subseteq \mathbb{R}$. We say that $f : X \to \mathbb{R}$ is **continuous on** A if f is continuous at every point $a \in A$. We say f is **continuous** if f is continuous on X.

Example 16.8 (Sequential continuity)**.** Let $f(x) = \sqrt{x}$. Let us prove that f is continuous on its domain $X = [0, \infty)$ using the definition of continuity in Definition 16.4. In this example, we will use the fact that if $\lim a_n = a$ and $a_n \geq 0$ for all n, then $\lim \sqrt{a_n} = \sqrt{a}$. (Exercise! Hint: Show that this is the same as proving $\lim(\sqrt{a_n} - \sqrt{a}) = 0$. Then "multiply by one".)

Let $a \in X$, and let (a_n) be a sequence such that $a_n \in X$ for all n (so $a_n \geq 0$ for all n) and $\lim a_n = a$. Then by the aforementioned exercise,

$$\lim f(a_n) = \lim \sqrt{a_n} = \sqrt{a} = f(a).$$

So f is continuous on its domain $X = [0, \infty)$.

Example 16.9 ($\varepsilon - \delta$ continuity)**.** Let us prove again that $f(x) = \sqrt{x}$ is continuous. This time, we will use Definition 16.5. Let $a \in X$, and let $\varepsilon > 0$. Let

$$\delta = \begin{cases} \varepsilon\sqrt{a} & \text{if } a > 0, \\ \varepsilon^2 & \text{if } a = 0. \end{cases}$$

Let $x \in X$ be arbitrary, and suppose that $|x - a| < \delta$. If $a = 0$, then $|f(x) - f(a)| = \sqrt{x} < \sqrt{\delta} = \varepsilon$, and we are done. If $a > 0$, then

$$|f(x) - f(a)| = |\sqrt{x} - \sqrt{a}| = \frac{|(\sqrt{x} - \sqrt{a})(\sqrt{x} + \sqrt{a})|}{|\sqrt{x} + \sqrt{a}|} = \frac{|x - a|}{\sqrt{x} + \sqrt{a}} < \frac{\varepsilon\sqrt{a}}{\sqrt{x} + \sqrt{a}} < \varepsilon.$$

One of the easiest way to determine if a function is continuous is to build it up from simpler functions which you already know are continuous and then use the properties you have proven about limits. Here are some familiar operations (f and g are functions and $k \in \mathbb{R}$):

$$\begin{aligned} f + g : &\quad (f + g)(x) = f(x) + g(x) \text{ (when } x \in \operatorname{dom}(f) \cap \operatorname{dom}(g)) \\ fg : &\quad (fg)(x) = f(x)g(x) \text{ (when } x \in \operatorname{dom}(f) \cap \operatorname{dom}(g)) \\ f/g : &\quad (f/g)(x) = f(x)/g(x) \text{ (when } x \in \operatorname{dom}(f) \cap \operatorname{dom}(g) \text{ and } g(x) \neq 0) \\ kf : &\quad (kf)(x) = k \cdot f(x) \\ |f| : &\quad (|f|)(x) = |f(x)| \\ f \circ g : &\quad (f \circ g)(x) = f(g(x)) \text{ (when } x \in \operatorname{dom}(g) \text{ and } g(x) \in \operatorname{dom}(f)) \end{aligned}$$

We can use the properties of limits of sequences covered so far to show that if f and g are continuous, then so are the above.

Proposition 16.10. *Let $a \in \mathbb{R}$, let g be continuous at a, and let f be continuous at $g(a)$. Then $f \circ g$ is continuous at a.*

Proof. Let (a_n) be sequence in the domain of g such that $\lim a_n = a$ and $g(a_n)$ is in the domain of f for all n. Then, since g is continuous at a, $\lim g(a_n) = g(a)$. Thus $(g(a_n))$ is a sequence in the domain of f such that $\lim g(a_n) = g(a)$. Since f is continuous at $g(a)$, it follows that $\lim f(g(a_n)) = f(g(a))$, as desired. □

Proposition 16.11. *Suppose f and g are continuous at $a \in \mathbb{R}$, and let $k \in \mathbb{R}$ be constant. Then $f + g$, fg, kf, and f/g (provided $g(a) \neq 0$) are continuous at a.*

Proof. These follow from the definition of continuity and the limit laws. For instance, let (a_n) be a sequence such that $\lim a_n = a$. Since f and g are continuous at a, we have that $\lim f(a_n) = f(a)$ and $\lim g(a_n) = g(a)$. Thus

$$\lim(f+g)(a_n) = \lim(f(a_n) + g(a_n)) = \lim f(a_n) + \lim g(a_n) = f(a) + g(a).$$

The others are similar. □

Corollary 16.12. *Let $k \geq 0$ be an integer, and let $a_0, a_1, \ldots, a_k \in \mathbb{R}$. Then $a_0 + a_1 x + \cdots + a_k x^k$ is continuous on $\mathbb{R}$. We call such functions* **polynomials**.

Proof. Exercise. (Hint: Proceed by induction on n and use the previous proposition.) □

Corollary 16.13. *If f is continuous at $a \in \mathbb{R}$, then so is $|f|$.*

Proof. Exercise. □

17. Properties of continuous functions

We will address the two most important properties of continuous functions. The importance of these cannot be overstated. First, a definition:

Definition 17.1. We say that a real-valued function f is **bounded** if there exists a real number M such that $|f(x)| \leq M$ for all $x \in \mathrm{dom}(f)$.

