Cournot Competition in Networked Markets

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The paper considers a model of competition among firms that produce a homogeneous good in a networked environment. A bipartite graph determines which subset of markets a firm can supply to. Firms compete *à la* Cournot and decide how to allocate their production output to the markets they are directly connected to. We provide a characterization of the production quantities at the unique equilibrium of the resulting game for any given network. Our results identify a novel connection between the equilibrium outcome and supply paths in the underlying network structure. We then proceed to study the impact of changes in the competition structure, e.g., due to a firm expanding into a new market or two firms merging, on firms' profits and consumer welfare. The modeling framework we propose can be used in assessing whether expanding in a new market is profitable for a firm, identifying opportunities for collaboration, e.g., a merger, joint venture, or acquisition, between competing firms, and guiding regulatory action in the context of market design and antitrust analysis.

Key words: Cournot competition; non-cooperative games; networks; horizontal mergers.

1. Introduction

Models of oligopolistic competition typically feature a number of firms operating in a single, well-defined, isolated market. In many settings though, firms compete with one another across several different markets. This is particularly prevalent in regional industries that have distinct geographical markets, such as electricity, natural gas, airlines, the cement industry, healthcare, and banking. For example, constraints imposed by the natural gas pipeline network and the electricity grid imply naturally that firms in these industries compete with one another in several distinct consumer markets. In fact, several papers aim to explore firms' strategic behavior in deregulated power markets by emphasizing the role of network structure on market outcomes (e.g., Borenstein, Bushnell, and Knittel (1999), Wu, Bose, Wierman, and Mohesenian-Rad (2013), Bose, Cai, Low, and Wierman (2014)). Similarly, airlines compete across several origin-destination pairs which can be viewed as distinct markets. In fact, the presence of strong multi-market contact effects has been empirically verified for the US airline industry (Kim and Singal (1993), Evans and Kessides (1994)).

As another example, cement, due to its high weight-to-price ratio, can only be economically transported by land or between export/import terminals by sea. Thus, the set of locations a cement plant can supply to is inherently limited by its distance to potential customers and access to waterways. Typically, cement firms own and operate several plants in different geographical locations competing this way in many consumer markets (Jans and Rosenbaum (1997) provide extensive empirical evidence supporting the effect of multi-market contact on prices in the cement industry). Regional competition is also starting to play an important role in healthcare. Networks of providers compete with one another over customers that may have a strong preference towards easy access to care. The past few years have witnessed an increasing number of mergers, acquisitions, and consolidations in the healthcare space that aim to enable providers expand their geographical reach. Finally, although we present our model and results for a single homogeneous good that is sold by a set of firms across different markets, our analysis is relevant for large conglomerate firms competing with one another in several distinct product markets.[1]

These examples motivate the study of oligopoly models in which firms strategically interact with one another across several markets. Interestingly, although, as mentioned above, there is strong empirical evidence that multi-market competition is prevalent in many industries and single-market models are inadequate to provide an accurate description of the strategic interactions in such environments, to the best of our knowledge, there is very little analytical work that explicitly accounts for the competition structure among firms. Our paper presents one step towards this direction. In particular, we consider a model where the competition structure is given by a general bipartite graph that represents the set of markets each firm can supply to. We emphasize that we do not require *any* assumptions on the number of firms and markets or the structure of the competition among them. Firms compete *à la* Cournot in each of the markets, i.e., price is determined as a function of the aggregate production quantity supplied to the market, and their cost of production is convex, thus their supply decisions in different markets are coupled.

We begin our analysis by showing that there exists a unique equilibrium for any given network and general concave inverse demand and convex cost functions. Then, we focus on linear inverse demands and quadratic costs and provide a characterization of the equilibrium flows, i.e., production quantities, in terms of supply paths in the network. This assumption which is prevalent in the literature is essential for clearly bringing out the role of the network structure in equilibrium

[1] Relatedly, an infographic created by French blog Convergence Alimentaire clearly illustrates the extent to which the entire consumer goods industry is dominated by ten multinational firms that compete with one another in several different product categories (http://www.convergencealimentaire.info/map.jpg).

decision making. In particular, we show that the *price-impact* matrix that can be written explicitly as a function of the underlying competition structure succinctly summarizes the effect of each firm-market pair on production quantities, firms' profits, and consumer welfare. Specifically, this matrix allows us to compute the impact on the price of any market that results from an increase in the quantity associated with a firm-market pair.

Armed with a characterization of equilibrium supply decisions in terms of the price-impact matrix, we explore the effect of changes in the network structure on firms' profits and consumer welfare. First, we study the question of a firm entering a new market. We show that entry may not be beneficial for either the firm or welfare on aggregate as such a move affects the entire vector of production quantities. The firm may face aggressive competition in its original markets after expanding into a new market. The extent to which a firm's competitors respond to the event of entry depends on the paths connecting the new market with the rest of the markets the firm supplies to and, thus, even distant markets (in a network sense) may have a first order impact on the firm's profits. We explicitly quantify the network effect on the competitors' response to entry and establish that the net benefit associated with expanding into a new market (even in the absence of fixed entry costs) decreases as the chain of competition increases and may in fact turn out to be negative.[2]

Furthermore, the effect on other firms and consumers also depends on their location in the network. A subset of firms and consumers may benefit while others may not. This is in stark contrast with standard results in Cournot oligopoly where entry directly implies more competition in the market and thus higher consumer welfare and lower profits for all the firms. Thus, our results have important implications for market design since regulatory measures that facilitate expansion to new markets may not necessarily lead to an increase in aggregate welfare.

Similarly, the effect of a merger between two firms on profits and overall welfare largely depends on the structure of competition in the original networked economy. In particular, we show that insights from analyzing mergers in a single market do not carry over in a networked environment. Market concentration indices are insufficient to correctly account for the network effect of a merger and one should not restrict attention only to the set of markets that the firms participating in the merger supply to. We highlight that even if two firms do not share a market in the original premerger economy they can exert market power by coordinating their supply decisions and lead to a decrease in consumer welfare unlike what traditional merger analysis would predict. Interestingly,

[2] There are several instances of firms that "spread themselves too thin" by entering seemingly profitable markets and as a result ended up facing fierce competition in their home markets. For example, many claim that Frontier airlines invited aggressive competition in its primary hub, Denver, by expanding to a large number of new routes (see also Bulow, Geanakoplos, and Klemperer (1985)). Similarly, Kmart was not able to fend off competition from its major rival, Walmart, presumably because of over-diversifying its product portfolio.

Kim and Singal (1993) highlight the importance of taking the competition structure into account when considering the welfare effects of a merger by empirically studying the wave of mergers that followed the Deregulation Act in the airline industry.

1.1. Related Literature

Bulow, Geanakoplos, and Klemperer (1985) serves as one of the main motivations behind our study. They analyze the strategic interactions between two firms that compete in two markets (a monopoly and a duopoly) and show that strategic complementarity and substitutability between the firms' actions determine the effect of exogenous changes in the markets on profits. We extend their environment by considering an arbitrary network of firms and markets and properly generalize the notion of strategic complementarity and substitutability to account for the network interactions among the firms. Our multi-market environment allows us to consider firms merging or expanding into new markets as well as to study the role of the competition structure in determining how welfare and profits are affected.

In addition to the papers discussed above, a different strand of literature studies bilateral trading of indivisible goods among agents in a network. Kranton and Minehart (2001) model competition among buyers of a single, indivisible good as an ascending price auction and study whether the resulting pattern of trades is efficient. Corominas-Bosch (2004) considers a non-cooperative bargaining game and provides conditions for the equilibrium of the bargaining game to coincide with the Walrasian outcome. More recently, Nakkas and Xu (2015) examine the role of heterogeneity in the buyers' valuations on the efficiency of realized trades as well as equilibrium pricing. Additionally, Ostrovsky (2008), Hatfield, Kominers, Nichifor, Ostrovsky, and Westkamp (2013), and Ashlagi, Braverman, and Hassidim (2014) study equilibrium stability in a model where agents can trade via bilateral contracts and a network determines the set of feasible relationships.

Relatedly, Manea (2011) study decentralized bargaining and provide insights on how an agent's bargaining power relates to her position in the network that represents the set of feasible trades. Nguyen (2015) analyzes a non-cooperative bargaining model with a general coalitional structure and provides a characterization of the set of equilibira via a convex program. Elliott (Forthcoming) studies markets in which there are heterogeneous gains from trade and relationship specific investments are necessary for trade to take place whereas Ashlagi, Kanoria, and Leshno (2015) study the effect of competition on the size of the core in large matching markets. Manea (2015) studies intermediation in markets that feature an underlying network structure and explores inefficiencies in trade that may be the result of hold-up or local competition. [3] Closest to the questions

[3] Several other papers, e.g., Blume, Easley, Kleinberg, and Tardos (2009) and Choi, Galeotti, and Goyal (2015), study trading among agents that are connected by a network structure.

we explore, Nava (2015) considers a model of quantity competition in which agents endogenously decide whether to assume the role of a buyer, seller, or intermediary. He establishes that in the limit, when the economy grows large, competition leads to (approximately) efficient outcomes.

Furthermore, our work is related to an earlier series of papers on spatial competition over bipartite networks, e.g., Kyparisis and Qiu (1990) and Qiu (1991). The setup studied in this line of work features firms supplying to the entire set of markets albeit at a different (transportation) cost per market. The main objective is to identify conditions under which an equilibrium exists and provide efficient algorithms to compute equilibria. Finally, independently from our contribution, Abolhassani, Bateni, Hajiaghayi, Mahini, and Sawant (2014) study a model of Cournot competition in networks similar to ours but their goal is to derive algorithms that compute the pure strategy Nash equilibria (as opposed to providing characterizations of equilibrium production quantities, profits, and welfare, which is the main focus of our study).

Also relevant are the recent contributions by Perakis and Roels (2007), Kluberg and Perakis (2012), Perakis and Sun (2014), and Federgruen and Hu (2015a) which study competition among firms that offer multiple products under various assumptions on the extent of substitution between the products. Their focus is on equilibrium existence and the extent of efficiency loss due to non-cooperative decision making. Cho (2013) studies horizontal mergers in multitier supply chains in which firms in tier i supply to all firms in tier $i-1$, i.e., the network between two tiers is complete, and interestingly concludes that the effect of a merger depends on whether the tier in which the merger occurs acts as a leader. Finally, Corbett and Karmarkar (2001), Adida and DeMiguel (2011), and Federgruen and Hu (2015b) study competition in multi-tier supply chains mainly showing the existence of equilibria and studying their efficiency properties. The papers cited above focus on structures where firms have access to all consumers, i.e., the network is complete. Thus, their focus is very different than ours which is establishing a connection between equilibrium outcomes and the underlying structure of competition among the firms.

Allon and Federgruen (2009) explore competition in service industries where providers cater to multiple customer segments, i.e., markets, using shared facilities. Although their model of competition is different than ours (they are looking at competition in prices and services level guarantees among queuing facilities), theirs is essentially another instance of multi-market competition (albeit one in which firms have access to all markets, i.e., the network is complete).

Our analysis of mergers extends the work of Farrel and Shapiro (1990) to the setting where firms compete with one another in multiple markets. Specifically, they study mergers in a single market, whereas our focus is on how the structure of competition affects the way profits and consumer welfare change after the merger. Importantly, our analysis establishes that the entire network structure plays a first-order role in determining whether a merger has a positive or negative effect on overall welfare and market concentration indices that have been widely used in antitrust analysis cannot accurately capture the welfare effect of a merger.

Finally, our paper is also related to a recent stream of papers that study games among agents that are embedded on a network structure. Ballester, Calvó-Armengol, and Zenou (2006) identify a close relation between an agent's equilibrium action in a game that features local positive externalities and her position in the network structure as captured by her Katz-Bonacich centrality. Candogan, Bimpikis, and Ozdaglar (2012) study the pricing problem of a monopolist that is selling a divisible good to a population of agents and provide a characterization of the optimal pricing policies as a function of the social network structure of agents.

2. Model

Consider an economy with n firms $F = \{f_1, \dots, f_n\}$ producing a perfectly substitutable good and competing in m markets $M = \{m_1, \dots, m_m\}$. A bipartite graph $G = (F \cup M, E)$, where E is a set of edges from the set of firms (F) to the set of markets (M), represents the subset of markets a firm can supply to. Finally, $F_i = \{m_k \in M \big| (f_i, m_k) \in E\}$ denotes the set of markets firm f_i supplies to and $M_k = \{f_i \in F \big| (f_i, m_k) \in E\}$ denotes the set of firms that supply to market m_k. An example of the economy described above is depicted in Figure 1.

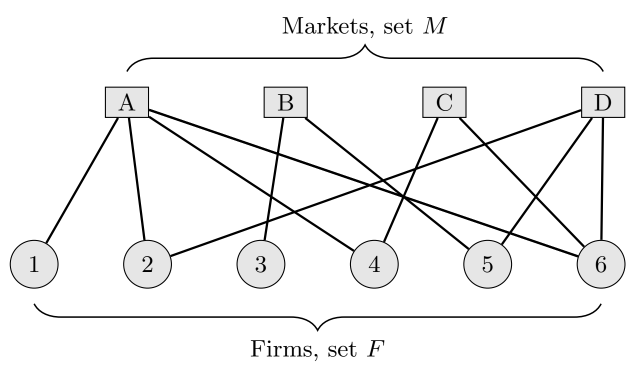


Figure 1 The structure of competition as a bipartite graph.

Firms compete in quantities, i.e., they decide how to allocate their aggregate production among the markets they supply to. Let q_{ik} denote the quantity firm f_i supplies to market m_k and $\mathbf{q_i}$ denote the vector of production quantities of firm f_i. Then, the price at market m_k is given by $\mathcal{P}_k\Big(\sum_{j \in M_k} q_{jk}\Big)$. We assume that $\mathcal{P}_k(\cdot)$ is a twice differentiable, strictly decreasing, and concave function for every k. Finally, the cost of production for firm f_i is given by $\mathcal{C}_i\Big(\sum_{j \in F_i} q_{ij}\Big)$, where $\mathcal{C}_i$ is a twice differentiable, strictly increasing, and convex function for every i. Thus, firm f_i's profit is given by the following expression:

$$\pi_i(\boldsymbol{q}) = \sum_{m_k \in F_i} q_{ik} \cdot \mathcal{P}_k\Big(\sum_{j \in M_k} q_{jk}\Big) - \mathcal{C}_i\Big(\sum_{j \in F_i} q_{ij}\Big).$$

Quite importantly, firm f_i's profit function is *not* separable in the markets it participates in for a general convex function $\mathcal{C}_i(\cdot)$ and the marginal profit from increasing q_{ik} is decreasing in firm f_i's aggregate production. This non-separability couples the markets a firm operates in, i.e., if costs were linear the environment would be equivalent to a set of markets that could be studied in isolation.[4] Given graph G firm f_i solves the following optimization problem:

$$\begin{aligned} \underset{\mathbf{q_i}}{\text{maximize}} \quad & \pi_i(\boldsymbol{q_i}, \boldsymbol{q_{-i}}) \\ \text{subject to} \quad & q_{ik} \geq 0 \qquad \text{for } k \in F_i \\ & q_{ik} = 0 \qquad \text{for } k \notin F_i, \end{aligned}$$

where $\mathbf{q_{-i}} = \{q_{jk} \text{ for } (f_j, m_k) \in E \text{ and } j \neq i\}$, denotes the vector of production quantities of its competitors. We denote the resulting game by $\mathcal{CG}\Big(\{\mathcal{P}_k\}_{1\leq k\leq m}, \{\mathcal{C}_i\}_{1\leq i\leq n}, G\Big)$. In section 3 we show that game $\mathcal{CG}$ has a unique equilibrium for general concave $\mathcal{P}_k$'s and convex $\mathcal{C}_i$'s. Then, we proceed to provide a characterization of the production quantities at equilibrium as a function of the underlying network structure. We state our characterization results for inverse linear demands and quadratic production costs as this allows us to bring out the role of graph G in the clearest and most transparent way, i.e., we consider

$$\mathcal{P}_k = \alpha_k - \beta_k \cdot \sum_{j \in M_k} q_{jk} \text{ and } \mathcal{C}_i = c_i \cdot \Big(\sum_{k \in F_i} q_{ik}\Big)^2,$$

where $\alpha_k, \beta_k, c_i > 0$. Although many of our qualitative insights carry over to the general concave-convex framework, it is worthwhile to note that both inverse linear demand functions and quadratic costs are fairly common assumptions in the literature (e.g., Singh and Vives (1984), Vives (2011) for general quantity competition models, and Yao, Adler, and Oren (2008), Bose, Cai, Low, and Wierman (2014) for studies specific to electricity networks).

Finally, given graph G, we define its *line graph* (denoted by $\mathcal{L}(G)$) as the graph that has a node corresponding to every edge in G and an edge between two nodes in $\mathcal{L}(G)$ if their corresponding edges in the original graph G share an endpoint (see Figure 2 for an illustrative example). The line graph helps keeping track of how the production quantities on different links influence one another.

3. Equilibrium Analysis

Lemma 1 establishes that game $\mathcal{CG}$ defined above has a unique equilibrium.

LEMMA 1. *Game* $\mathcal{CG}\Big(\{\mathcal{P}_k\}_{1\leq k\leq m}, \{\mathcal{C}_i\}_{1\leq i\leq n}, G\Big)$ *has a unique Nash equilibrium when* $\{\mathcal{P}_k(\cdot)\}_{1\leq k\leq m}$ *are twice differentiable, concave, and strictly decreasing, and* $\{\mathcal{C}_i(\cdot)\}_{1\leq i\leq n}$ *are twice differentiable, convex, and increasing.*

[4] Note that although, in the interest of analytical tractability, we incorporate this feature through a convex cost function (which is a standard assumption in many oligopoly models), our analysis provides qualitative insights for settings in which this coupling arises due to the firm having limited resources that it shares among its markets and/or fixed capital to finance its operations.

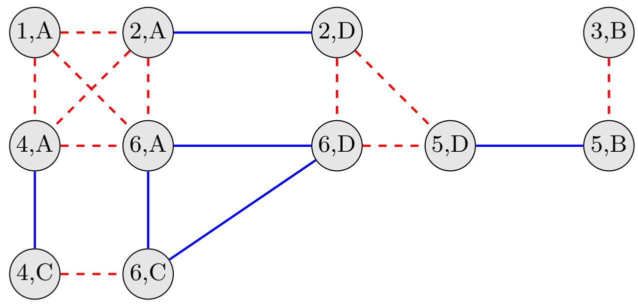


Figure 2 **The line graph associated with the economy depicted in Figure 1. Each node corresponds to a firm-market pair in the original graph G. Red dashed edges connect nodes (firm-market pairs) that share a market whereas blue solid edges connect nodes that share a firm. Finally, the weight of a blue solid edge is equal to $2c_i$, where c_i is the cost parameter of the corresponding firm and the weight of a red dashed edge is equal to β_k. For example, the weight of the edge connecting nodes $(5,B)$ and $(5,D)$ is equal to $2c_5$ and the weight of the edge connecting nodes $(2,A)$ and $(6,A)$ is equal to β_A.**

We provide a proof of Lemma 1 that is based on Rosen (1965). Harker (1986) establishes a similar result using a variational inequality characterization.

The remainder of the section provides a characterization of the production quantities at equilibrium as a function of the structure of competition among the firms. To clearly bring out the role of network G, we focus on linear inverse demands and quadratic production costs. We also let $\mathcal{CG}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{c}, G)$ denote the corresponding game where $\boldsymbol{\alpha} = [\alpha_1, \cdots, \alpha_m]^T$, $\boldsymbol{\beta} = [\beta_1, \cdots, \beta_m]^T$, and $\boldsymbol{c} = [c_1, \cdots, c_n]^T$. Finally, we let $\bar{\boldsymbol{\alpha}}$ denote a $|E| \times 1$ column vector such that for every link $(i,k) \in E$ we have $\bar{\alpha}_{ik} = \alpha_k$.

First, we provide Lemma 2 which states that the unique equilibrium of $\mathcal{CG}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{c}, G)$ is given by the solution to a *linear complementarity problem.* In particular, let $LCP(\boldsymbol{w}, Q)$ denote the problem of finding vector $\boldsymbol{z} \geq \mathbf{0}$ such that $Q\boldsymbol{z} + \boldsymbol{w} \geq \mathbf{0}$ and $\boldsymbol{z}^T(Q\boldsymbol{z} + \boldsymbol{w}) = \mathbf{0}$. Then, we have

Lemma 2. *The unique equilibrium $\boldsymbol{q}$ of $\mathcal{CG}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{c}, G)$ is given by the unique solution of $LCP(-\boldsymbol{\alpha}, D)$, where D is the following $|E| \times |E|$ matrix*

$$D_{ik,j\ell} = \begin{cases} 2(\beta_k + c_i) & \text{if } i = j, k = \ell \\ 2c_i & \text{if } i = j, k \neq \ell \\ \beta_k & \text{if } i \neq j, k = \ell \\ 0 & \text{otherwise.} \end{cases} \quad . \tag{1}$$

Let $\boldsymbol{q}_E^*$ denote the column vector of equilibrium production quantities for each firm-market pair, i.e., edges of graph G, where the edges are ordered lexicographically. Note that a subset of edges may carry zero flow, i.e., $q_{ik}^* = 0$ even though $(f_i, m_k) \in E$. The results that follow are stated for the set of *active* edges, i.e., the subset of edges for which the corresponding production quantities are strictly positive. Lemma 3 below states that we can ignore the inactive edges, which we denote by

$Z(q_E^*) = \{(f_i, m_k) \big| [q_E^*]_{ik} = 0\}$, without any loss of generality since they are strategically redundant and play no role in determining the equilibrium.

Lemma 3. *Consider game $\mathcal{CG}(\boldsymbol{a}, \boldsymbol{\beta}, \boldsymbol{c}, G)$ and let $\boldsymbol{q}_E^*$ denote the vector of production quantities at equilibrium for this game. Also, let $G' = (F \cup M, E')$ with $E' = E \setminus Z(q_E^*)$ and $\boldsymbol{q'}$ be the equilibrium of game $\mathcal{CG}(\boldsymbol{a}, \boldsymbol{\beta}, \boldsymbol{c}, G')$. Then, $\boldsymbol{q}^*_{E \setminus Z(\boldsymbol{q}^*)} = \boldsymbol{q}'$.*

For the remainder of the paper we focus on networks G for which the resulting equilibria only have active edges. Finally, although not critical for the results, we state Theorem 1 for symmetric games $\mathcal{CG}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{c}, G)$, i.e., games for which firms have the same production technology ($c_i = c_j = c$, for all $f_i, f_j \in F$), and markets have the same demand slope ($\beta_k = \beta_\ell = \beta$, for all $m_k, m_\ell \in M$). This way asymmetry between the firms arises only due to the structure of G. Before stating the result we define $|E| \times |E|$ matrix W as

$$w_{i_1k_1, i_2k_2} = \begin{cases} 2c \text{ if } i_1 = i_2, k_1 \neq k_2 \\ \beta \;\; \text{if } i_1 \neq i_2, k_1 = k_2 \\ 0 \;\; \text{otherwise.} \end{cases} \tag{2}$$

Note that matrix W is a weighted adjacency matrix associated with the line graph $\mathcal{L}(G)$ of graph G. In particular, the rows and columns of W correspond to the links in graph G. If two links are connected in G via a firm, then their weight in W is $2c$ whereas if they are connected via a market, then their weight is β. For the links which do not share a node in G, the weight is 0. The non-zero entries of W are equal to the change in the marginal profit from production corresponding to a firm-market pair that results from an infinitesimal increase in the quantity corresponding to another firm-market pair. For example, for links originating from the same firm ($i_1 = i_2$) this value is equal to $2c$ (thus capturing the increase in the marginal cost of production for the firm) whereas for edges ending up in the same market it is equal to β (thus capturing the marginal decrease in the market's price). We obtain the following characterization:

Theorem 1. *The unique Nash equilibrium of the symmetric game $\mathcal{CG}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{c}, G)$ is given by*

$$\boldsymbol{q}^* = [I + \gamma W]^{-1} \gamma \bar{\boldsymbol{\alpha}}, \tag{3}$$

where $\gamma = (2(c+\beta))^{-1}$. Furthermore, if $\lambda_{\max}(\gamma W) < 1$, Expression (3) can be rewritten as

$$\boldsymbol{q}^* = \left[\sum_{k=0}^{\infty} (\gamma W)^{2k} - \sum_{k=0}^{\infty} (\gamma W)^{2k+1} \right] \gamma \bar{\boldsymbol{\alpha}}. \tag{4}$$

Theorem 1 implies that production quantity q_{ik}^*, i.e., the quantity that firm f_i supplies to market m_k at equilibrium, can be written as a weighted sum of the sizes of all the markets (vector $\boldsymbol{\alpha}$), where the weights are given by matrix $[\mathrm{I}+\gamma W]^{-1}$. In particular, the weights depend on the location of link (i,k) within the network through the paths that start from link (i,k) and end up in the nodes representing the different markets.

Consider the production quantity that firm f_i supplies to market m_k, i.e., q_{ik}. The weight that corresponds to market m_ℓ in the expression for q_{ik} is increasing (decreasing) with the weights of even (odd) paths from link (i,k) to market m_ℓ.[5] This insight is most apparent in Equation (4), as the even (odd) power terms correspond directly to paths of even (odd) length. In informal terms, Equation (4) is driven by the intuition that *"the enemy of my enemy is my friend."* [6]

Bulow, Geanakoplos, and Klemperer (1985) find that the main driver in determining how changes in one market affect a firm's prospects in a second market is whether competitors view their actions as strategic substitutes or complements. Theorem 1 provides a way to relate the degree of strategic substitutability or complementarity between the actions of firms in two markets with the supply paths that connect them, thus appropriately generalizing these notions to a networked environment. Furthermore, the theorem implies that in order to determine whether a pair of actions are strategic substitutes or complements, one may need to consider the entire competition structure (and not just focus on the pair in isolation).

We can further exploit the simplicity of the equilibrium characterization in Equation (4) by using the weighted adjacency matrix W of the line graph $\mathcal{L}(G)$. We show that the equilibrium production quantities are closely related to the following measure of centrality of the nodes in graph $\mathcal{L}(G)$.

Definition 1. Given a weighted adjacency matrix W and a scalar ρ, the *Katz-Bonacich centrality* of the nodes in the network is defined as the following vector

$$\boldsymbol{b}(W,\rho) = \sum_{t=0}^{\infty} (\rho W)^t \mathbf{1}.$$

When all markets have the same size $\alpha_1 = \cdots = \alpha_m = \alpha$, we obtain the following corollary.

Corollary 1. *Suppose that* $\lambda_{\max}(\gamma W) < 1$*. Then, the unique Nash equilibrium of game* $\mathcal{CG}(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{c},G)$ *when markets are symmetric is given by*

$$\boldsymbol{q}^* = \boldsymbol{b}(W,-\gamma)\gamma\alpha. \tag{5}$$

[5] For a formal proof of this fact, refer to Proposition 7 in the Appendix.

[6] As a side remark, note that $\lambda_{\max}(\gamma W) < 1$ ensures that the infinite sum in Equation (4) converges. Lemma 5 in the Appendix provides conditions under which $\lambda_{\max}(\gamma W) < 1$. The conditions essentially require that the network is sufficiently sparse.

It is important to note that equilibrium production quantities in Corollary 1 are written in terms of centrality vector $\boldsymbol{b}(W, -\gamma)$ that features a negative scalar. Thus, unlike traditional notions of centrality for which nodes that are connected to central nodes are themselves central, this intuition does not hold in our setting. In the context of trading/bargaining networks having many direct connections (trading opportunities) contributes to centrality (bargaining power), however if one's connections themselves have many connections, centrality is reduced (as the agent's potential trading partners have many outside options). Similarly, in multi-market competition a firm's direct connections (and more generally paths of odd length) contribute negatively to the firm's profits while paths of length two (and more generally paths of even length) have a positive contribution.

Finally, we conclude this section with an alternative characterization of the equilibrium production quantities that highlights their dependence on the "importance" of a firm-market pair, as captured by the *price-impact* matrix Λ defined below. This characterization also illustrates that even distant links (in a network sense) may have a considerable impact on the price that the product is sold in a market. In particular, let Λ denote the following $|E| \times m$ matrix

$$\Lambda_{ik,\ell} = -\beta \sum_{j \in M_\ell} \frac{\psi_{j\ell,ik}}{\psi_{ik,ik}} \quad \forall (i,k) \in E \text{ and } m_\ell \in M. \tag{6}$$

As becomes apparent from Proposition 1 that follows in Section 4, entry (ik, ℓ) of matrix Λ is equal to the change in market m_ℓ's price that results from a marginal increase in the production that firm f_i supplies to market m_k. One can view the entries of Λ as a measure of the firms' market power, i.e., the larger their absolute values, the higher market power the corresponding firms have in the underlying networked environment as changes in their actions have a large impact on market prices (and consequently firms' profits and consumer surplus).

COROLLARY 2. *The equilibrium production quantities can be expressed as*

$$\boldsymbol{q}^* = -V\Lambda\boldsymbol{\alpha}\frac{\gamma}{\beta},$$

where $V = Diag(\Psi)$.

Interestingly, since $V \geq 0$ and $\boldsymbol{\alpha} \geq 0$, Corollary 2 implies that $\boldsymbol{q}^* \propto -\Lambda$.

4. Changing the Structure of Competition

This section explores the effects on the firms' profits and consumer welfare of changes in the structure of competition among the firms, i.e., changes in graph G. In particular, we study changes in welfare when a firm enters a new market as well as when two firms merge and choose their production quantities with the goal of maximizing their joint profit. Our main focus is on highlighting the role of the underlying network structure and identifying how insights derived from the analysis of a single market differ due to significant network effects. For most of the section, we focus on the case where firms supply to all the markets they operate in at equilibrium. This is a natural assumption in this setting and allows us to express equilibrium quantities in closed form. Furthermore, it is a ubiquitous assumption in the literature that studies games on networks as it allows focusing on the interplay between the agents' strategic interactions and the underlying network structure (e.g., Bulow, Geanakoplos, and Klemperer (1985) and Bramoulle, Kranton, and D'Amours (2014)).

Next, we describe how firms adjust their production quantities in response to an infinitesimal shock dq_{ik} in q_{ik}, i.e., the production quantity that firm f_i supplies to market m_k. This shock may be the result of a change in market m_k, e.g., an increase in the market size or a tax break, or a change in the cost structure of firm f_i. Note that the requirement of the shock being sufficiently small is necessary in order to ensure that the set of active edges, i.e., firm-market pairs for which the equilibrium quantities are positive, remains unchanged in the equilibrium after the shock.

Proposition 1. *Consider an exogenous shock dq_{ik} in the quantity firm f_i supplies to market m_k. Then, in the new equilibrium firms adjust their production quantities according to the following expression*

$$dq_{j\ell} = \frac{\psi_{ik,j\ell}}{\psi_{ik,ik}} dq_{ik} \quad \forall (j,\ell) \in E, \tag{7}$$

where recall that $\Psi = [I + \gamma W]^{-1}$.