The first result is used (implicitly) in every optimization problem in calculus. We will call upon the Bolzano-Weierstrass theorem Theorem 13.6, which, as you may recall, relied decisively on the least upper bound property!

Theorem 17.2 (Extreme value theorem)**.** *Let $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a continuous function.*

(1) f is a bounded function.
(2) f assumes its maximum and minimum values on $[a, b]$; in other words, there exist $x_0, y_0 \in [a, b]$ such that $f(x_0) \leq f(x) \leq f(y_0)$ for all $x \in [a, b]$.

It is typical in math to say that if $\sup E \in E$ (resp. $\inf E \in E$), we say that E has a maximum (resp. minimum). If $\sup E \notin E$ (resp. $\inf E \notin E$), then using the word "maximum" (resp. "minimum") no longer makes sense.

Proof. (1) Suppose to the contrary that f is not bounded $[a, b]$. Then to each $n \in \mathbb{N}$ there corresponds an $x_n \in [a, b]$ such that $|f(x_n)| > n$. Thus $\lim_{n\to\infty} |f(x_n)| = \infty$.

By the Bolzano-Weierstrass theorem, (x_n) has a subsequence (x_{n_k}) which converges to some number x_0. Note that since each $x_n \in [a, b]$, each $x_{n_k} \in [a, b]$ as well; thus by HW4, it follows that $x_0 = \lim_{k\to\infty} x_{n_k} \in [a, b]$. Since f is continuous on $[a, b]$, it follows that $\lim_{k\to\infty} f(x_{n_k}) = f(x_0)$. Because f is continuous at x_0, it follows that $\lim_{k\to\infty} |f(x_{n_k})| = |f(x_0)|$.

But since (x_{n_k}) is also a subsequence of (x_n) and $\lim_{n\to\infty} |f(x_n)| = \infty$, we must also have $\lim_{k\to\infty} |f(x_{n_k})| = \infty$, a contradiction. Thus f is bounded on $[a, b]$, as desired.

(2) Since f is bounded on $[a, b]$, we may define $M := \sup\{f(x) : x \in [a, b]\}$ by the least upper bound property. For each $n \in \mathbb{N}$, there exists $y_n \in [a, b]$ such that $M - \frac{1}{n} < f(y_n) \leq M$ (otherwise, M would not be the supremum). Hence $\lim f(y_n) = M$ by the "squeeze theorem" from HW4. By Bolzano-Weierstrass, there is a convergent subsequence (y_{n_k}) converging to a limit $y_0 \in [a, b]$.

Let $y_0 = \lim_{k\to\infty} y_{n_k}$. Since f is continuous at y_0, we have $f(y_0) = \lim_{k\to\infty} f(y_{n_k})$. Since $(f(y_{n_k}))_{k\in\mathbb{N}}$ is a subsequence of the convergent sequence $(f(y_n))_{n\in\mathbb{N}}$, it follows that $\lim_{k\to\infty} f(y_{n_k}) = \lim_{n\to\infty} f(y_n) = M$ by Proposition 13.4. Thus $f(y_0) = M$ by the definition of continuity. Thus f assumes its maximum at y_0.

One can play the same game for $-f$, and we deduce that $-f$ achieves its maximum at some $x_0 \in [a, b]$, and thus f achieves its minimum at x_0.

□

Note that Theorem 17.2 is false if $[a, b]$ is replaced by (a, b). For instance, $f(x) = 1/x^2$ is continuous but unbounded on $(0, 1)$. The function x^4 is continuous on $(-1, 1)$, but it does not achieve a maximum value on $(-1, 1)$.

Theorem 17.3 (Intermediate value theorem)**.** *Let $I \subseteq \mathbb{R}$ be an interval, and suppose that $f : I \to \mathbb{R}$ is continuous. Let $a, b \in I$ with $a < b$. For every y in between $f(a)$ and $f(b)$ (so either $f(a) < y < f(b)$ or $f(b) < y < f(a)$, depending on the signs of $f(a)$ and $f(b)$), there exists at least one $x \in (a, b)$ such that $f(x) = y$.*

Proof. Without loss of generality, we may assume that $f(a) < y < f(b)$ (the other case is similar). Let $S = \{x \in [a, b] \colon f(x) < y\}$. Then S is nonempty because $a \in S$, and S is bounded above by b. Thus $x_0 := \sup S$ exists as a real number.

For each $n \in \mathbb{N}$, the number $x_0 - 1/n$ is not an upper bound for S, so there exists $b_n \in S$ such that $x_0 - 1/n < b_n \leq x_0$. By the squeeze theorem, $\lim b_n = x_0$. Since f is continuous and since $f(b_n) < y$, we have by HW4

$$f(x_0) = f(\lim b_n) = \lim f(b_n) \leq y.$$

On the other hand, for each $n \in \mathbb{N}$, we can let $a_n = \min\{b, x_0 + 1/n\}$. Then for each n, $a_n \notin S$. Thus $f(a_n) \geq y$. By the squeeze theorem, $\lim a_n = x_0$. Thus by HW4,

$$f(x_0) = f(\lim a_n) = \lim f(a_n) \geq y.$$

By the trichotomy, it must be true that $f(x_0) = y$. □

Here is a neat corollary of the intermediate value theorem; it is an example of a *fixed point theorem*, and it's quite surprising at first glance that it should even be true (considering how weak the hypotheses are.)