In particular, Proposition 1 implies that a change in the quantity firm f_i supplies to market m_k has ripple effects to the entire network, i.e., it affects the supply decisions of the entire set of competitors (even those that are not directly in competition with firm f_i). Note that since Ψ is a symmetric, positive semidefinite matrix with positive diagonal entries, according to Proposition 1 firms f_i and f_j view their actions in markets m_k and m_ℓ respectively as strategic complements or substitutes depending on the sign of $\psi_{ik,j\ell}$. The latter is equal to the difference of the sums of the weights of even and odd paths from link (i,k) to link (j,ℓ). Thus, matrix Ψ allows us to determine the level of strategic complementarity or substitutability between the actions corresponding to two firm-market pairs and relate it to supply paths in the network structure of competition. This result extends Bulow, Geanakoplos, and Klemperer (1985) by identifying the role of the structure of competition on whether firms view their actions as strategic substitutes or complements.

The following example clearly demonstrates the second order network effects associated with a change in a firm's output.

Example 1. Consider the game defined over the graph in Figure 3(a) and assume that the production quantity corresponding to edge $(1, A)$ decreases by ϵ_1 (e.g., due to small changes in firm 1's cost of production). Then, firm f_2 would respond by increasing its output in market A by ϵ_2. Next, consider the game defined over Figure 3(b) and assume again that the production quantity corresponding to edge $(1, A)$ decreases by ϵ_1. Then, clearly firm f_2 would increase its supply to market A. One would expect that the increase is smaller than ϵ_2 in this case due to the fact that firm f_2 is also supplying to market B (and production costs are convex). However, this is not the case: firm f_2 ends up responding more aggressively to the same change in firm f_1's output. In Figure 3(a), firm f_2 can increase its supply to market A only by increasing its production and hence its marginal cost whereas in Figure 3(b) firm f_2 can divert supply from market B to market A, without increasing its marginal cost.

It is worthwhile to note that this aspect of multi-market competition has been empirically demonstrated in Jans and Rosenbaum (1997). As they show for the U.S. cement industry, a firm's ability to divert production from a market that it has *"enough control over the price"* to another market allows the firm to respond more aggressively to its competitors.

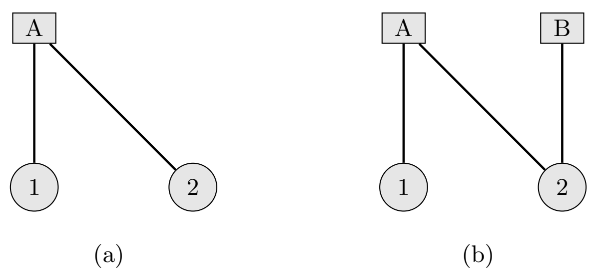


Figure 3 **A change in firm f_1's production quantity leads to different responses from f_2.**

Thus, as Example 1 illustrates, the response from a firm's competitors to changes in the quantity it supplies to any of its markets may be amplified relative to the case of a single-isolated market due to network effects.

4.1. Expanding into a New Market

This section explores the question of how equilibrium production quantities change when firm f_i enters market m_k which in our setting is equivalent to adding edge (i, k) to graph G. Entry has a direct effect on firm f_i's profit as the firm has to adjust the allocation of its production to the markets it supplies to. In addition, there is a second order effect that relates to how changes in firm f_i's production quantities across its markets affect the actions of its competitors and their propagation through the network.

The following example illustrates how the competition structure among the firms may affect the profits associated with entry and consequently determine whether entry is beneficial for the firm.

Example 2. Let firm f_1 enter market B for the two network structures illustrated in Figure 4. Then, in the first case (Figure 4(a)) it is profitable for firm f_1 to enter market B, whereas in the second (Figure 4(b)) it is not. This is due to the fact that when firm f_1 enters market B, it shifts part of its production to B and thus decreases its supply to market A. Firm f_2, on the other hand, responds to a decrease in the level of competition in A by increasing its supply to this market which results in a decrease in the profit for firm f_1 in market A. The increase in f_2's output in A depends on the level of strategic complementarity between edges (f_1, B) and (f_2, A). Utilizing the results from Section 3, we obtain that firm f_2 responds more aggressively, i.e., increases its supply to market A by a higher amount, in the network of Figure 4(b) and as a consequence entry is *less* profitable for f_1 in this network (for the parameters of the example it is actually *not* profitable for firm f_1 to enter market B). It is important to note that the difference between the two networks depicted in Figure 4 is *not local* for firm f_1. It is actually straightforward to extend this example in such a way that the difference between the two networks is arbitrarily far (in terms of network distance) from firm f_1, i.e., it involves firms and markets that seemingly should not affect firm f_1's profits. Thus, one has to take into account the *entire* network topology in order to determine the effect on the firm's profits of expanding into a new market. Interestingly, if the economy takes the form of a chain (e.g., Figure 4) then the profits associated with expanding on a new market decrease with the length of the chain.

(a) (b)

Figure 4 **In both games $\boldsymbol{\alpha} = [0.5, 1, 1]^T$, $\boldsymbol{\beta} = \mathbf{1}$, and $\boldsymbol{c} = \mathbf{1}$. In (a) adding edge $(1, B)$ leads to an increase in firm f_1's aggregate profit, however in (b) the profit decreases.**

The remainder of this subsection expands on the discussion above and provides a characterization of the change in firm f_i's profits when it enters market m_k, i.e., edge (i,k) is added to graph G. We restrict attention to edges (i,k) that have positive marginal profit for firm f_i. As one would expect, this is without any loss of generality according to Lemma 4 below.

Lemma 4. *If edge (i,k) has a negative marginal profit for $\boldsymbol{q}^*$, i.e.,*

$$\mathcal{P}_k = \alpha_k - \beta_k \Big(\sum_j q^*_{jk} \Big) < 2c, \tag{8}$$

then the equilibrium does not change if firm f_i enters market m_k.

The lemma follows directly from the fact that the optimality conditions for the actions of all agents remain unchanged if the new edge (i,k) is such that inequality (8) holds and thus the equilibrium remains the same.

Proposition 2 below provides a characterization of how a firm's profits change as a result of the firm entering a new market.

Proposition 2. *Consider firm f_i entering market m_k and let q'_{ik} denote the production quantity that f_i supplies to m_k in the resulting post-entry equilibrium. Then,*

(i) the aggregate production of firm f_i in the post-entry equilibrium is higher than its production in the pre-entry equilibrium.

(ii) the quantity firm f_i supplies to each of its existing markets (other than market m_k) is lower in the post-entry equilibrium.

(iii) if the set of active links remains the same, then firm f_i's profits change as follows

$$\Delta\pi_i = \underbrace{q'_{ik}(\mathcal{P}_k + \Lambda'_{ik,k}q'_{ik})}_{*} + \underbrace{q'_{ik}\sum_{\ell\in F_i}\left(\frac{\psi'_{i\ell,ik}}{\psi'_{ik,ik}}\mathcal{P}_\ell + \Lambda'_{ik,\ell}\left(\frac{\psi'_{i\ell,ik}}{\psi'_{ik,ik}}q'_{ik} + q_{i\ell}\right)\right)}_{\bullet} - \underbrace{\Delta\mathcal{C}_i}_{\ddagger}, \tag{9}$$

where $\Psi' = [I+\gamma W']^{-1}$, $\Lambda'_{ik,\ell} = -\beta\sum_{j\in M_\ell}\frac{\psi'_{j\ell,ik}}{\psi'_{ik,ik}}$ for every market m_ℓ, and W' denotes the $(|E|+1)\times(|E|+1)$ weight matrix defined over graph $G\cup(i,k)$.

Proposition 2 states that the effect of entry on firm f_i's profits can be decomposed in three terms. First, term $(*)$ illustrates the *direct effect* of entry as $q'_{ik}(\mathcal{P}_k + \Lambda'_{ik,k}q'_{ik})$ is equal to the profit that firm f_i obtains in market m_k (since the price at market k in the post-entry equilibrium is equal to $\mathcal{P}_k + \Lambda'_{ik,k}q'_{ik}$). Note that when firm f_i enters market m_k its competitors respond by decreasing the production quantities they supply to m_k. However, in total the price in market m_k decreases in the post-entry equilibrium and this is summarized by the fact that $\Lambda'_{ik,k} < 0$. Larger values of $\Lambda'_{ik,k}$ indicate that firm f_i obtains a larger direct profit when entering market m_k.[7]

On the other hand, term $(\bullet)$ captures the change in firm f_i's profits from its operations in markets other than m_k. Note that according to part (ii), firm f_i decreases its supply to each of its existing markets, i.e., markets other than m_k, and its profits from supplying to those markets are lower in the post-entry equilibrium. Thus, term $(\bullet)$ is always negative and its exact value depends on how aggressively firm f_i's competitors respond to the event of entry by increasing their production to markets other than m_k.

The extent to which competitors respond to the event of firm f_i entering market m_k is encapsulated in the price-impact entry $\Lambda'_{ik,\ell}$ for $m_\ell \in F_i$. A small value for $\Lambda'_{ik,\ell}$ implies that firms respond aggressively in market m_ℓ. This may affect firm f_i's profit adversely, and thus make entry less profitable than a setting where network effects were absent.[8] For instance, entry for firm f_1 to

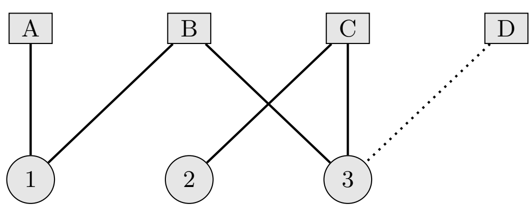
[7] Large values of $\Lambda'_{ik,k}$ indicate that firm-market pair $f_i - m_k$ has a high level of strategic substitutability with the rest of the links supplying to market m_k.

[8] Small values for $\Lambda_{ik,\ell}$, $\ell \neq k$ imply high degree of strategic complementarity between the quantity firm f_i supplies to market m_k and the quantities f_i's competitors supply to market m_ℓ.

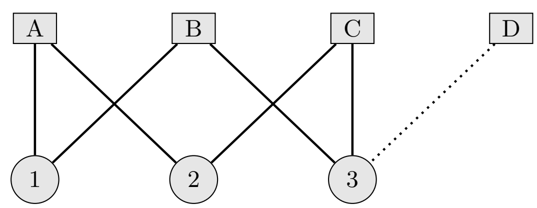
market B is profitable for the example depicted in Figure 4(a) whereas it is unprofitable for the one in Figure 4(b). The difference between the two examples is edge (f_2, C) which creates an additional path of even length between edges (f_1, B) and (f_2, A), thus increasing the level of strategic complementarity between them (and leading to a more aggressive response from f_2 in Figure 4(b)).

Finally, term (‡) is equal to the difference in production costs before and after entry and as we show in the first part of Proposition 2 it is always positive, since the aggregate quantity that firm f_i supplies to the markets increases in the post-entry equilibrium.

The following example based on Figure 5 further illustrates the intuition behind Proposition 2. In this example, the difference between the two networks is a single edge, i.e., the one connecting firm f_2 with market A. Edge (f_2, A) increases the level of strategic complementarity between (f_2, C) and (f_3, D), however it decreases the level of strategic complementarity between (f_1, B) and (f_3, D)[9]. Market B is much larger than C $(\alpha_2 > \alpha_3)$and thus the latter effect dominates the former making it profitable for firm f_3 to enter market D only for the setting of Figure 5(b).



(a)



(b)

Figure 5 For both examples, $\boldsymbol{\alpha} = [1, 4, 1, 1.8]^T$, $\boldsymbol{\beta} = [1, 2, 1, 1]^T$, and $\boldsymbol{c} = [1, 1, 1]^T$. Entering market D is not profitable for firm f_3 for the setting in Figure 5(a), whereas it is profitable for the one in Figure 5(b).

Having established that entering a new market may not be profitable for a firm due to an aggressive response from its competitors in the rest of the firm's markets, a natural question that arises is with regards to how entry may affect consumer surplus and aggregate welfare. First, let us formally define aggregate consumer surplus in this environment.

DEFINITION 2. The aggregate consumer surplus for game $\mathcal{CG}(\boldsymbol{\alpha}, \beta, c, G)$ is defined as the sum of the consumer surplus in all the markets

$$CS \triangleq \sum_{k=1}^{m} \frac{(\alpha_k - \mathcal{P}_k)^2}{2\beta}.$$

[9] Edge (f_2, A) creates a path of odd length between edges (f_3, D) and (f_1, B): $(f_3, D) \to (f_3, C) \to (f_2, C) \to (f_2, A) \to (A, f_1) \to (f_1, B)$.

In a single Cournot market adding a new competitor is always beneficial for consumers. However, as the example in Figure 6(c) illustrates this is no longer true in a multi-market competition setting as increasing the level of competition (by having firm 2 compete in market A) leads to a decrease in the aggregate consumer surplus. The reason behind this can be roughly explained as follows: a new edge (i, k) may "spread" the competition along the network structure, i.e., firms shift (part of) their production away from an area of the network where consumers benefit from intense competition to an area where competition is less intense.

Proposition 3 below provides a characterization of how consumer surplus changes in the event of firm f_i entering market m_k. The result follows by noting that if we add edge (i, k) to G, and the set of active links remains the same then the price in market ℓ in the resulting equilibrium is given by

$$\mathcal{P}'_\ell = \mathcal{P}_\ell + \Lambda'_{ik,\ell} q'_{ik}.$$

So the price in any market m_ℓ with positive $\Lambda'_{ik,\ell}$ increases, whereas the price in the rest of the markets decrease.

Proposition 3. *Consider firm f_i entering market m_k and let q'_{ik} denote the production quantity that f_i supplies to m_k in the resulting equilibrium. If the set of active links remains the same, then the aggregate consumer surplus changes as follows*

$$\Delta CS = -\frac{q'_{ik}}{2\beta} \sum_{\ell=1}^{m} \Lambda'_{ik,\ell} \big(2(\alpha_\ell - \mathcal{P}_\ell) - \Lambda'_{ik,\ell} q'_{ik}\big). \tag{10}$$

Proposition 3 implies that if link (i, k) has a positive price-impact in markets where the difference between market size and price, i.e. $\alpha_\ell - \mathcal{P}_\ell$, is large then increasing firm f_i's production quantity in market m_k might lead to a decrease in the aggregate consumer surplus.

A natural question that arises at this point is with regards to the competition structure that maximizes social welfare defined as the sum of the firms' profits and the aggregate consumer surplus. The preceding discussion illustrates that standard intuition from single Cournot markets no longer applies and increasing the extent to which firms compete with one another may actually lead to a decrease in aggregate consumer surplus and welfare.

However, we are able to show that in a symmetric economy, i.e., when all markets have the same size, the network that maximizes social welfare is complete. This is no longer true though when markets have different sizes as then a social planner may find a sparser structure optimal.

Proposition 4 states that in a symmetric economy the complete network maximizes aggregate social welfare.

Proposition 4. *In a symmetric economy, i.e., when firms and markets are symmetric, the complete network maximizes social welfare.*

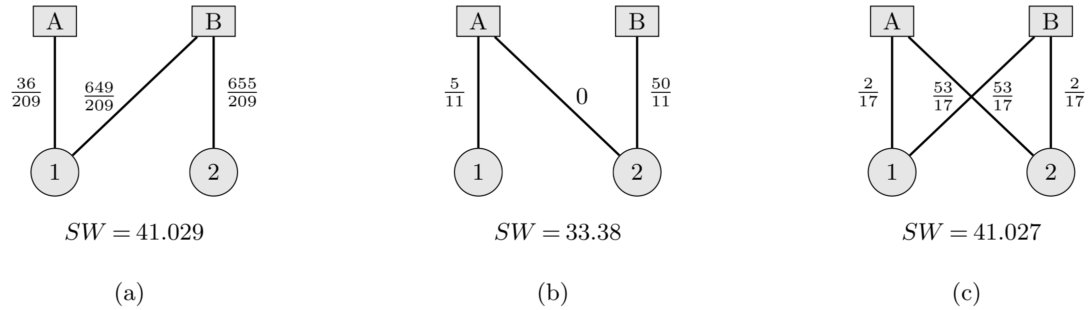


Figure 6 In this example, there are two firms and two markets where $\alpha_A = 1$ and $\alpha_B = 10$. Also $\beta = 1$ for both markets and $c = 0.1$ for both firms.

Finally, the example in Figure 6 illustrates that when the economy is asymmetric, then the complete network may not maximize social welfare as the social planner finds it optimal to direct competition to the markets with large size.

4.2. Horizontal Mergers

This subsection studies horizontal mergers between firms in the networked environment described in Section 2. First, we study the effects of the merger on third parties and then analyze how the firms participating in the merger, the "insiders",[10] coordinate their production levels and supply decisions. Our analysis illustrates that the impact of a merger is shaped by the network structure of the competition among the firms and that the intuition we have from the analysis of markets in isolation does not hold true.

As the effect on the insiders' profits should presumably be positive (otherwise the firms would have no incentive to initiate the merger), the analysis is mostly concerned with the profits of outsiders and the welfare of consumers in the equilibrium that is established after the merger. Farrel and Shapiro (1990) study the same question in a single Cournot market and provide general conditions under which mergers that are profitable for insider firms also lead to an increase in aggregate welfare. However, their results do not generalize in a networked environment.

In particular, much of the antitrust analysis in real-world markets is centered around changes in the level of market concentration that can be attributed to the merger. A reasonable way to extend the analysis to a networked environment is to consider each of the markets in which both firms participate and conclude that a merger should be allowed when the predicted change in concentration in any of those markets is not too high. However, such an approach would essentially treat each market in isolation and potentially overlook (significant) second order network effects. Example 3 below illustrates this pitfall.

[10] We borrow this terminology from Farrel and Shapiro (1990).

Example 3. In the three market environment depicted in Figure 7, considering market A in isolation (which is the only market that both insider firms supply to) would likely lead to a favorable response regarding a potential merger between firms f_3 and f_4, since the market is sufficiently competitive. However, this reasoning is somewhat misleading. Firms f_5 and f_6 would react to less (more) aggressive competition in markets A (C) respectively and potentially create a captive market in B. This second order network effect illustrates that considering each market in isolation may be incomplete and motivates our discussion on mergers in a networked environment.

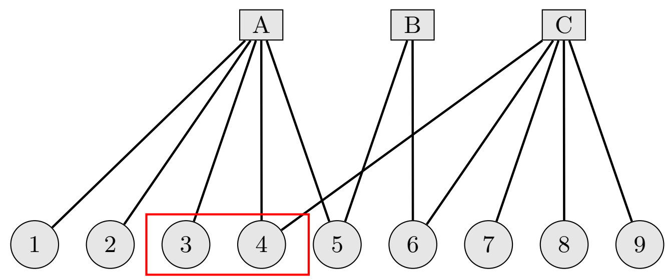


Figure 7 Let $c=1$, $\beta=1$, $\alpha_A=1$, $\alpha_B=0.3$, and $\alpha_C=1$. Firms f_5 and f_6 supply the same quantity to market B before the merger due to symmetry. If firms f_3 and f_4 merge then (i) their joint production decreases in market A; (ii) f_4 increases its production in C; (iii) f_6 shifts a fraction of its production from C to B; (iv) f_5 finds that market A is more profitable than B. Thus, although consumer welfare in markets A and C does not decrease substantially, competition in market B is significantly lower in the post-merger equilibrium and thus the overall effect of the merger on welfare may be negative.

As should be evident from the example above, measuring the overall effect of a merger on total welfare is not a straightforward task when firms compete across several markets. One would potentially need to study how changes in firms' actions propagate across the network. The goal in the remainder of this section is to provide some insights towards this direction. First, following Farrel and Shapiro (1990), we impose no assumptions on how a merger affects the insider firms, i.e., their production costs, as this is typically hard to observe or predict. Instead, Proposition 5 provides an expression for the change in the production quantities of outsider firms in response to a given change in the output of insiders. Let $\mathcal{I}$ denote the set of insider firms and assume that their merger results in a change of their total output in market m_k given by $\Delta q_{\mathcal{I},k}$. Moreover, let $\mathcal{O}$ and $G^{\mathcal{O}} = G \setminus \mathcal{I}$ denote the rest of the firms (the outsiders), and their subnetwork respectively. Also let $W^{\mathcal{O}}$ be equal to $W_{E(G^{\mathcal{O}}),E(G^{\mathcal{O}})}$, i.e., the sub-matrix of W corresponding to the rows and columns of the outsiders links.

Proposition 5. *Assume that insider firms, i.e., firms in set $\mathcal{I}$, change the total output they supply to market m_ℓ by $\Delta q_{\mathcal{I},\ell}$. Then, the production quantity that outsider firm f_i supplies to market*

m_k changes as follows in the post-merger equilibrium as long as the set of active edges remains the same

$$\Delta q_{ik} = -\frac{\beta}{2(c+\beta)} \sum_{m_\ell \big| m_\ell \in F_n \text{ for } n \in \mathcal{I}} \; \sum_{j \in \mathcal{O} \text{ and } j \in M_\ell} \psi^{\mathcal{O}}_{ik,j\ell} \Delta q_{\mathcal{I},\ell},$$

where $\Psi^{\mathcal{O}} = [I + \gamma W^{\mathcal{O}}]^{-1}$.

Proposition 5 provides a relation between the post-merger production quantities of the insider firms, i.e., the firms that participate in the merger, and those of the outsider firms. This relation can be helpful for assessing the overall effect of a merger on welfare as it provides a closed form expression for the changes in both prices and market concentration[11]. Concretely, the regulator can use this relation to provide a set of constraints on the post-merger equilibrium supply of insider firms that any merger has to satisfy. For instance, such constraints were imposed in the merger between US-airways and American Airlines in 2013.[12] In particular, the Department of Justice, as a condition for allowing the merger, required that the two airlines gave up landing and takeoff slots and gates at *"seven key constrained airports."*, thus effectively limiting their post-merger presence in those airports. The slots were to go to low cost airlines, *"resulting in more choices and more competitive airfares for consumers."*. We strongly believe that Proposition 5 can be useful in such a setting as it allows one to quantify the effects of a merger taking also the network interactions into account.

Finally, for the remainder of this section we consider the benchmark setting in which mergers do not result in cost synergies between the two insider firms f_i and f_j, i.e., their production functions do not change after the merger. Rather they benefit from jointly deciding how much to supply to the markets they participate to maximize their aggregate profit, denoted by $\pi_{ij}(\cdot)$, i.e.,

$$\pi_{ij}(\hat{\boldsymbol{q}}) = \pi_i(\hat{\boldsymbol{q}}) + \pi_j(\hat{\boldsymbol{q}}),$$

where $\hat{\boldsymbol{q}}$ denotes the vector of post-merger equilibrium production quantities. This formulation makes the analysis equivalent to the case of a multi-unit firm, owning more than one production plants. Firms of this nature are common and they are the natural results of mergers in several industries, including cement and concrete (Hortaçsu and Syverson (2007)).

Moreover, we focus on the (more challenging) case when firms f_i and f_j can only supply to the markets they originally could supply to, i.e., firm f_i can only produce for its original markets irrespective of the markets firm f_j supplies to. This formulation of a merger as a "change in

[11] In particular, following Farrel and Shapiro (1990), this proposition enables us to study the *external* effect of a merger on consumers and outsider firms.

[12] For more information, see http://www.justice.gov/opa/pr/2013/November/13-at-1202.html.

ownership" of the participating firms is a better fit for regional industries with high transportation costs, e.g., gas, electricity, and cement. Note that the complementary case that firms can supply to each other's markets is simpler to analyze and we comment on how results differ in the two settings as we describe our results.

Note that the post-merger payoff structure is different from our original framework. In particular, the firm that results from the merger between f_i and f_j chooses its production quantities in order to maximize $\pi_{ij}(\cdot)$. Thus, Proposition 1 does not apply directly to guarantee equilibrium uniqueness. In fact, there are examples where the number of equilibria may be infinite. However, we can show that an equilibrium always exists and it is generically unique. Moreover, when there are multiple equilibria, they are all equivalent in the sense that they result in the same prices for all the markets as well as the same profits for all the firms.

Definition 3. Post-merger equilibria $\boldsymbol{q}$ and $\boldsymbol{q'}$ are called equivalent if and only if for every market m_k the following holds

$$q_{ik} + q_{jk} = q'_{ik} + q'_{jk},$$

where i, j are the insider firms.

Note that given the aggregate supply of insider firms in a market the response of outsider firms is unique. Therefore, Definition 3 implies that if two post-merger equilibria $\boldsymbol{q}$ and $\boldsymbol{q}'$ are equivalent, then for any outsider firm f_t and every market m_k we have $q_{tk} = q'_{tk}$. The following theorem extends our equilibrium existence and uniqueness results from Section 3 to the setting where firms f_i, f_j merge their operations, i.e., they choose their production quantities so as to maximize their joint profit.

Theorem 2. *Suppose that firms f_i and f_j merge and their joint profit is given by*

$$\pi_{ij}(\hat{\boldsymbol{q}}) = \pi_i(\hat{\boldsymbol{q}}) + \pi_j(\hat{\boldsymbol{q}}).$$

Then the following hold,

(i) A post-merger equilibrium always exists.

(ii) If f_i and f_j do not share a market, then the post-merger equilibrium is unique and coincides with the pre-merger equilibrium.

(iii) If $\lambda_{\min}(W) \neq -(2c+\beta)$ the post-merger equilibrium is unique.

(iv) If multiple equilibria exist, they are all equivalent.

Given that the post-merger equilibrium exists and it is essentially unique, our goal in the remainder of this section is to provide a characterization of its properties. In particular, we show that it is equivalent to the equilibrium of a game that has the form described in Section 2 and it is directly related to the original pre-merger game. Specifically, as we formally state in Proposition 6 it turns out that the post-merger equilibrium can be computed as the equilibrium of one of two distinct games (see also Figure 8 for illustrations of both types of post-merger equilibria).

PROPOSITION 6. *Any post merger equilibrium is equivalent to the equilibrium of one of the following two games:*

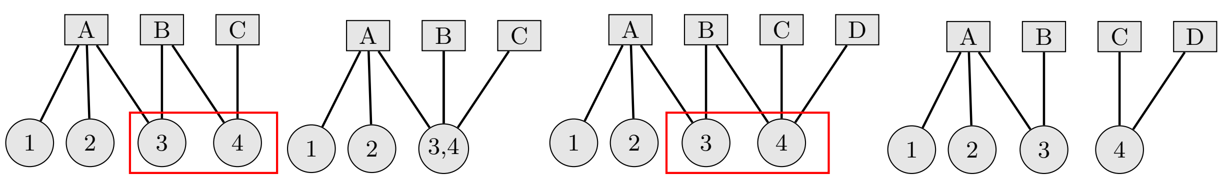
(i) Firms f_i and f_j can be thought of a single firm f_{ij} (single node in the competition graph) that is connected to the union of the markets that f_i and f_j were originally connected to. The cost function of f_{ij} is given by the following expression

$$\mathcal{C}_{ij}(x) = 2\mathcal{C}\left(\frac{x}{2}\right).$$

(ii) Firm f_i and f_j operate so as to maximize their own profit (as in the pre-merger environment) and the competition graph is given as

$$G = \{F \cup M, E - \{(f_i, m_k) \big| (f_i, m_k) \wedge (f_j, m_k) \in E\}\},$$

i.e., the competition graph is the same as the original G without the links from f_i to the markets it shares with f_j.



(a) Pre-merger (b) Post-merger (c) Pre-merger (d) Post-merger

Figure 8 For this example $\alpha = \beta = c = 1$. The equilibria of the mergers corresponding to Figures 8(a), 8(c) are the same as the equilibria of the games in Figures 8(b), 8(d).

Proposition 6 highlights that firms after a merger will necessarily behave in one of two distinct ways. In particular, part (i) of the Proposition describes a post-merger equilibrium outcome where the insider firms share the production equally and can be thought of as a single firm that has access to the union of the markets the original firms had access to. This case arises when the market access for the insider firms is similar in the pre-merger economy and they benefit from splitting their aggregate production to minimize their joint cost. Part (ii) instead states that when the firms have disparate market access, the post-merger equilibrium is equivalent to the outcome of the competition where the insider firms act independently but only one of them has access to the markets they share in the original network. The optimality of segregating their original markets is due to the difference in their supply connections. When one of the firms supplies to many markets whereas the other only to a few, it is optimal to have only the latter supply to the markets which they both participate in. The example in Figure 8 demonstrates the two cases; 8(a) where the insider firms have the same number of connections and 8(c) where one supplies to more markets than the other. This is a nice characterization result that allows us to compute the supply quantities, profits, and welfare associated with the post-merger equilibrium in a straightforward way given our characterization results from Section 3.

5. Conclusion

This paper studies a model of competition in a networked environment. A bipartite graph determines the set of potential supply relationships. We provide a characterization of the unique equilibrium that highlights the relation between production quantities and supply paths in the underlying network structure. Using this characterization we derive several comparative statics results regarding the effect on quantities, prices, and welfare of changes in the network structure that may be the result of a firm expanding into a new market or a merger between two firms. Our results illustrate that qualitative insights from the analysis of a single market do not generalize when firms compete with one another in multiple markets. The modeling framework we propose nicely complements the extensive empirical work that establishes the relevance of multi-market competition in firms' decision making and it may have significant implications in assessing whether expanding in a new market is profitable for a firm, identifying opportunities for collaboration, e.g., a merger, strategic alliance, or acquisition,[13] between competing firms, and guiding regulatory action in the context of market design and antitrust analysis.

[13] For a recent contribution on the study of revenue sharing in airline alliances refer to Hu et al. (2013). Incorporating an explicit competition structure and looking at alliances other than the grand coalition is a fruitful direction for future research.