Corollary 17.4. *If $f : [0, 1] \to [0, 1]$ is continuous, then f has a fixed point. That is, there exists a point $x_0 \in [0, 1]$ such that $f(x_0) = x_0$.*

Proof. Let $g(x) = f(x) - x$. Then g is also continuous on $[0, 1]$. Since $0 \leq f(x) \leq 1$, we have that $-x \leq g(x) \leq 1 - x$. Thus $g(0) \geq 0$ and $g(1) \leq 0$. It follows from the intermediate value theorem that there is a point $x_0 \in [0, 1]$ such that $g(x_0) = 0$. In other words, $f(x_0) - x_0 = 0$. □

18. Uniform continuity

In order to motivate our next topic, we will rely on standard (though currently unproven by us) properties of the sine function $\sin(x)$ ($|\sin(x)| \leq 1$, $\sin(\pi n) = 0$ for all $n \in \mathbb{Z}$, continuous on $\mathbb{R}$). Let us consider two examples.

Example 18.1. Let $f(x) = x\sin(1/x)$ if $x \neq 0$ and $f(0) = 0$. Let us show that f is continuous at each point $x_0 \in \mathbb{R}$. If $x_0 \neq 0$, then $1/x$ is continuous at $x = x_0$, so $\sin(1/x)$ is continuous at x_0 (compositions of continuous functions are continuous), hence $x\sin(1/x)$ is continuous at x_0 (products of continuous functions are continuous). If $x_0 = 0$, then given $\varepsilon > 0$, let $\delta = \varepsilon$. The for all $x \in \mathbb{R} - \{0\}$ such that $|x - 0| < \delta$, we have

$$|f(x) - f(0)| = |x\sin(1/x)| \leq |x| < \delta = \varepsilon.$$

If $x = 0$, then $|f(x) - f(0)| = 0 < \varepsilon$.

Example 18.2. Let $g(x) = \sin(1/x)$ if $x \neq 0$ and $g(0) = 0$. By the above argument, we know that $g(x)$ is continuous on $\mathbb{R} - \{0\}$, but $g(x)$ is discontinuous at $x = 0$. Let

$$a_n = 2/(\pi(4n-3)), \qquad b_n = 1/(\pi n).$$

Note that $\lim a_n = \lim b_n = 0$. But $g(a_n) = \sin(\frac{\pi(4n-3)}{2}) = \sin(\frac{\pi}{2}) = 1$ for all $n \geq 1$ while $g(b_n) = \sin(\pi n) = 0$ for all $n \geq 1$. Thus g fails to be continuous at $x = 0$.

Both f and g are continuous on $(0, 1]$ since they are built out of functions which are continuous on that interval. But we showed that f is continuous at 0, but g is not. Also, since $(g(a_n))$ and $(g(b_n))$ converged to different values, there is no way to define $g(0)$ so that g is continuous at 0. So g is not continuous on $[0, 1]$. In many (applied) problems, this sort of distinction in endpoint behavior is important to understand. What is the key difference between these examples? The answer lies in a property called *uniform continuity*.

Definition 18.3. A function $f : X \to \mathbb{R}$ is **uniformly continuous** on X if for all $\varepsilon > 0$ there exists a $\delta > 0$, depending only on ε, such that

$$|f(x) - f(y)| < \varepsilon \qquad \text{whenever} \quad x, y \in X \text{ and } |x - y| < \delta.$$

In Definition 16.5, the value of δ could depend on a; we didn't care. Now, we care! Could we choose a value of δ that doesn't depend on a for all a in the domain? If the answer is yes, then f is uniformly continuous on its domain.

Example 18.4. Let $f : [-4, 2] \to \mathbb{R}$ be given by $f(x) = x^2$. Given $\varepsilon > 0$, let $\delta = \varepsilon/8$. Note that if $x, y \in [-4, 2]$ and $|x - y| < \delta$, then

$$|f(x) - f(y)| = |x^2 - y^2| = |x - y| \cdot |x + y| \leq |x - y| \cdot (|x| + |y|) \leq 8|x - y| < 8\delta = \varepsilon.$$

Since the ε does **not** depend on x and y, f is uniformly continuous on $[-4, 2]$.

Proposition 18.5. *If f is continuous on a closed interval $[a, b]$, then f is uniformly continuous on $[a, b]$.*

Proof. Suppose to the contrary that f is continuous on $[a, b]$ but not uniformly continuous on $[a, b]$. There exists $\varepsilon > 0$ such that for every $\delta > 0$, we can find $x_\delta, y_\delta \in [a, b]$ such that

$$|x_\delta - y_\delta| < \delta \qquad \text{but} \qquad |f(x_\delta) - f(y_\delta)| \geq \varepsilon.$$

Thus for every $n \geq 1$, there exist $x_n, y_n \in [a, b]$ such that

$$|x_n - y_n| < 1/n \qquad \text{but} \qquad |f(x_n) - f(y_n)| \geq \varepsilon. \tag{18.1}$$

By Bolzano-Weierstrass, since (x_n) and (y_n) are bounded, there exist convergent subsequences (x_{n_k}) and (y_{n_k}). Since $|x_n - y_n| < 1/n$, we have that $\lim x_{n_k} = \lim y_{n_k}$; let's call the limit L. Since f is continuous at L, we have

$$\lim_{k\to\infty} f(x_{n_k}) = \lim_{k\to\infty} f(y_{n_k}) = f(L), \qquad \text{so} \qquad \lim_{k\to\infty}(f(x_{n_k}) - f(y_{n_k})) = 0.$$

But this contradicts (18.1). Thus f is uniformly continuous on $[a, b]$. □

To make the connection between uniformly continuous functions and Cauchy sequences even more solid, we have the following proposition.

Proposition 18.6. *If $f : X \to \mathbb{R}$ is uniformly continuous on X and (a_n) is a Cauchy sequence in X, then $(f(a_n))$ is a Cauchy sequence.*

Proof. Let (a_n) be a Cauchy sequence in X, and let $\varepsilon > 0$. Since f is uniformly continuous on X, there exists a $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon \qquad \text{whenever } x, y \in X \text{ and } |x - y| < \delta. \tag{18.2}$$

We now use this δ as the ε in the definition of a Cauchy sequence: The value of δ above produces a threshold $N > 0$ such that if $m, n \geq N$, then

$$|a_m - a_n| < \delta.$$

Now, we take $x = a_m$ and $y = a_n$ in (18.2) to conclude that $|f(a_m) - f(a_n)| < \varepsilon$. Thus $(f(a_n))$ is a Cauchy sequence. □

When working with functions $f : X \to \mathbb{R}$ with $a, b \in \mathbb{R}$ and $a < b$, it is often important that we be able to "extend" f to a larger domain.

Definition 18.7. Let $f : \mathrm{dom}(f) \to \mathbb{R}$ be a function. We say that $\widetilde{f}$ is an **extension of** f if $\mathrm{dom}(f) \subseteq \mathrm{dom}(\widetilde{f})$ and $f(x) = \widetilde{f}(x)$ for all $x \in \mathrm{dom}(f)$.

Example 18.8. The function f in Example 18.1 is an extension of the function $x \sin(1/x)$. The function g in Example 18.2 is an extension of $\sin(1/x)$.

Proposition 18.9. *Let $a < b$. A function $f : (a, b) \to \mathbb{R}$ is uniformly continuous if and only if it can be extended to a continuous function $\widetilde{f} : [a, b] \to \mathbb{R}$.*

Proof. ($\Leftarrow$) Suppose f can be extended to a continuous function $\widetilde{f} : [a, b] \to \mathbb{R}$. By Proposition 18.5, $\widetilde{f}$ is uniformly continuous on $[a, b]$. Thus f is uniformly continuous on (a, b) since $\tilde{f}$ is the same as f on (a, b).

($\Rightarrow$) Suppose that f is uniformly continuous on (a, b). We just need to define $\widetilde{f}(a)$ and $\widetilde{f}(b)$ to make $\widetilde{f} : [a, b] \to \mathbb{R}$ continuous. We will do this for $\widetilde{f}(a)$ ($\widetilde{f}(b)$ is analogous).

There is really only one reasonable way to define $\widetilde{f}(a)$:

$$\widetilde{f}(a) := \lim f(a_n) \text{ for any sequence } (a_n) \text{ in } (a, b) \text{ with } \lim a_n = a.$$

There are two hurdles: (1) How do we know such a limit exists? (2) If we pick two different sequences (a_n) and (a'_n) whose limits are a, do we obtain the same $\widetilde{f}(a)$?

Solution to (1): Note that if $\lim a_n = a$, then $(f(a_n))$ is a Cauchy sequence by Proposition 18.6. Thus $\lim f(a_n)$ exists.

Solution to (2): Define $(z_n) = (a_1, a'_1, a_2, a'_2, a_3, a'_3, \ldots)$. It should be clear that $\lim z_n = a$ as well, so $\lim f(z_n)$ exists (by the solution to (1)) while $(f(a_n))$ and $(f(a'_n))$ are subsequences of $(f(z_n))$. Thus the subsequences must converge to the same limit, namely $\widetilde{f}(a)$. □

19. Uniform continuity; limits of functions

Note that we extended $f:(a,b)\to\mathbb{R}$ to $\widetilde{f}:[a,b]\to\mathbb{R}$ by defining

$$\widetilde{f}(a) := \lim f(a_n) \text{ for any sequence } (a_n) \text{ in } (a,b) \text{ with } \lim a_n = a.$$

If one had to "predict" what $\widetilde{f}(a)$ ought to be based only on what one knows about the behavior of f on (a,b). This is essentially the idea behind limits of functions. We will start by defining left-hand and right-hand limits, then we'll define the usual (two-sided) limit.

Definition 19.1. Let $f : X \to \mathbb{R}$ be a function, and let $a \in X$.

(1) Suppose there exists $b \in \mathbb{R}$ such that $b > a$ and $(a,b) \subseteq X$. We say that

$$\lim_{x\to a^+} f(x) = L$$

if

$$\lim_{n\to\infty} f(a_n) = L \qquad \text{for any sequence } (a_n) \text{ in } (a,b) \text{ converging to } a.$$

Here, L (which could be real or $\pm\infty$) is called the **right-hand limit** of f at a.

(2) Suppose there exists $c \in \mathbb{R}$ such that $c < a$ and $(c,a) \subseteq X$. We say that

$$\lim_{x\to a^-} f(x) = L$$

if

$$\lim_{n\to\infty} f(a_n) = L \qquad \text{for any sequence } (a_n) \text{ in } (c,a) \text{ converging to } a.$$

Here, L (which could be real or $\pm\infty$) is called the **left-hand limit** of f at a.