Appendix : Proofs

Proof of Lemma 1

First, note that each agent's action space is convex, bounded, and closed. Second, we show that when the strategies of other firms are fixed to $\boldsymbol{q}_{-i}$, then π_i is concave in $\boldsymbol{q}_i$. This follows by showing that the Hessian matrix below is negative semi-definite. Define the $|F_i| \times |F_i|$ Hessian matrix H_i as follows:

$$H_i = \begin{bmatrix} \frac{\partial^2 \pi_i}{\partial q_{i,1}, \partial q_{i,1}} & \frac{\partial^2 \pi_i}{\partial q_{i,1} \partial q_{i,2}} & \cdots & \frac{\partial^2 \pi_i}{\partial q_{i,1} \partial q_{i,m}} \\ & \vdots & \vdots & \\ \frac{\partial^2 \pi_i}{\partial q_{i,m}, \partial q_{i,1}} & \frac{\partial^2 \pi_i}{\partial q_{i,m}, \partial q_{i,2}} & \cdots & \frac{\partial^2 \pi_i}{\partial q_{i,m}, \partial q_{i,m}} \end{bmatrix}.$$

Also define the $|F_i| \times (m+1)$ matrix V as follows:

$$V_{ik,\ell} = \begin{cases} \sqrt{-\left(2\mathcal{P}'_k\left(\sum_{j\in M_k} q_{jk}\right) + q_{ik}\mathcal{P}''\left(\sum_{j\in M_k} q_{jk}\right)\right)} & \text{if } \ell = k \\ \sqrt{\mathcal{C}''\left(\sum_{j\in F_i} q_{ij}\right)} & \text{if } \ell = m+1 \\ 0 & \text{otherwise.} \end{cases}$$

Then, we obtain that

$$H_i = -VV^T,$$

and one can conclude that H_i is negative semi-definite. Thus, this is a concave game and has an equilibrium according to Rosen (1965). To prove uniqueness, it is sufficient to show that the $|E| \times |E|$ matrix $[G(x,r) + G^T(x,r)]$ defined below is negative definite for every x in the action space and some fixed and positive r. Let $r = 1$ for the remainder of the proof. Then, for every vector of actions $\mathbf{q}$ we have

$$G(\mathbf{q},1) = \begin{bmatrix} \frac{\partial^2 \pi_1}{\partial q_{1,1} \partial q_{1,1}} & \frac{\partial^2 \pi_1}{\partial q_{1,1} \partial q_{1,2}} & \cdots & \frac{\partial^2 \pi_1}{\partial q_{1,1} \partial q_{n,m}} \\ & \vdots & \vdots & \\ \frac{\partial^2 \pi_n}{\partial q_{n,m} \partial q_{1,1},} & \frac{\partial^2 \pi_n}{\partial q_{n,m} \partial q_{1,2}} & \cdots & \frac{\partial^2 \pi_n}{\partial q_{n,m} \partial q_{n,m}} \end{bmatrix}.$$

Each row of $G(\mathbf{q},1)$ corresponds to an edge in the original competition graph. The entries corresponding to edge (i,k) are equal to the cross derivatives of π_i with respect to q_{ik} and $q_{j\ell}$ for all $(j,\ell) \in E$, i.e.,

$$\frac{\partial^2 \pi_i}{\partial q_{ik} \partial q_{j\ell}} = \begin{cases} 2\mathcal{P}'_k\left(\sum_{j\in M_k} q_{jk}\right) - \mathcal{C}''_i\left(\sum_{j\in F_i} q_{ij}\right) + q_{ik}\mathcal{P}''_k\left(\sum_{j\in M_k} q_{jk}\right) & \text{if } i=j \text{ and } k=\ell \\ -\mathcal{C}''_i\left(\sum_{j\in F_i} q_{ij}\right) & \text{if } i=j \text{ and } k\neq\ell \\ \mathcal{P}'_k\left(\sum_{j\in M_k} q_{jk}\right) + q_{ik}\mathcal{P}''_k\left(\sum_{j\in M_k} q_{jk}\right) & \text{if } i\neq j \text{ and } k=\ell \\ 0 & \text{otherwise.} \end{cases}$$

Let $\eta_k \equiv -\mathcal{P}'_k\left(\sum_{j\in M_k} q_{jk}\right)$ and $2\theta_i \equiv \mathcal{C}''_i\left(\sum_{j\in F_i} q_{ij}\right)$ and define $|E| \times |E|$ matrices Ω and Φ as follows

$$\Omega_{ik,j\ell} = \begin{cases} 2(\eta_k + \theta_i) & \text{if } i=j, k=\ell \\ 2\theta_i & \text{if } i=j, k\neq l \\ \eta_k & \text{if } i\neq j, k=\ell \\ 0 & \text{otherwise.} \end{cases} \quad \text{and } \Phi_{ik,j\ell} = \begin{cases} (q_{ik}+q_{jk})\mathcal{P}''_k\left(\sum_{j\in M_k} q_{jk}\right) & \text{if } k=\ell \\ 0 & \text{otherwise.} \end{cases}.$$

Then, $G(\boldsymbol{q},1) + G^T(\boldsymbol{q},1) = -2\Omega + \Phi$. So to complete the proof, it is sufficient to show that Ω is positive definite and Φ is negative semi-definite. We show the positive definiteness of Ω by providing a full rank matrix R such that $\Omega = R^T R$. In particular, let R be a $(|E|+m+n)\times|E|$ matrix that can be written in the form of a block matrix as follows

$$R = \begin{bmatrix} A \\ B \end{bmatrix},$$

where A is an $|E|\times|E|$ diagonal matrix with

$$A_{ik,j\ell} = \begin{cases} \sqrt{\eta_k} & \text{if } i=j, k=\ell \\ 0 & \text{otherwise,} \end{cases}$$

and B is an $(m+n)\times|E|$ matrix with

$$B_{t,(i,k)} = \begin{cases} \sqrt{2\theta_i} & \text{if } t \le n, t = i \\ \sqrt{\eta_k} & \text{if } t > n, t = n+k \\ 0 & \text{otherwise.} \end{cases}$$

Matrix R is full rank since A is a diagonal matrix with non-zero entries on its diagonal. It is also straightforward to check that $\Omega = R^T R$. This implies that Ω is positive-definite. Furthermore, matrix Φ can be decomposed as follows

$$\Phi = \Phi^1 + \Phi^2 + \cdots + \Phi^m,$$

where Φ^k is a matrix that corresponds to market m_k. For every two edges (i,k) and (j,k) we have

$$\Phi^k_{ik,jk} = q_{ik}\mathcal{P}_k^{''}\Big(\sum_{j\in M_k} q_{jk}\Big) + q_{jk}\mathcal{P}_k^{''}\Big(\sum_{j\in M_k} q_{jk}\Big),$$

and so

$$\Phi^k = \mathcal{P}_k^{''}\Big(\sum_{j\in M_k} q_{jk}\Big)\left(\boldsymbol{q}_k \mathbf{1}_k^T + \mathbf{1}_k \boldsymbol{q}_k^T\right),$$

where $\boldsymbol{q}_k$ is a $|E|\times 1$ vector, with $q_{k_{j\ell}} = q_{j\ell}$ if $\ell = k$ and 0 otherwise. Moreover, $\mathbf{1}_k$ is a $|E|\times 1$ vector, with$1_{k_{j\ell}} = 1$ if $\ell = k$ and 0 otherwise. Thus, given that $\mathcal{P}(\cdot)$ is concave, matrix Φ^k is negative semi-definite for every k and this completes the proof. Q.E.D.

Proof of Lemma 2

First, note that the marginal profit associated with a firm-market pair has to be non-positive at equilibrium, i.e.,

$$\frac{\partial \pi_i}{\partial q_{ik}}\Big|_{q^*_{ik}} = \alpha_k - \beta_k q^*_{ik} - \beta_k \sum_{j\in\mathcal{M}_k} q^*_{jk} - 2c_i \sum_{l\in\mathcal{F}_i} q^*_{i\ell} \le 0. \tag{11}$$

This set of equations can be rewritten in matrix form as

$$-\bar{\boldsymbol{\alpha}} + D\boldsymbol{q}^* \ge 0, \tag{12}$$

where recall that D is a $|E|\times|E|$ matrix defined in Equation (1).

Second, every firm-market pair for which the corresponding production quantity is strictly positive at equilibrium has to satisfy Equation (11) with equality, i.e., if $q^*_{ik} > 0$ then $\frac{\partial \pi_i}{\partial q_{ik}}|_{q^*_{ik}} = 0$. So given that we assume that all edges in E are active at equilibrium, we obtain the following

$$\boldsymbol{q}^*(-\bar{\boldsymbol{\alpha}} + D\boldsymbol{q}^*) = 0. \tag{13}$$

Finally, the supply at a firm-market pair has to be non-negative at equilibrium

$$\boldsymbol{q}^* \geq 0. \tag{14}$$

Conditions (12), (13), and (14) constitute a linear complementarity problem $LCP(-\bar{\boldsymbol{\alpha}}, D)$. According to the results in Samelson, Thrall, and Wesler (1958), problem $LCP(-\bar{\boldsymbol{\alpha}}, D)$ has a unique solution if and only if all the principal minors of D are positive. Positive definite matrices satisfy this condition and thus what remains to be shown is that D is positive definite for every graph G, which follows by arguments similar to the ones we used for the proof of the positive definiteness of matrix Ω in Theorem 1. Thus, the equilibrium of game $\mathcal{CG}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{c}, G)$ can be characterized as the unique solution for $LCP(-\bar{\boldsymbol{\alpha}}, D)$. Q.E.D.

Proof of Lemma 3

Note that as we showed in Lemma 2 the equilibrium of game $\mathcal{CG}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{c}, G)$ is the unique solution of a linear complementarity problem. Furthermore, note that production quantities that take value zero do not have any effect in the linear complementarity problem, i.e., the solution of the LCP remains the same even when they are omitted. Thus, the production quantities that take strictly positive values in the solution of the linear complementarity problem corresponding to $\mathcal{CG}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{c}, G)$ take the same values in the solution of the linear complementarity problem corresponding to $\mathcal{CG}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{c}, G')$. Q.E.D.

Proof of Theorem 1

For every active edge (i,k) it should be the case that $\frac{\partial \pi_i(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{c}, G)}{\partial q^*_{ik}} = 0$, which implies that

$$q^*_{ik} = \frac{\alpha_k - 2c\sum_{\ell \in F_i, \ell \neq k} q^*_{i\ell} - \beta \sum_{j \in M_k} q^*_{jk}}{2(\beta + c)} = \frac{\alpha_k}{2(\beta + c)} - \sum_{(j,\ell) \in E(G^*)} (\gamma W)_{ik, j\ell} q^*_{j\ell}. \tag{15}$$

This further implies that

$$\boldsymbol{q}^* = \gamma \bar{\boldsymbol{\alpha}} - \gamma W \boldsymbol{q}^* \Rightarrow \boldsymbol{q}^* = [\mathrm{I} + \gamma W]^{-1} \gamma \bar{\boldsymbol{\alpha}}. \tag{16}$$

To show the second part of the theorem, first note that Expression (16) can be rewritten as

$$\boldsymbol{q}^* = [\mathrm{I} - (-\gamma W)]^{-1} \gamma \bar{\boldsymbol{\alpha}}.$$

Matrix $[\mathrm{I} - (-\gamma W)]^{-1}$ can be rewritten as the power series of matrix $(-\gamma W)$ if and only if the spectral radius of $(-\gamma W)$ is less than 1, i.e., if and only if

$$-1 < \lambda_{\min}(\gamma W) \leq \lambda_{\max}(\gamma W) < 1.$$

The final step in the proof involves showing that for every game $\lambda_{\min}(\gamma W) > -1$. To this end, define the $(n+m) \times |E|$ edge incident matrix B for graph G as follows

$$B_{v,(i_t,k_t)} = \begin{cases} \sqrt{\frac{2c}{2(c+\beta)}} & \text{if } v \leq n,\ i_t = v, \\ \sqrt{\frac{\beta}{2(c+\beta)}} & \text{if } v > n,\ k_t = v - n, \\ 0 & \text{otherwise.} \end{cases}$$

Then, it is straightforward to see that

$$\gamma W = B^T B - \frac{2c + \beta}{2(c+\beta)} \mathrm{I}.$$

Note that $B^T B$ is a positive semidefinite matrix and thus all of its eigenvalues are non-negative. Furthermore, $\frac{2c+\beta}{2(c+\beta)} < 1$ thus we conclude that

$$\lambda_{\min}(\gamma W) \geq -\frac{2c+\beta}{2(c+\beta)} > -1.$$

This concludes the proof since to be able to rewrite $[\mathrm{I} - (-\gamma W)]^{-1}$ as in Expression (1) it suffices that $\lambda_{\max}(\gamma W) < 1$. Q.E.D.

Finally, to complete the picture we state and prove Lemma 5 below that provides a sufficient condition for $\lambda_{\max}(\gamma W) < 1$. The condition essentially implies that graph G is sufficiently sparse. We chose to state the corollary for when the Cournot game is symmetric, but it readily extends to asymmetric games.

Lemma 5. *Consider a symmetric $\mathcal{CG}$ game. Then, $\lambda_{\max}(\gamma W) < 1$ if one of the following two conditions holds*

(i) The marginal cost of production is sufficiently low, i.e., $2c < \beta$. Furthermore, each market can have at most 2 suppliers and each firm supply to at most 3 markets.

(ii) The marginal cost of production is sufficiently high, i.e., $2c \geq \beta$. Furthermore, each firm can supply to at most 2 markets and each market can have at most 2 suppliers.

Proof: We provide a proof for the case where $2c < \beta$ (the proof for the other case is identical). Consider the line graph $\mathcal{L}(G)$. Without loose of generality we can assume that G is connected and as a result $\mathcal{L}(G)$ is also connected. Thus W is an irreducible matrix with non-negative entries, and by Perron-Frobenius theorem, we have

$$\lambda_{\max}(\gamma W) \leq \max_{(i,k)\in E} \gamma \sum_{(j,\ell)\in E} w_{ik,j\ell}.$$

Now consider the case where $2c < \beta$. In this case if each market have at most 2 supplies and each firm has at most 3 suppliers, one can see that for any link (i,k) we have $\gamma \sum_{(j,\ell)\in E} w_{ik,j\ell} \leq 2\frac{2c}{2(c+\beta)} + \frac{\beta}{2(c+\beta)} < 1$. Q.E.D.

Lemma 5 shows that sparsity is a sufficient condition to have $\lambda_{\max}(\gamma W) < 1$. In the following lemma we show that sparsity is also a necessary condition.

Lemma 6. *Consider a symmetric $\mathcal{CG}$ game. Then, $\lambda_{\max}(\gamma W) < 1$ only if one of the following two conditions holds*

(i) If the marginal cost of production is sufficiently high, i.e., $2c \geq \beta$ then each firm can supply to at most 10 markets and the total number of links connecting firms to markets is smaller than $4n$.

(ii) If the marginal cost of production is sufficiently low, i.e., $2c < \beta$ then, each market can have at most 5 suppliers and the total number of links connecting firms to markets is smaller than $3m$.

Proof: We provide a proof for the case when $2c \geq \beta$ (the proof for the case when $\beta > 2c$ is identical). Consider the line graph $\mathcal{L}(G)$ and remove the edges that have weight β (recall that edges in $\mathcal{L}(G)$ have weight equal to $2c$ or β). Let W' denote the adjacency matrix of the resulting matrix. Every edge in the remaining line graph has weight $2c$ so we have

$$W' = 2c \times H,$$

where H is a binary adjacency matrix. Note that $\lambda_{\max}(\gamma W') \le \lambda_{\max}(\gamma W) \le 1$, since removing edges always decreases the maximum eigenvalue of a matrix. Furthermore,

$$\lambda_{\max}(\gamma W') = \frac{2c}{2(c+\beta)}\lambda_{\max}(H) \ge \frac{1}{3}\lambda_{\max}(H).$$

Therefore, $\lambda_{\max}(\gamma W) < 1$ implies that $\lambda_{\max}(H) \le 3$. Finally, note that for every unweighted graph corresponding to adjacency matrix H we have

$$\max\left\{\sqrt{deg_{\max}(H)}, \overline{deg}(H)\right\} \le \lambda_{\max}(H),$$

where $deg_{\max}(H)$, $\overline{deg}(H)$ are the maximum and average degrees of the graph corresponding to H respectively. Thus, we obtain that $deg_{\max}(H) \le 9$. Finally, this implies that each firm can supply to at most 10 markets (since it can have at most 9 direct neighbors in the line graph $\mathcal{L}(G)$). Similarly, $\overline{deg}(H) \le 3$ implies that on average each firm can supply to at most 4 markets. This concludes the proof of the lemma.

Finally, Proposition 7 below confirms a basic feature of the equilibrium described in Theorem 1: equilibrium production corresponding to a firm-market pair increases (decreases) with the weights of even (odd) paths from the edge corresponding to the pair to any of the markets.

PROPOSITION 7. *Consider the unique Nash equilibrium $\boldsymbol{q}^*$ of game $\mathcal{CG}(\boldsymbol{a}, \boldsymbol{\beta}, \boldsymbol{c}, G)$. Then, the quantity firm f_i supplies to market m_k at equilibrium, i.e., q^*_{ik} is increasing (decreasing) with the weights of even (odd) paths from edge (i,k) to a market m_ℓ.*

We provide a proof for this proposition after the proof of Proposition 1 as we need a lemma that is stated as part of the proof of Proposition 1.