(3) If

$$\lim_{x\to a^+} f(x) = L = \lim_{x\to a^-} f(x),$$

then we have the **two-sided limit**

$$\lim_{x\to a} f(x) = L.$$

Note that in all cases, the sequences (a_n) always lie in intervals that do not contain a.

Some remarks are in order:

(1) Note that f need not be defined at a for $\lim_{x\to a} f(x)$ to exist.
(2) $\lim_{x\to a} f(x) = f(a)$ if and only if f is defined on an open interval (c,b) containing a and f is continuous at a.
(3) When $\lim_{x\to a} f(x)$ exists, it is unique.

Notation 19.2 (Nonstandard!)**.** We take $\lim_{x\to a^*}$ to be shorthand for $\lim_{x\to a^+}$ or $\lim_{x\to a^-}$ if $*$ is one of the symbols $+$ or $-$, respectively, or the two-sided limit $\lim_{x\to a}$ if $*$ is no symbol.

Proposition 19.3. *Let f,g be functions for which*

$$\lim_{x\to a^*} f(x) = L, \qquad \lim_{x\to a^*} g(x) = M$$

for some $L, M \in \mathbb{R}$.

(1) $\lim_{x\to a^*}(f+g)(x) = L + M$
(2) $\lim_{x\to a^*}(fg)(x) = LM$
(3) $\lim_{x\to a^*}(f/g)(x) = L/M$ *provided that* $M \neq 0$.

(4) Let $c < a < b$, and let g be a function defined on $(c,a)\cup(a,b)$. Let $\lim_{x\to a} g(x) = L \in \mathbb{R}$. Let $Y = \{g(x)\colon x \in (c,a)\cup(a,b)\}\cup\{L\}$. If $f\colon Y \to \mathbb{R}$ is continuous at L, then

$$\lim_{x\to a}(f\circ g)(x) = f(L).$$

It is crucial here (as we will see when we start computing derivatives) that the behavior <u>at $x = a$</u> is **completely irrelevant** for computing limits of functions. The goal of computing a limit is to judge based on the "data" you have on f (its values, whether f is continuous, etc.) available in intervals of the shape (c,a) or (a,b) and make a *prediction* for what $f(a)$ ought to look like. You can make a prediction based on data to the left of a (namely $\lim_{x\to a^-} f(x)$) or based on date to the right of a (namely $\lim_{x\to a^+} f(x)$). If your prediction from the left and your prediction from the right match, then you have a good prediction for what $f(a)$ ought to equal (namely $\lim_{x\to a} f(x)$). However, sometimes predictions fail to match reality! Sometimes $f(a)$ is different from $\lim_{x\to a} f(x)$; sometimes $f(a)$ may not exist at all. When your "two-sided" prediction for $f(a)$ coincides with the actual value of $f(a)$, then we have continuity at $x = a$.

We had two definitions of continuity (sequential and δ-ε), and we showed that the two definitions are equivalent. The notion of the limit of a function also has two definitions.

Definition 19.4. Let f be a function defined on the set $(c,a)\cup(a,b)$, where $c < a < b$. Then $\lim_{x\to a} f(x) = L$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \qquad \text{whenever } 0 < |x-a| < \delta.$$

There are also corresponding definitions for $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$, which the reader can write down.

Proposition 19.5. *Definitions 19.1 and 19.4 are equivalent.*

Proof. Exercise. □

Example 19.6. Let $a \in \mathbb{R}$, and let us prove that

$$\lim_{x\to a}\frac{x^2-a^2}{x-a} = 2a.$$

To begin, let $\varepsilon > 0$ be given, and let $\delta =$??? (not sure what this should be yet, so let's do some work first). Now, if $|x-a| < \delta$, then

$$\Big|\frac{x^2-a^2}{x-a} - 2a\Big| = \Big|\frac{(x-a)(x+a)}{x-a} - 2a\Big| = |x+a-2a| = |x-a| < \delta.$$

Since we want the above display to be less than ε, we may take $\delta = \varepsilon$.

We can also do this computation using the fact that, for limits, we only consider values of $f(x)$ for x close to a, but never for x *equal to* a. So we can write

$$\lim_{x\to a}\frac{x^2-a^2}{x-a} = \lim_{x\to a}\frac{(x-a)(x+a)}{x-a} = \lim_{x\to a}(x+a)$$

since the limit only cares about $x \in (c,a)\cup(a,b)$, but not about $x = a$. Now, since $x+a$ is continuous at a, the limit as $x \to a$ of $x+a$ equals $a+a$ (the limit equals the evaluation when taking the limit of a function at a point of continuity).

20. Limits of functions; the derivative

Example 20.1. Let's do Exercise 20.16 from Ross: Suppose $\lim_{x\to a^+} f(x) = L$ and $\lim_{x\to a^+} g(x) = M$. If there exists a number $b > a$ such that $f(x) \le g(x)$ for all $x \in (a, b)$, then $L \le M$.

Proof. Given $\varepsilon > 0$, there exist $\delta_1, \delta_2 > 0$ such that

$$|f(x) - L| < \frac{\varepsilon}{2} \quad \text{when } 0 < x - a < \delta_1 \qquad \text{and} \qquad |g(x) - M| < \frac{\varepsilon}{2} \quad \text{whenever } 0 < x - a < \delta_2.$$

(The condition above usually looks like $|x - a| < \delta$, but since this is the right-hand limit we are only thinking about $x > a$. Thus $|x - a| = x - a$.) Now, if $0 < x - a < \min\{\delta_1, \delta_2, b - a\}$, then $a < x < b$ and

$$L - \frac{\varepsilon}{2} < f(x) < L + \frac{\varepsilon}{2} \qquad \text{and} \qquad M - \frac{\varepsilon}{2} < g(x) < M + \frac{\varepsilon}{2}$$

Since $a < x < b$, our hypothesis on f and g tells us that $f(x) \le g(x)$. Thus

$$L - \frac{\varepsilon}{2} < f(x) \le g(x) < M + \frac{\varepsilon}{2}.$$

Therefore $L < M + \varepsilon$ for all $\varepsilon > 0$, which implies that $L \le M$. □

We now begin our discussion of the derivative. Derivatives should be quite familiar to you from your calculus courses. I expect that you can differentiate all functions that are built out of elementary functions (polynomials, trig functions, logs, exponents) using algebra and composition. So I will place little focus on these sorts of things. Instead we'll give a rigorous definition of the derivative and use it to prove the properties that will be useful to us.

Definition 20.2. Let $f : X \to \mathbb{R}$ be a function, and let $a \in X$ such that there is an interval $(c, b) \subseteq X$ containing a. (Remember that limits and derivatives are *local* ideas, so we need an open interval around a to talk about the derivative.) We say that f is **differentiable at** a if the limit

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{h\to 0} \frac{f(a + h) - f(a)}{h}$$

exists and is finite. We call this limit the **derivative of** f **at** a, which we write as $f'(a)$.

Example 20.3. The calculation in Example 19.6 shows that the derivative of $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is differentiable at a for any $a \in \mathbb{R}$. Thus we can say that f is **differentiable on** $\mathbb{R}$. Moreover, for each $x \in \mathbb{R}$, we have that $f'(x) = 2x$.

Proposition 20.4. *If f is differentiable at a, then f is continuous at a.*

Proof. Recall from the last lecture that f is continuous at a if and only if f is defined on an open interval containing a. This holds here because differentiability requires f to be defined on an open interval containing a. Now, remember that when computing limits of functions as $x \to a$, we only care about the behavior of f near a, but not at a. So we are justified in writing

$$\lim_{x\to a} f(x) = \lim_{x\to a} \left((x - a)\frac{f(x) - f(a)}{x - a} + f(a) \right) = f(a) + \lim_{x\to a} (x - a)\frac{f(x) - f(a)}{x - a}.$$

It is clear that $\lim_{x\to a}(x - a) = 0$, and by the hypothesis that f is differentiable at a, we have

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a} \in \mathbb{R}.$$

Thus

$$\lim_{x\to a} f(x) = f(a) + \Big(\lim_{x\to a}(x-a)\Big)\Big(\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\Big) = f(a) + 0\cdot(\text{real number}) = f(a),$$

as desired. □

The following result is straightforward to prove using our development of limits.

Proposition 20.5. *Let f and g be functions which are differentiable at a, and let $c \in \mathbb{R}$. Then $f+g$ and cf are differentiable at a. We have the formulae $(cf)'(a) = c\cdot f'(a)$ and $(f+g)'(a) = f'(a) + g'(a)$.*

The arithmetic of derivatives for products, quotients, and compositions of functions is much more interesting, as you perhaps already know.

Proposition 20.6 (Product rule)**.** *Let f and g be functions that are differentiable at a. Then fg is differentiable at a and $(fg)'(a) = f(a)g'(a) + f'(a)g(a)$.*

Proof. The "add zero" trick is hopefully familiar by now:

$$\begin{aligned}\lim_{x\to a}\frac{f(x)g(x)-f(a)g(a)}{x-a} &= \lim_{x\to a}\frac{f(x)g(x)-f(x)g(a)+f(x)g(a)-f(a)g(a)}{x-a}\\ &= \lim_{x\to a} f(x)\frac{g(x)-g(a)}{x-a} + g(a)\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\\ &= f(a)g'(a) + f'(a)g(a),\end{aligned}$$

as desired. (Note that we relied on the fact that differentiability implies continuity.) □

Proposition 20.7 (Quotient rule)**.** *Let f and g be functions that are differentiable at a. Suppose that $g(a) \neq 0$. Then f/g is differentiable at a and*

$$(f/g)'(a) = \frac{g(a)f'(a) - f(a)g'(a)}{[g(a)]^2}.$$

Proof. Given our hypotheses, this is straightforward from the identity

$$\frac{\frac{f(x)}{g(x)} - \frac{f(a)}{g(a)}}{x-a} = \frac{1}{g(x)g(a)}\Big(g(a)\frac{f(x)-f(a)}{x-a} - f(a)\frac{g(x)-g(a)}{x-a}\Big).$$

(We use the fact that differentiability implies continuity, but where do we use it?) □

Example 20.8. Let $n \in \mathbb{N}$. We will prove by induction that $(x^n)' = nx^{n-1}$ for all $x \in \mathbb{R}$. Note that $(x)' = 1$. Now, assume that $(x^n)' = nx^{n-1}$ for some $n \in \mathbb{N}$. By the product rule, $(x^{n+1})' = (x\cdot x^n)' = (x)'x^n + x\cdot(x^n)' = x^n + x\cdot nx^{n-1} = (n+1)x^n$, as desired. Similarly, one can prove by induction that if $n \in \mathbb{N}$, then $(x^{-n})' = -nx^{-n-1}$. Since $(1)' = 0$, we have established that $(x^n)' = nx^{n-1}$ for all $n \in \mathbb{Z}$.