Proof of Corollary 2

According to Theorem 1, we have $\boldsymbol{q}^* = \left[\mathrm{I} + \gamma W\right]^{-1} \gamma \bar{\boldsymbol{\alpha}}$. So, for any link (i,k) we have:

$$\begin{aligned} q^*_{ik} &= \gamma \sum_{(j,\ell)\in E} \psi_{ik,j\ell}\alpha_\ell = \gamma \sum_{\ell \in M} \alpha_\ell \sum_{j\in M_\ell} \psi_{ik,j\ell} \\ &= -\frac{\gamma}{\beta}\psi_{ik,ik} \sum_{\ell\in M} \alpha_\ell \sum_{j\in M_\ell} -\beta \frac{\psi_{ik,j\ell}}{\psi_{ik,ik}} \\ &= -\frac{\gamma}{\beta}\psi_{ik,ik} \sum_{\ell\in M} \alpha_\ell \Lambda_{ik,\ell}. \end{aligned} \tag{17}$$

The claim follows by writing Equation (17) in a matrix form. Q.E.D.

Proof of Proposition 1

Consider an exogenous change dq_{ik} in the production quantity corresponding to link (i,k) and denote by $dq_{j\ell}$ the change in the production quantity corresponding to link (j,ℓ) in the new equilibrium. The first order optimality conditions imply the following equation for any link $(j,\ell) \ne (i,k)$

$$q_{j\ell} + dq_{j\ell} = \frac{\alpha_\ell}{2(\beta + c)} - \sum_{(j_1,\ell_1)\in E} (\gamma W)_{j\ell, j_1\ell_1}(q_{j_1\ell_1} + dq_{j_1\ell_1}). \tag{18}$$

By subtracting Equation (15) from Equation (18), we get the following equation:

$$dq_{j\ell} = -\sum_{(j_1,\ell_1)\in E\setminus(i,k)} (\gamma W)_{j\ell,j_1\ell_1} dq_{j_1\ell_1} - (\gamma W)_{j\ell,ik} dq_{ik}. \tag{19}$$

Let $\tilde{W} = W_{E\setminus(i,k),E\setminus(i,k)}$ and let $\boldsymbol{\zeta}$ denote a vector such that for every link $(j,\ell) \neq (i,k)$, we have $\zeta_{j\ell} = \gamma w_{j\ell,ik}$. Then, we can rewrite Equation (19) in a matrix form as follows:

$$d\boldsymbol{q} = -[\mathrm{I} + \gamma \tilde{W}]^{-1}\boldsymbol{\zeta} dq_{ik}. \tag{20}$$

Finally, let $\tilde{\Psi}' = [\mathrm{I} + \gamma\tilde{W}]^{-1}$. In order to make the calculations easier we define matrix $\tilde{\Psi}$ which constructed by attaching one additional row and column corresponding to link (i,k), to the matrix $\tilde{\Psi}'$. The entries of the new row and column are all zero except the entry on diagonal which is equal to 1. The following lemma relates matrices Ψ and $\tilde{\Psi}$.

LEMMA 7. *Let $\tilde{W} = W_{E\setminus(i,k),E\setminus(i,k)}$. Then, we have*

$$\Psi = \tilde{\Psi} + \frac{\boldsymbol{\Gamma}}{1-C}, \tag{21}$$

where

$$\Gamma_{j_1\ell_1,j_2,\ell_2} = \begin{cases} \left(\sum_{(j,\ell)\in E} \zeta_{j\ell}\tilde{\psi}_{j\ell,j_1\ell_1}\right)\left(\sum_{(j,\ell)\in E} \zeta_{j\ell}\tilde{\psi}_{j\ell,j_2\ell_2}\right) & \text{if } (j_1,\ell_1) \neq (i,k) \text{ and } (j_2,\ell_2) \neq (i,k), \\ -\sum_{(j,\ell)\in E} \zeta_{j\ell}\tilde{\psi}_{j\ell,j_2\ell_2} & \text{if } (j_1,\ell_1) = (i,k) \text{ and } (j_2,\ell_2) \neq (i,k), \\ -\sum_{(j,\ell)\in E} \zeta_{j\ell}\tilde{\psi}_{j\ell,j_1\ell_1} & \text{if } (j_1,\ell_1) \neq (i,k) \text{ and } (j_2,\ell_2) = (i,k), \\ C & \text{if } (j_1,\ell_1) = (i,k) \text{ and } (j_2,\ell_2) = (i,k), \end{cases} \tag{22}$$

and $C = \sum_{(j_1,\ell_1)\in E}\sum_{(j_2\ell_2)\in E} \zeta_{j_2\ell_2}\tilde{\psi}_{j_2\ell_2,j_1\ell_1}\zeta_{j_1\ell_1}$.

Proof: Note that we have:

$$\begin{aligned}
\Psi &= [\mathrm{I} + \gamma W]^{-1} \\
&= [\mathrm{I} + \gamma\tilde{W} + e_{ik}\boldsymbol{\zeta}^T + \boldsymbol{\zeta} e_{ik}^T]^{-1} \\
&= \left(\tilde{\Psi} - \frac{\tilde{\Psi}\boldsymbol{\zeta} e_{ik}^T\tilde{\Psi}}{1 + e_{ik}^T\tilde{\Psi}\boldsymbol{\zeta}}\right) - \frac{\left(\tilde{\Psi} - \frac{\tilde{\Psi}\boldsymbol{\zeta} e_{ik}^T\tilde{\Psi}}{1+e_{ik}^T\tilde{\Psi}\boldsymbol{\zeta}}\right) e_{ik}\boldsymbol{\zeta}^T\left(\tilde{\Psi} - \frac{\tilde{\Psi}\boldsymbol{\zeta} e_{ik}^T\tilde{\Psi}}{1+e_{ik}^T\tilde{\Psi}\boldsymbol{\zeta}}\right)}{1 + \boldsymbol{\zeta}^T\left(\tilde{\Psi} - \frac{\tilde{\Psi}\boldsymbol{\zeta} e_{ik}^t\tilde{\Psi}}{1+e_{ik}^T\tilde{\Psi}\boldsymbol{\zeta}}\right) e_{ik}} \\
&= \left(\tilde{\Psi} - \tilde{\Psi}\boldsymbol{\zeta} e_{ik}^T\tilde{\Psi}\right) - \frac{\left(\tilde{\Psi} - \tilde{\Psi}\boldsymbol{\zeta} e_{ik}^T\tilde{\Psi}\right) e_{ik}\boldsymbol{\zeta}^T\left(\tilde{\Psi} - \tilde{\Psi}\boldsymbol{\zeta} e_{ik}^T\tilde{\Psi}\right)}{1-C} \\
&= \tilde{\Psi} + \frac{\boldsymbol{\Gamma}}{1-C}
\end{aligned}$$

where the second equality follows from applying the Sherman-Morrisson formula twice.
Equation (21) implies that $\psi_{ik,ik} = 1 + \frac{C}{1-C} = \frac{1}{1-C}$. Also, again according to Equation (21), we obtain

$$\psi_{ik,j\ell} = \psi_{j\ell,ik} = \frac{\Gamma_{ik,j\ell}}{1-C} = \Gamma_{ik,j\ell}\psi_{ik,ik}. \tag{23}$$

Finally using Equations (20) and (23) we obtain

$$dq_{j\ell} = -\sum_{(j_1,\ell_1)\in E} \tilde{\psi}_{j\ell,j_1\ell_1}\zeta_{j_1\ell_1} dq_{ik} = \frac{\psi_{j\ell,ik}}{\psi_{ik,ik}} dq_{ik},$$

which concludes the proof of the Proposition. Q.E.D.

Proof of Proposition 7

Note that according to Theorem 1, at equilibrium $q^*_{ik} = \gamma \sum_{\ell \in M} \alpha_\ell \sum_{j \in M_\ell} \psi_{ik,j\ell}$. So in order to prove the Proposition, it is enough to show the following claim:

Claim. *In any network and for every two arbitrary links (i,k) and (j,ℓ), $\psi_{ik,j\ell}$ is increasing (decreasing) with the weights of even (odd) paths from edge (i,k) to edge (j,ℓ).*

Proof: We prove this claim by induction on the number of edges in the network. Obviously the claim holds for the trivial network with only one link. Now assume that the claim holds for any network with $|E|-1$ links, and we will prove it for a network with $|E|$ links. Remove an arbitrary link (i,k), and let $\tilde{W} = W_{E\setminus(i,k),E\setminus(i,k)}$, then according to Lemma 7 we have

$$\Psi = \tilde{\Psi} + \frac{\Gamma}{1-C},$$

where Γ is a matrix defined in equation (22). According to the induction hypothesis entries of $\tilde{\Psi}$ satisfy the claim. Also note that according to the proof of Lemma 7, $\psi_{ik,ik} = \frac{1}{1-C}$, and since Ψ is a positive definite matrix, and thus all the diagonal entries are positive, we should have $\frac{1}{1-C} > 0$. So it is enough to have the entries of Γ satisfy the claim as well i.e. for every two arbitrary links (i,k) and (j,ℓ), $\Gamma_{ik,j\ell}$ is increasing (decreasing) with the weights of even (odd) paths from edge (i,k) to edge (j,ℓ), which is obviously hold by the definition of Γ.

Proof of Lemma 4

Let $\boldsymbol{q}$ and $\boldsymbol{q}'$ denote the pre- and post-entry equilibrium respectively. Also recall that according to Lemma 2, the equilibrium of any Cournot game is the unique solution to $LCP(-\bar{\boldsymbol{\alpha}}, D)$. Then if $\alpha_k - \beta_k\left(\sum_{j\in M_k} q^*_{jk}\right) < 2c$, having $q'_{j\ell} = q_{j\ell}, \forall (j,\ell) \neq (i,k)$ and $q'_{ik} = 0$ is the solution for LCP and thus is an equilibrium. Q.E.D.

Proof of Proposition 2

We prove each part of the proposition separately. For each firm f_i we let s_i and s'_i denote the total production of firm f_i in the pre- and post-entry equilibrium respectively. Also for each market m_k we let d_k and d'_k denote the total quantity supplied to market m_k in the pre- and post-entry equilibrium.

Part (i) The following builds on the proof of Lemma 3.2 in Qiu (1991). Define sets M_1, F_1, M_2 and F_2 as follows:

$$M_1 \triangleq \{m_\ell : d'_\ell \leq d_\ell\}, \quad M_2 \triangleq \{m_\ell : d'_\ell > d_\ell\},$$

$$F_1 \triangleq \{f_j : s'_j \leq s_j\}, \quad F_2 \triangleq \{f_j : s'_j > s_j\}.$$

In order to simplify the exposition below, we define for every $i,k \in \{1,2\}$ the following quantities:

$$Q_{ik} \triangleq \sum_{j\in F_i}\sum_{\ell \in M_k} q_{j\ell}, \quad Q'_{ik} \triangleq \sum_{j\in F_i}\sum_{\ell \in M_k} q'_{j\ell}.$$

For the sake of contradiction assume that $f_i \in F_1$. Note that since for every $f_j \in F_1$ and $m_\ell \in M_1$ we have $q'_{j\ell} \geq q_{j\ell}$ [14] we obtain

$$Q'_{1,1} \geq Q_{1,1}. \tag{24}$$

[14] Note that according to the first order optimality conditions we have

$$\alpha_\ell - \beta_\ell d_\ell - 2c_j s_j - \beta q_{j\ell} = 0,$$
$$\alpha_\ell - \beta_\ell d'_\ell - 2c_j s'_j - \beta q'_{j\ell} = 0,$$

and since $s'_j \leq s_j$ and $d'_\ell \leq d_\ell$, we have $q'_{j\ell} \geq q_{j\ell}$.

Similarly, since for every $f_j \in F_2$ and $m_\ell \in M_2$ we have $q'_{j\ell} \leq q_{j\ell}$, we have

$$Q'_{2,2} \leq Q_{2,2}. \tag{25}$$

Also since the total quantity supplied to each of the markets in M_1 did not increase we have:

$$Q'_{1,1} + Q'_{2,1} \leq Q_{1,1} + Q_{2,1}, \text{ which implies that } Q'_{2,1} \leq Q_{2,1}. \tag{26}$$

Similarly, since in the post-entry equilibrium the total supply of each firm in F_2 has increased, we obtain

$$Q_{2,1} + Q_{2,2} < Q'_{2,1} + Q'_{2,2} \text{ which implies that } Q'_{2,1} > Q_{2,1},$$

and it contradicts inequality (26). This in turn implies that we should have $F_2 = \emptyset$. Furthermore, in the post-entry equilibrium the total quantity supplied to each of the markets in M_2 has increased in the post-entry equilibrium, and thus $Q_{1,2} + Q_{2,2} < Q'_{1,2} + Q'_{2,2}$, which in turn implies that $Q'_{1,2} > Q_{1,2}$. Combining the last inequality with inequality (25) contradicts the fact that the total output of F_1 does not increase in the post entry equilibrium. So we must have $f_i \in F_2$. Similarly we can prove that $m_i \in M_2$.

Part (ii) Let us first prove the following lemma, which is central in the analysis for this part.

LEMMA 8. *Let* $\boldsymbol{q}$, $\boldsymbol{q}'$ *denote the pre- and post-entry equilibrium. Then, there cannot be a cycle on the edges of* G *in which for every two consecutive edges* (f_i, m_k) *and* (f_j, m_ℓ)*, i.e., edges for which either* $f_i = f_j$ *or* $m_k = m_\ell$*, we have* $q'_{ik} > q_{ik}$ *and* $q'_{j\ell} < q_{j\ell}$*.*

Proof: Assume that firm f_j increases the quantity it supplies to market m_{ℓ_1} whereas it decreases the quantity it supplies to market m_{ℓ_2}, i.e., $q'_{j\ell_1} > q_{j\ell_1}$ and $q'_{j\ell_2} < q_{j\ell_2}$. The first order optimality conditions for the firm's optimization problem imply the following equations:

$$\begin{array}{ll} \mathcal{P}_{\ell_1} - 2c_j s_j = \beta q_{j\ell_1}, & \mathcal{P}_{\ell_2} - 2c_j s_j = \beta q_{j\ell_2} \\ \mathcal{P}'_{\ell_1} - 2c_j s'_j = \beta q'_{j\ell_1}, & \mathcal{P}'_{\ell_2} - 2c_j s'_j = \beta q'_{j\ell_2} \end{array}.$$

The equations above further imply

$$\begin{cases} \mathcal{P}_{\ell_1} - \mathcal{P}_{\ell_2} = \beta(q_{j\ell_1} - q_{j\ell_2}) \\ \mathcal{P}'_{\ell_1} - \mathcal{P}'_{\ell_2} = \beta(q'_{j\ell_1} - q'_{j\ell_2}) \end{cases}.$$

However, by the assumption that $q'_{j\ell_1} > q_{j\ell_1}$ and $q'_{j\ell_2} < q_{j\ell_2}$, we obtain $\mathcal{P}'_{\ell_1} - \mathcal{P}'_{\ell_2} > \mathcal{P}_{\ell_1} - \mathcal{P}_{\ell_2}$. Now, for the sake of contradiction assume that such cycle $(i_1, \ell_1), (i_2, \ell_1), \ldots, (i_t, \ell_t), (i_1, \ell_t)$ exists. Then,

$$\begin{cases} \mathcal{P}'_{\ell_1} - \mathcal{P}'_{\ell_t} > \mathcal{P}_{\ell_1} - \mathcal{P}_{\ell_t} \\ \mathcal{P}'_{\ell_2} - \mathcal{P}'_{\ell_1} > \mathcal{P}_{\ell_2} - \mathcal{P}_{\ell_1} \\ \mathcal{P}'_{\ell_3} - \mathcal{P}'_{\ell_2} > \mathcal{P}_{\ell_3} - \mathcal{P}_{\ell_2} \\ \vdots \\ \mathcal{P}'_{\ell_t} - \mathcal{P}'_{\ell_{t-1}} > \mathcal{P}_{\ell_t} - \mathcal{P}_{\ell_{t-1}} \end{cases},$$

which in turn leads to a contradiction.

Before going on to complete the proof of Part (ii) we define sets $\bar{M}_1$, $\bar{F}_1$, $\bar{M}_2$, $\bar{F}_2$ as follows

$$\begin{array}{ll} \bar{M}_1 \triangleq \{m_k : d'_k \leq d_k\}, & \bar{M}_2 \triangleq \{m_\ell \neq m_k : d'_\ell > d_\ell\}, \\ \bar{F}_1 \triangleq \{f_i : s'_i \leq s_i\}, & \bar{F}_2 \triangleq \{f_j \neq f_i : s'_j > s_j\}. \end{array}$$

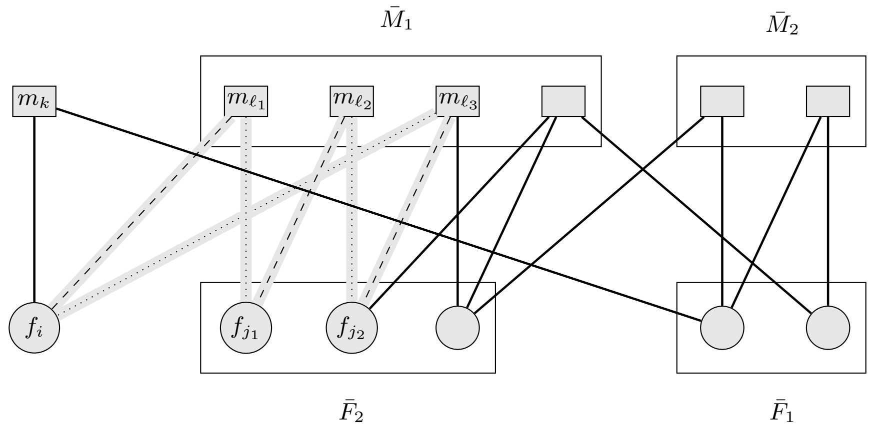


Figure 9 In this figure firm f_i enters market m_k. In this figure flows of the links in the shadowed cycle $f_i \to m_{\ell_1} \to f_{j_1} \to m_{\ell_2} \to f_{j_2} \to m_{\ell_3} \to f_i$ are decreasing/increasing alternatively.

For the sake of contradiction assume that firm f_i increases its output to market $m_{\ell_1} \neq m_k$. Note that we must have $m_{\ell_1} \in \bar{M}_1$, so since the total quantity supplied to markets in $\bar{M}_1$ does not increase, there must exist another link (j_1, ℓ_1) such that the flow in (j_1, ℓ_1) decreases, i.e., $q'_{j_1\ell_1} < q_{j_1\ell_1}$. Similarly, we must have $f_{j_1} \in \bar{F}_2$, and since the total quantity produced by the firms in $\bar{F}_2$ increases there must exist another link (j_1, ℓ_2) for which $m_{\ell_2} \in \bar{M}_1$ and $q'_{j_1\ell_2} > q_{j_1\ell_2}$.

Using the same argument as before (since the total quantity supplied to each of the markets in $\bar{M}_1$ has increased in the post-entry equilibrium), there must exist a firm f_r such that the quantity it supplies to market m_{ℓ_2} decreases in the post-entry equilibrium. Since the environment has a finite number of firms and markets the argument above implies that there cannot exist a cycle where the change in the flow of consecutive links between the pre- and post-entry equilibrium has alternating signs (Figure 9 provides a graphical illustration of the proof).

Part (iii) Note that by directly applying Proposition 1 we obtain the quantities in the new equilibrium as follows:

$$q'_{j\ell} = q_{j\ell} + \frac{\psi'_{j\ell,ik}}{\psi'_{ik,ik}} q'_{ik} \quad \forall (j,\ell) \in G. \tag{27}$$

So if for each firm f_j and market m_ℓ we denote by $\mathcal{C}'_j$ and $\mathcal{P}'_\ell$ the production cost of firm f_j and the price at market m_ℓ in the post-entry equilibrium respectively, then the profit for firm f_i in the post-entry equilibrium is given by

$$\begin{aligned}
\pi_i(\boldsymbol{q'}) &= q'_{ik}\mathcal{P}'_k + \sum_{\ell \in F_i} q'_{i\ell}\mathcal{P}'_\ell - \mathcal{C}'_i \\
&= q'_{ik}\mathcal{P}'_k + \sum_{\ell \in F_i} (q_{i\ell} + \frac{\psi'_{il,ik}}{\psi'_{ik,ik}} q'_{ik})(\mathcal{P}_\ell + \Lambda'_{ik,\ell} q'_{ik}) - \mathcal{C}'_i \\
&= \pi_i + q'_{ik}\left(\mathcal{P}_k + \Lambda'_{ik,k} q'_{ik}\right) + q'_{ik} \sum_{\ell \in F_i} \left(\frac{\psi'_{i\ell,ik}}{\psi'_{ik,ik}} \mathcal{P}_\ell + \Lambda'_{ik,\ell} \left(\frac{\psi'_{i\ell,ik}}{\psi'_{ik,ik}} q'_{ik} + q_{i\ell} \right) \right) - \Delta\mathcal{C}_i,
\end{aligned}$$

which completes the proof. Q.E.D.

Proof of Proposition 3

The price in market m_ℓ in the post-entry equilibrium is equal to

$$\mathcal{P}'_\ell = \mathcal{P}_\ell + \Lambda'_{ik,\ell} q'_{ik}.$$

So the aggregate consumer surplus CS' in the post-entry equilibrium will be given by

$$\begin{aligned} CS' &= \sum_{\ell=1}^{m} \frac{(\alpha_\ell - \mathcal{P}'_\ell)^2}{2\beta} \\ &= \sum_{\ell=1}^{m} \frac{(\alpha_\ell - \mathcal{P}_\ell - \Lambda'_{ik,\ell} q'_{ik})^2}{2\beta} \\ &= CS - \frac{q'_{ik}}{2\beta} \sum_{\ell=1}^{m} \Lambda'_{ik,\ell} \left(2(\alpha_\ell - \mathcal{P}_\ell) - \Lambda'_{ik,\ell} q'_{ik}\right). \end{aligned}$$

Thus, we conclude that $\Delta CS = -\frac{q'_{ik}}{2\beta} \sum_{\ell=1}^{m} \Lambda'_{ik,\ell} \left(2(\alpha_\ell - \mathcal{P}_\ell) - \Lambda'_{ik,\ell} q'_{ik}\right)$. Q.E.D.

Proof of Proposition 4

In order to simplify the exposition, we let $q(G) = \sum_{(i,k)\in G} q_{ik}(G)$ for any network G. In other words, $q(G)$ denotes the total quantity supplied by the firms in network G. The expression for the aggregate welfare corresponding to a network G is as follows

$$\begin{aligned} \text{Social Welfare} &= \sum_i \pi_i + CS \\ &= \sum_i \left(\sum_{k\in F_i} q_{ik}(G) \Big(\alpha_k - \beta \sum_{j\in M_k} q_{jk}(G)\Big) - c\Big(\sum_{k\in F_i} q_{ik}(G)\Big)^2 \right) + \sum_k \frac{(\alpha_k - \mathcal{P}_k)^2}{2\beta} \\ &= \sum_k \sum_{i\in M_k} q_{ik}(G)\alpha_k - \frac{\beta}{2} \sum_k \Big(\sum_{i\in M_k} q_{ik}(G)\Big)^2 - c \sum_i \Big(\sum_{k\in F_i} q_{ik}(G)\Big)^2. \end{aligned}$$

Among all networks G and vector of quantities that satisfy the following condition:

$$\alpha - \beta \sum_{j\in M_k} q_{jk}(G) - 2c \sum_{\ell\in F_i} q_{i\ell}(G) - \beta q_{ik}(G) \geq 0 \ \ \forall (i,k) \in G, \tag{28}$$

we consider a network and quantities that maximize the social welfare:

$$SW(G) = \alpha q(G) - \frac{\beta}{2} \sum_k \Big(\sum_{j\in M_k} q_{jk}(G)\Big)^2 - c \sum_i \Big(\sum_{\ell\in F_i} q_{i\ell}(G)\Big)^2.$$

Note that in the equilibrium for network G, the condition (28) has to be satisfied with equality. So quantity $SW(G)$ is an upper bound for the maximum social welfare at equilibrium. The proof of the proposition consists of two parts:

Part (i) In the first part of the proof, we show that for any given graph G different than the complete network, the total supply $q(G)$ is less than the total supply that corresponds to the equilibrium in the complete network. First, we show that for any $(i,k) \notin G$ we have:

$$\alpha - \beta \sum_{j\in M_k} q_{jk}(G) - 2c \sum_{\ell\in F_i} q_{i\ell}(G) \geq 0. \tag{29}$$

For the sake of contradiction assume that there exists $(i,k) \notin G$ such that inequality (29) does not hold. Then, we show that we can change the vector of quantities so that aggregate welfare increases. Note that

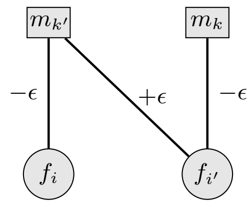


Figure 10

for every $f_{i'} \in M_k$, we have $\sum_{\ell \in F_{i'}} q_{i'\ell}(G) < \sum_{\ell \in F_i} q_{i\ell}(G)$ [15]. So there exists a market $m_{k'} \in F_i$ such that $q_{i'k'}(G) < q_{ik'}(G)$, potentially such that $(i', k') \notin G$. Assume that we decrease $q_{ik'}(G)$ and $q_{i'k}(G)$ by ϵ and increase $q_{i'k'}(G)$ by ϵ (see Figure 10). For ϵ sufficiently small, condition (28) still holds and aggregate welfare changes as follows

$$\begin{aligned}\Delta SW(G) &= \beta \sum_{j \in M_k} q_{jk}(G)\epsilon - \frac{\beta}{2}\epsilon^2 + 2c \sum_{\ell \in F_i} q_{i\ell}(G)\epsilon - c\epsilon^2 - \alpha\epsilon \\ &= \epsilon \left(\beta \sum_{j \in M_k} q_{jk}(G) + 2c \sum_{\ell \in F_i} q_{i\ell}(G) - \alpha \right) - \epsilon^2 \left(\frac{\beta}{2} + c \right).\end{aligned}$$

We have assumed that

$$\alpha - \beta \sum_{j \in M_k} q_{jk}(G) - 2c \sum_{\ell \in F_i} q_{i\ell}(G) < 0,$$

and thus we conclude that for small enough ϵ, we have $\Delta SW(G) > 0$. This contradicts our choice of G as the networks that maximizes social welfare.

Finally in order to show that $q(G)$ is less than or equal to the total supply in the equilibrium at the complete network, suppose that we distribute quantity $q(G)$ equally among all markets, all firms, and all firm-market pairs in the complete network. We only need to show that:

$$\alpha - \beta \frac{q(G)}{m} - 2c \frac{q(G)}{n} - \beta \frac{q(G)}{mn} \geq 0.$$

To see why the above inequality holds, note that we have:

$$\begin{aligned}mn \left(\alpha - \beta \frac{q(G)}{m} - 2c \frac{q(G)}{n} - \beta \frac{q(G)}{mn} \right) &= mn\alpha - n\beta q(G) - m2cq(G) - \beta q(G) \\ &= mn\alpha - n \sum_k \beta \sum_{j \in M_k} q_{jk}(G) - m \sum_i 2c \sum_{\ell \in F_i} q_{i\ell}(G) - \beta \sum_{(i,k) \in G} q_{ik}(G).\end{aligned}$$

Now by using inequality (28) the above expression is lower bounded by

$$\begin{aligned}&\sum_{f_i \in F, m_k \in M} \big(\alpha - \beta \sum_{j \in M_k} q_{jk}(G) - 2c \sum_{\ell \in F_i} q_{i\ell}(G)\big) - \sum_{(i,k) \in G} \big(\alpha - \beta \sum_{j \in M_k} q_{jk}(G) - 2c \sum_{\ell \in F_i} q_{i\ell}(G)\big) \\ &= \sum_{(i,k) \notin G} \big(\alpha - \beta \sum_{j \in M_k} q_{jk}(G) - 2c \sum_{\ell \in F_i} q_{i\ell}(G)\big) \geq 0,\end{aligned}$$

where the last inequality follows from inequality (29).

[15] Note that link $(i', k) \in G$ and thus according to condition (28) we have

$$\alpha - \beta \sum_{j \in M_k} q_{jk}(G) - 2c \sum_{\ell \in F_{i'}} q_{i'\ell}(G) - \beta q_{i'k}(G) \geq 0.$$

Also since for sake of contradiction we have assumed that $\alpha - \beta \sum_{j \in M_k} q_{jk}(G) - 2c \sum_{\ell \in F_i} q_{i\ell}(G) < 0$, we must have $\sum_{\ell \in F_{i'}} q_{i'\ell}(G) < \sum_{\ell \in F_i} q_{i\ell}(G)$. Otherwise, increasing $q_{ik}(G)$ would have increased social welfare, contradicting our choice of G.

Part (ii) Next we show that the equilibrium corresponding to the complete network leads to higher aggregate welfare. Let H be the complete network, and assume that the total supply at the equilibrium of H is equal to $q(H) = q(G) + x$. Recall that according to part (i), we have $x \geq 0$. According to the first order optimality condition we have:

$$\alpha - \beta \frac{q(G)+x}{m} - 2c \frac{q(G)+x}{n} - \beta \frac{q(G)+x}{mn} = 0. \tag{30}$$

Now let us write the social welfare that corresponds to the equilibrium of the complete network

$$\begin{aligned} SW(H) &= \alpha\big(q(G)+x\big) - \frac{\beta}{2} \sum_{k \in M} \left(\frac{q(G)+x}{m}\right)^2 - c \sum_{i \in F} \left(\frac{q(G)+x}{n}\right)^2 \\ &= \alpha q(G) - \frac{\beta}{2} \sum_k \left(\frac{q(G)}{m}\right)^2 - c \sum_i \left(\frac{q(G)}{n}\right)^2 + \alpha x - \frac{\beta}{2} \sum_k \frac{2q(G)x + x^2}{m^2} - c \sum_i \frac{2q(G)x + x^2}{n^2}. \end{aligned} \tag{31}$$

Note that we have

$$\alpha q(G) - \frac{\beta}{2} \sum_k \left(\frac{q(G)}{m}\right)^2 - c \sum_i \left(\frac{q(G)}{n}\right)^2 \geq SW(G), \tag{32}$$

so by combining equation (31) and inequality (32), it is enough to show that

$$\alpha x - \frac{\beta}{2} \sum_k \frac{2q(G)x + x^2}{m^2} - c \sum_i \frac{2q(G)x + x^2}{n^2} > 0. \tag{33}$$

We can rewrite expression (33) as

$$\begin{aligned} \alpha x - \frac{\beta}{2} \sum_k \frac{2q(G)x + x^2}{m^2} - c \sum_i \frac{2q(G)x + x^2}{n^2} &= \alpha x - \frac{\beta}{2} \frac{2q(G)x + x^2}{m} - c \frac{2q(G)x + x^2}{n} \\ &= x \left(\alpha - \beta \frac{q(G) + \frac{x}{2}}{m} - 2c \frac{q(G) + \frac{x}{2}}{n} \right) > 0, \end{aligned}$$

and conclude that the inequality holds because of equation (30). Q.E.D.