21. The Chain Rule. Setup for Cauchy's Mean Value Theorem.

The key idea of differentiability is as follows: Take a function f, and look at f near a point of interest, say $x = a$. If f is differentiable at a, then if you zoom in arbitrarily close to a, then f looks linear. In other words, if f is defined on an open interval containing a, then near a, f is well-approximated by a line. Let's make this notion precise. Let f be differentiable at a. By our limit laws,

Lemma 21.1. *Let f be defined on an open interval I containing a. Then f is differentiable at a with derivative $f'(a)$ if and only if there exists a function $\varepsilon(x)$ defined on I such that $\lim_{x\to a} \varepsilon(x) = 0$ and $f(x) = f(a) + (f'(a) + \varepsilon(x))(x - a)$.*

In Lemma 21.1, you can think of the $\varepsilon(x)$ as like a "fudge factor" that tends to zero as $x \to a$. This notion will be useful in our proof of the chain rule.

Proposition 21.2 (Chain rule)**.** *If f is differentiable at a and g is differentiable at $f(a)$, then the composition $g \circ f$ is differentiable at a and*

$$(g \circ f)'(a) = g'(f(a)) \cdot f'(a).$$

Proof. We first need to check that $g \circ f$ is defined on an open interval containing a; this is Exercise 28.13 in Ross (and the solution is on p. 390). Be sure to look at this.

By our hypotheses and Lemma 21.1, there exist functions ε_1 and ε_2 such that

$$f(x) = f(a) + [f'(a) + \varepsilon_1(x)] \cdot (x - a), \tag{21.1}$$

$$g(s) = g(f(a)) + [g'(f(a)) + \varepsilon_2(s)] \cdot (s - f(a)), \tag{21.2}$$

where

$$\lim_{x\to a} \varepsilon_1(x) = 0, \tag{21.3}$$

$$\lim_{s\to f(a)} \varepsilon_2(s) = 0, \qquad \text{which is logically equivalent to} \qquad \lim_{x\to a} \varepsilon_2(f(x)) = 0. \tag{21.4}$$

Since f is differentiable, hence continuous, at a, the logical equivalence follows from a change of variables.

Using (21.2) with $s = f(x)$, we obtain

$$g(f(x)) = g(f(a)) + [g'(f(a)) + \varepsilon_2(f(x))] \cdot (f(x) - f(a)).$$

We now insert for $f(x)$ the expression (21.1) to obtain

$$\begin{aligned} g(f(x)) &= g(f(a)) + [g'(f(a)) + \varepsilon_2(f(x))] \cdot [f(a) + (f'(a) + \varepsilon_1(x)) \cdot (x - a) - f(a)] \\ &= g(f(a)) + [g'(f(a)) + \varepsilon_2(f(x))] \cdot [f'(a) + \varepsilon_1(x)] \cdot (x - a) \\ &= g(f(a)) + [g'(f(a))f'(a) - \varepsilon_3(x)] \cdot (x - a), \end{aligned}$$

where

$$\varepsilon_3(x) = g'(f(a))\varepsilon_1(x) + f'(a)\varepsilon_2(f(x)) + \varepsilon_1(x)\varepsilon_2(f(x)).$$

By Lemma 21.1, the proof is finished once we show that $\lim_{x\to a} \varepsilon_3(x) = 0$. But this follows from (21.3), (21.4), and the limit laws. □

One of the most important results in the theory of differentiable functions is the mean value theorem (and Cauchy's generalization). It plays a key role in the proofs of the Fundamental Theorem of Calculus as well as Taylor's Theorem, and it is implicitly used in most single-variable optimization problems. Before proving it (next class), we require some setup.

Definition 21.3. Let $f : X \to \mathbb{R}$ be a function with $[a, b] \subseteq X$. We say that f has a **local maximum** at $c \in [a, b]$ if there exists a $\delta > 0$ such that $f(x) \leq f(c)$ for all $x \in (c - \delta, c + \delta)$. We say that f has a **local minimum** at $c \in [a, b]$ if there exists a $\delta > 0$ such that $f(x) \geq f(c)$ for all $x \in (c - \delta, c + \delta)$.

Lemma 21.4. *Let f be defined on $[a, b]$. If f has a local maximum or a local minimum at $c \in (a, b)$, and if $f'(c)$ exists, then $f'(c) = 0$.*

Proof. By Lemma 21.1, there exists a function $\varepsilon(x)$ such that $\lim_{x \to c} \varepsilon(x) = 0$ and

$$f(x) - f(c) - f'(c) \cdot (x - c) = \varepsilon(x) \cdot (x - c).$$

Suppose to the contrary that $f'(c) \neq 0$. Thus we deduce that there exists $\delta > 0$ so that

$$|\varepsilon(x)| \cdot |x - c| \leq \frac{|f'(c)| \cdot |x - c|}{2} \qquad \text{whenever } |x - c| < \delta.$$

Thus if $|x - c| < \delta$, then $f(x) - f(c)$ and $f'(c) \cdot (x - c)$ have the same sign. This sign changes depending on whether $x - c$ is positive or negative. But this contradicts our hypothesis that $f(c)$ is a local maximum or a local minimum. Thus our assumption that $f'(c) \neq 0$ was false, so $f'(c) = 0$ as desired. $\square$

22. The mean value theorem

Theorem 22.1 (Rolle's theorem)**.** *Let f be a function which is continuous on $[a,b]$ and differentiable on (a,b). If $f(a)=f(b)$, then there is a point $c\in(a,b)$ such that $f'(c)=0$.*

Proof. By Theorem 17.2, f achieves its maximum and its minimum on $[a,b]$. If both the maximum and the minimum are achieved along the boundary of $[a,b]$, then the condition $f(a)=f(b)$ ensures that f is constant, hence $f'(x)=0$ for all $x\in(a,b)$. Otherwise, f has a local maximum or a local minimum somewhere in (a,b). By Lemma 21.4, the derivative at a local maximum or a local minimum must be zero. □

Theorem 22.2 (Cauchy's mean value theorem)**.** *Let f and g be functions which are continuous on $[a,b]$ and differentiable on (a,b). There is a point $c\in(a,b)$ such that $[f(b)-f(a)]\cdot g'(c)=[g(b)-g(a)]f'(c)$.*

Proof. For $x\in[a,b]$, define $h(x)=[f(b)-f(a)]\cdot g(x)-[g(b)-g(a)]\cdot f(x)$. Then h is continuous on $[a,b]$, differentiable on (a,b), and satisfies

$$h(a)=f(b)g(a)-f(a)g(b)=h(b).$$

By Rolle's theorem, there exists $c\in(a,b)$ such that

$$0=h'(c)=[f(b)-f(a)]\cdot g'(c)-[g(b)-g(a)]\cdot f'(c),$$

as desired. □

Cauchy's mean value theorem is a key component in proving l'Hospital's rule (see below). In most situations, we use the following highly useful corollary.

Corollary 22.3 ("The" mean value theorem)**.** *Let f be a function which is continuous on $[a,b]$ (with $a<b$) and differentiable on (a,b). There is a point $c\in(a,b)$ such that*

$$f'(c)=\frac{f(b)-f(a)}{b-a},\qquad \text{or equivalently}\qquad f(b)-f(a)=(b-a)f'(c).$$

Proof. Take $g(x)=x$ in Cauchy's mean value theorem. □

The mean value theorem is simply amazing! Put another way, if f is differentiable on (a,b), then the mean value states that the "fudge factor" $\varepsilon(x)$ in Lemma 21.1 must be *exactly* zero at some point $c\in(a,b)$, yielding a clean expression for $f'(c)$ in terms of $f(a)$ and $f(b)$. This means that the behavior of the derivative (a local notion, remember we need to "zoom in" to think about limits) has a very direct and concrete impact on the function itself.

Here are some incredibly useful consequences of the mean value theorem. A common theme is that the behavior of the derivative of a function (which, if you recall, only depends on how the function behaves *locally*, very close a given point) can have a strong impact on the global (big-picture) behavior of the function.

Theorem 22.4. *Suppose that f is differentiable on (a,b).*

(1) If $f'(x)\geq 0$ (resp. >0) for all $x\in(a,b)$, then f is monotonically (resp. strictly) increasing on (a,b).
(2) If $f'(x)=0$ for all $x\in(a,b)$, then f is constant.
(3) If $f'(x)\leq 0$ (resp. <0) for all $x\in(a,b)$, then f is monotonically (resp. strictly) decreasing on (a,b).

Proof. All conclusions can be read off from the equation

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(x),$$

which is valid, for *each* pair of numbers $x_1, x_2 \in (a, b)$, for *some* $x \in (x_1, x_2)$. □

Corollary 22.5. *Suppose f and g are differentiable on (a, b) and $f'(x) = g'(x)$ for every $x \in (a, b)$. Then there exists a constant $c \in \mathbb{R}$ such that $f(x) = g(x) + c$ for every $x \in (a, b)$.*

Proof. Apply Theorem 22.4(2) to the function $f - g$. □

Here is another application of the mean value theorem.

Example 22.6. Let $f(x) = x^3 + 3x + 1$. By the most recent HW, f has at least one real root since f is a degree 3 polynomial with real coefficients. Since f is continuous and $f(0) = 1$ and $f(-1) = -3$, the Intermediate Value Theorem tells us that one of these roots lies in the interval $(-1, 0)$.

Now, suppose that f has two distinct real roots a and b. Then $f(a) = f(b) = 0$, so by Rolle's theorem or the Mean Value Theorem, there exists $c \in (a, b)$ such that $f'(c) = 0$. But $f'(x) = 3x^2 + 3 > 0$ for all $x \in \mathbb{R}$, a contradiction. Thus f has *exactly one* real root.

Cauchy's mean value theorem is rather helpful in evaluating certain tricky limits.

Theorem 22.7. *Suppose f, g are real and differentiable on (a, b) and $g'(x) \neq 0$ for all $x \in (a, b)$. (The endpoints could be infinite). Suppose that*

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = A \qquad \left(\text{resp. } \lim_{x\to b} \frac{f'(x)}{g'(x)} = A\right).$$

(One could take A to be infinite.) If

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \qquad \left(\text{resp. } \lim_{x\to b} f(x) = \lim_{x\to b} g(x) = 0\right)$$

or

$$\lim_{x\to a} |g(x)| = \infty \qquad \left(\text{resp. } \lim_{x\to b} |g(x)| = \infty\right),$$

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = A \qquad \left(\text{resp. } \lim_{x\to b} \frac{f(x)}{g(x)} = A\right).$$

Analogues exist as $x \to b$, or if $g(x) \to -\infty$. This relies

Proof. See Ross. □