Proof of Proposition 5

The first order optimality conditions for any outsider firm f_j in the post-merger equilibrium imply that

$$q_{ik} + \Delta q_{ik} = \frac{\alpha_\ell}{2(\beta + c)} - \sum_{(j,\ell) \in E} (\gamma W)_{ik,j\ell} (q_{j\ell} + \Delta q_{j\ell}). \tag{34}$$

Subtracting Equation (15), i.e., the equation that corresponds to the first order optimality conditions for firm f_j in the original equilibrium, from Equation (34) we obtain

$$\Delta q_{ik} = - \sum_{(j,\ell) \in E} (\gamma W)_{ik,j\ell} \Delta q_{j\ell} = - \sum_{(j,\ell) \in E, j \in \mathcal{O}} (\gamma W)_{ik,j\ell} \Delta q_{j\ell} - \gamma \beta \Delta q_{\mathcal{I},k}. \tag{35}$$

Let us define vector $\boldsymbol{\eta}$ as follows:

$$\eta_{j\ell} = \Delta q_{\mathcal{I},\ell} \quad \forall (j,\ell) \in E \text{ where } j \in \mathcal{O},$$

then we can rewrite Equation (35) as follows

$$\Delta \boldsymbol{q}^{\mathcal{O}} = -\gamma W^{\mathcal{O}} \Delta \boldsymbol{q}^{\mathcal{O}} - \beta \gamma \boldsymbol{\eta}.$$

Thus we conclude that the changes in the production output for the outsider firms are given by

$$\Delta \boldsymbol{q}^{\mathcal{O}} = -[\mathrm{I} + \gamma W^{\mathcal{O}}]^{-1} \beta \gamma \boldsymbol{\eta}.$$

Finally by recalling that $\gamma = \frac{1}{2(c+\beta)}$, we get

$$\Delta q_{ik} = - \frac{\beta}{2(c+\beta)} \sum_{m_\ell \big| m_\ell \in F_n \text{ for } n \in \mathcal{I}} \; \sum_{j \in \mathcal{O} \text{ and } j \in M_\ell} \psi^{\mathcal{O}}_{ik,j\ell} \Delta q_{\mathcal{I},\ell}.$$

Q.E.D.

Proof of Theorem 2

First, we prove the following lemma, which states that any equilibrium in the post-merger game is a solution to an appropriately defined linear complementarity problem.

LEMMA 9. *Strategy profile $\boldsymbol{q}$ is an equilibrium for the game that results when firms f_i and f_j merge if and only if $\boldsymbol{q}$ is a solution of the linear complementarity problem$LCP(-\boldsymbol{\alpha}, D')$, where D' is a $|E| \times |E|$ matrix defined as follows*

$$D'_{i_1k,j_1\ell} = \begin{cases} 2(\beta+c) & \text{if } i_1 = j_1, k = \ell \\ 2c & \text{if } i_1 = j_1, k \neq \ell \\ \beta & \text{if } i_1 \neq j_1, k = \ell \text{ and } \{i_1, j_1\} \neq \{i, j\} \text{ or } k \notin F_i \cap F_j \\ 2\beta & \text{if } i_1 \neq j_1, k = \ell \text{ and } \{i_1, j_1\} = \{i, j\} \text{ and } k \in F_i \cap F_j \\ 0 & \text{otherwise.} \end{cases}$$

Proof: We follow the same approach as in the proof of Lemma 2. The first order optimality conditions imply that any equilibrium strategy profile has to be a solution of the linear complementarity problem $LCP(-\bar{\boldsymbol{\alpha}}, D')$. Next, we show that the other direction holds as well, i.e., we show that any solution that satisfies the first order optimality conditions is an equilibrium. To this end, first consider firm $f_{i_1} \neq f_i, f_j$. Define the $|F_{i_1}| \times |F_{i_1}|$ Hessian matrix H^{i_1} for f_{i_1}'s optimization problem as follows

$$H^{i_1}_{i_1k,i_1\ell} = \begin{cases} -2(\beta+c) & \text{if } k = \ell \\ -2c & \text{if } k \neq \ell \end{cases} .$$

It is straightforward to see that H^{i_1} is a negative definite matrix and thus outsider firm f_{i_1} has no profitable deviation. Similarly, for the firm that results from the merger of f_i and f_j, let H^{ij} denote the Hessian matrix associated with its optimization problem, i.e.,

$$H^{ij}_{i_1k,j_1l} = \begin{cases} -2(\beta+c) & \text{if } i_1 = j_1, k = \ell \\ -2\beta & \text{if } i_1 \neq j_1, k = \ell \\ -2c & \text{if } i_1 = j_1, k \neq \ell \\ 0 & \text{otherwise.} \end{cases}.$$

Hessian H^{ij} is negative semi-definite (and furthermore higher order conditions involve $\mathbf{0}$ matrices), thus we conclude that the insider firms have no incentive to deviate from their strategy and the joint strategy profile that corresponds to the solution of the linear complementarity problem is an equilibrium.

Given Lemma 9 we can proceed with the proofs of parts (i)-(iv) of the Theorem. First, for part (ii) note that when the two insider firms f_i and f_j do not share any markets, then $D' = D$ (matrix D was defined in Equation (1)) and therefore the linear complementarity problem corresponding to the post-merger game is exactly the same as the one corresponding to the game before the merger. This implies that the equilibrium is unique and it is the same as the equilibrium for the pre-merger game.

Then, we turn our attention to parts (i) and (iii). Lemma 9 implies that if D' is positive semi-definite, then $LCP(-\bar{\boldsymbol{\alpha}}, D')$ has a solution (albeit not necessarily unique). On the other hand, if D' is positive definite then $LCP(-\bar{\boldsymbol{\alpha}}, D')$ has a unique solution. Note that D' and D are related as follows

$$D' = D + X,$$

where X is the following $|E| \times |E|$ matrix

$$X_{i_1k,j_1\ell} = \begin{cases} \beta \text{ if } \{i_1, j_1\} = \{i, j\} \text{ and } k = \ell \in F_i \cap F_j \\ 0 \text{ otherwise.} \end{cases}$$

According to Weyl's theorem and noting that $\lambda_{\min}(X) = -\beta$, we obtain

$$\lambda_{\min}(D') \geq \lambda_{\min}(D) + \lambda_{\min}(X) = \lambda_{\min}(D) - \beta.$$

So if we show that $\lambda_{\min}(D)$ is at least equal to β then matrix D' is positive semi-definite and thus an equilibrium always exists. Furthermore, in the case that $\lambda_{\min}(D) > \beta$, D' is positive definite and thus the equilibrium is unique. Note that

$$D = \frac{1}{\gamma} B^T B + \beta \mathrm{I},$$

where matrix B is defined in Equation (5). So since matrix $\frac{1}{\gamma} B^T B$ is symmetric positive semi-definite, one can conclude that $\lambda_{\min}(D) \geq \beta$, and equilibrium always exists. Finally, note that

$$\frac{1}{\gamma} B^T B = W + (2c + \beta)\mathrm{I},$$

and thus again by Weyl's theorem we obtain

$$\lambda_{\min}(\frac{1}{\gamma} B^T B) \geq \lambda_{\min}(W) + (2c + \beta).$$

So if $\lambda_{\min}(W) \neq -(2c+\beta)$ then $\lambda_{\min}(\frac{1}{\gamma} B^T B) > 0$, matrix D' is positive definite, and thus the equilibrium is unique. This concludes the proof of parts (i) and (iii) of the Theorem, since the condition $\lambda_{\min}(W) \neq -(2c+\beta)$ holds generically.

Next we show the last part of the Theorem. We say that the post-merger equilibrium $\boldsymbol{q}$ is *balanced* if the aggregate production of firm f_i is equal to the aggregate production of firm f_j. Also, we call a post-merger equilibrium *connected* if both firms f_i and f_j supply a strictly positive production quantity to at least one of the markets they share. The proof follows from the following three lemmas.

Lemma 10. *All balanced equilibria are equivalent.*

Proof: Consider a balanced equilibrium $\boldsymbol{q}$ and let

$$G' = \{F \cup M, E \cup \{(f_i, m_k) \big| m_k \in F_j\} \cup \{(f_j, m_k) \big| m_k \in F_i\}\},$$

denote the network that results from G when we add the links from both firms f_i and f_j to all the markets that at least one of them participates in the original networked economy represented by graph G. Then, we claim that vector $\boldsymbol{q}'$ with

$$q'_{j\ell} = \begin{cases} 0 \ \text{ if } (j, \ell) \text{ is a link in } G' \text{ but not in } G, \\ q_{j\ell} \text{ otherwise,} \end{cases}$$

is an equilibrium for the game defined over G'. Consider first a market $m_v \in F_i \cap F_j$, i.e., a market that the two insider firms share in the original network G. Note that

$$\frac{\partial \pi_{ij}}{\partial q_{iv}} = \alpha_v - 2\beta q_{iv} - 2\beta q_{jv} - \beta \sum_{u \in M_v, u \neq i,j} q_{uv} - 2c \sum_{\ell \in F_i} q_{i\ell} \leq 0,$$

$$\frac{\partial \pi_{ij}}{\partial q_{jv}} = \alpha_v - 2\beta q_{jv} - 2\beta q_{iv} - \beta \sum_{u \in M_v, u \neq i,j} q_{uv} - 2c \sum_{\ell \in F_j} q_{j\ell} \leq 0,$$

and since the equilibrium is balanced, we have $\frac{\partial \pi_{ij}}{\partial q_{iv}} = \frac{\partial \pi_{ij}}{\partial q_{jv}}$. Next consider a market m_v such that $m_v \in F_j$ but $m_v \notin F_i$. Since $\boldsymbol{q}$ is an equilibrium for the game defined over G we have

$$\frac{\partial \pi_{ij}}{\partial q_{jv}} = \alpha_v - 2\beta q_{jv} - \beta \sum_{u \in M_v, u \neq i,j} q_{uv} - 2c \sum_{\ell \in F_j} q_{j\ell} \leq 0.$$

Finally, consider the first order condition corresponding to link (i, v) in network G'

$$\frac{\partial \pi_{ij}}{\partial q_{iv}} = \alpha_v - 2\beta q_{iv} - 2\beta q_{jv} - \beta \sum_{u \in M_v, u \neq i,j} q_{uv} - 2c \sum_{\ell \in F_i} q_{i\ell},$$

Since $\boldsymbol{q}$ is a balanced equilibrium, we have

$$\frac{\partial \pi_{ij}}{\partial q_{iv}}|_{\boldsymbol{q}'} = \frac{\partial \pi_{ij}}{\partial q_{jv}}|_{\boldsymbol{q}'} \leq 0,$$

so there is no incentive for firm f_i to produce in market m_v. Similarly, we obtain that firm f_j has no incentive to produce in market m_ℓ which is such that $m_\ell \in F_i$ but $m_\ell \notin F_j$. Putting all this together, we conclude that vector $\boldsymbol{q}'$ is an equilibrium of the game defined over G'.

The second case we need to consider is when the two insider firms participate in all of one another's markets in the original pre-merger economy. Then, we can rewrite the first order conditions that correspond to their optimization problem after the merger as the first order conditions of a single firm that has cost parameter equal to $\frac{c}{2}$. The cost of production is convex, and thus it is straightforward to see that at any equilibrium the aggregate output of firm f_i must be equal to the aggregate output of firm f_j. For any market $v \in F_i \cup F_j$, define $q_{xv} = q_{iv} + q_{jv}$, and replace firms f_i and f_j with a single firm f_x connected to $F_x \triangleq F_i \cup F_j$ with cost parameter $\frac{c}{2}$. Now consider the first order optimality conditions for firm f_x

$$\frac{\partial \pi_x}{\partial q_{xv}} = \alpha_v - \beta q_{xv} - \beta \sum_{u \in M_v} q_{uv} - c \sum_{\ell \in F_x} q_{x\ell},$$

and note that at any equilibrium $\boldsymbol{q}'$, we have

$$\frac{\partial \pi_x}{\partial q_{xv}}|_{\boldsymbol{q}'} = \frac{\partial \pi_{ij}}{\partial q_{iv}}|_{\boldsymbol{q}'} = \frac{\partial \pi_{ij}}{\partial q_j}|_{\boldsymbol{q}'} \leq 0.$$

This implies that any equilibrium $\boldsymbol{q}'$ for the post-merger game in a network that insider firms share all their markets is equivalent to the unique equilibrium for the case when we replace them by a single firm f_x and can be derived by decomposing q_{xv} into q_{iv} and q_{jv}. Consequently, all post-merger equilibria for the case when insider firms share all their markets are equivalent. Finally, since we can convert any equilibrium $\boldsymbol{q}$ of the original network to an equilibrium $\boldsymbol{q}'$ in a network where insider firms share all their markets, we conclude that all equilibria $\boldsymbol{q}$ in the original network are equivalent.

LEMMA 11. *Every connected equilibrium is balanced.*

Proof: Assume that firms f_i and f_j both supply strictly positive production quantity to markets $k_i \in F_i \cap F_j$ and $k_j \in F_i \cap F_j$ respectively. Then due to the strict convexity of their productions costs, the aggregate supply of firms f_i and f_j should be equal in the post-merger equilibrium. Otherwise, if for example $\sum_{m_k \in F_i} q_{ik} < \sum_{m_k \in F_j} q_{jk}$, the two firms can reduce their production costs without decreasing their aggregate supply in any of the markets by decreasing q_{jk_j} by (sufficiently small) ϵ while increasing q_{ik_j} by the same amount. So, in any connected equilibrium, the aggregate supply of both insider firms should be the same.

Lemma 12. *If there exists a connected equilibrium, then all equilibria are balanced.*

Proof: Let $\boldsymbol{q}_1$ be a connected equilibrium and $\boldsymbol{q}_2$ be any other equilibrium. Since the solution space of a linear complementarity problem is convex, $\boldsymbol{q}_3 = \gamma\boldsymbol{q}_2 + (1-\gamma)\boldsymbol{q}_1$ is also a solution to the linear complementarity problem for every $\gamma \in (0,1)$. Now, note that since $\boldsymbol{q}_1$ is connected, equilibrium $\boldsymbol{q}_3$ should also be connected. Thus, according to Lemmas 10 and 11, we conclude that $\boldsymbol{q}_1$ and $\boldsymbol{q}_3$ are equivalent and balanced and as a result equilibrium strategy profiles $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ are also equivalent and balanced.

To conclude the proof it is sufficient to show that even when there is no connected equilibrium in the post-merger game, then still all equilibria are equivalent. To this end, assume that there is no connected post-merger equilibrium. This implies that there cannot exist two equilibria $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ such that in $\boldsymbol{q}_1$ firm f_i supplies a positive production quantity to a shared market whereas in $\boldsymbol{q}_2$ firm f_j supplies a positive production quantity to a shared market. If this was the case, a convex combination of $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ (which would also be an equilibrium strategy profile) would be connected. Therefore, it has to be the case that one of the firms, say f_i, does not supply to any of markets it shares with firm f_j. Equivalently, we can consider the network that results from removing all links from firm f_i to the markets it shares with firm f_j, since the equilibria in the post-merger game remain the same. However, according to part (ii) of the Theorem, in this case the post-merger equilibrium is unique and it is same as the equilibrium in the pre-merger game. Q.E.D.

Proof of Proposition 6

The proof follows directly from the proof of Theorem 2. In particular, according to Lemma 12, if there exists a connected equilibrium then all equilibria are balanced and equivalent. Therefore, since in all balanced equilibria, the two firms produce the same quantity on aggregate, it is equivalent to view them as a single firm which is connected to the $F_i \cup F_j$ — union of the markets that f_i and f_j originally participate in — and its cost function is equal to $C_{ij}(x) = 2C(x/2)$.

On the other hand, if there exists no connected equilibrium, then by definition one of the firms, say f_i does not supply at equilibrium to any of the markets that the insider firms share. Thus, we can remove the links from firm f_i to the markets that the insider firms share without affecting the equilibrium strategy profile. Part (ii) of Theorem 2 implies that the post-merger equilibrium in this case coincides with the pre-merger equilibrium for a graph G in which the two insider firms do not share any markets. Q.E.D.

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