The economic efficiency of simple pricing mechanisms in two-sided markets

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Abstract. We study the price of anarchy of a simple budget-balanced trading mechanism for multiple divisible goods in markets containing both producers and consumers (i.e. two-sided markets). Each producer is asked to submit a linear pricing function that specifies a per-unit price $p(d)$ as a function of the demand d that it faces. Consumers then buy their preferred resource amounts at these prices.
We prove that having three producers for every resource guarantees the price of anarchy is bounded. In general, the price of anarchy depends heavily on the level of horizontal and vertical competition in the market, on producers' cost functions, and on the elasticity of consumer demand. We show how these characteristics affect economic efficiency and in particular, we find that the price of anarchy equals 2/3 in a perfectly competitive market, 3/4 in a monopsony, and $2\epsilon(2-\epsilon)/(4-\epsilon)$ in a monopoly where consumer valuations have a fixed elasticity of ϵ. These results hold in markets with multiple goods, particularly in bandwidth markets over arbitrary graphs.
Variants of this mechanism have been commercially deployed in the past; our results suggest how to modify these systems, so that they are provably efficient. On the theory side, our work demonstrates that near-optimal efficiency can be achieved within two-sided markets by very simple and natural pricing mechanisms. It also generalizes several results for fixed-demand and fixed-supply markets.

1 Introduction

We consider the problem of designing a budget-balanced mechanism that enables the trading of divisible goods between self-interested producers and consumers. In particular, we study a *pricing mechanism* that sets prices on goods based on user inputs and then lets the users trade in their own best interests. Such mechanisms are often used in practice, for example in power engineering or in networking. They also model real-world trading within markets.

Specifically, we analyze the following simple pricing rule for a market of Q consumers and R producers. When there is only one good, each producer $r \in R$

is asked to provide a linear *pricing function* $p_r(f_r) = \gamma_r f_r$ with slope $\gamma_r > 0$, which specifies the per-unit price producer r will charge if its total demand is f_r. In other words, if a consumer buys x units from r, he or she will pay r a sum of $p_r(f_r)x = (\gamma_r f_r)x$. Although γ_r specifies a pricing function, for brevity we will often refer to γ_r as a "price". After seeing the producers' prices, each consumer $q \in Q$ chooses a resource amount d_{qr} to buy from producer r and pays $p_r(f_r)d_{qr}$ for it. Because producers input scalar pricing information and consumers input resource quantities, we call this mechanism *Bertrand-Cournot* (somewhat stretching the usual terminology).

When multiple goods are traded, we identify the market with a multigraph $G = (V, E)$. Each consumer q owns a source-sink pair $(s_q, t_q) \in V$, and each producer r operates on an edge $e_r \in E$. For simplicity, we assume there is a one-to-one relationship between edges and producers. As in the single-good mechanism, producer r inputs a linear pricing function $p_r(f_r)$. Consumers, on the other hand, buy edge capacities from producers on each each edge in order to send an (s_q, t_q)-flow on G. Capacities therefore correspond to goods. Specifically, each consumer q directly submits for each (s_q, t_q)-path p the size d_{qp} of the flow it wishes to send over p, and pays $\sum_{p\in P} \sum_{e\in p} p_e(f_e)d_{qp}$ where f_e is the total demand faced by the producer at edge e. This is precisely the cost of the capacity needed to send the chosen flow.

Our primary motivation for studying this mechanism is to understand the economic efficiency of simple and natural pricing mechanisms. The simplest such mechanism one can imagine would be to simply ask producers for prices and consumers for resource amounts. Unfortunately, this method was shown to be highly inefficient in most market settings (Roughgarden and Chawla, 2008). The mechanism we consider is a slight variant of the above, and yet it is provably efficient.

Representing the market as a graph makes our mechanism directly applicable to real-world markets for goods such as transportation, bandwidth, and electricity. Perhaps more interestingly, this graphical structure allows us to study the effects of horizontal and vertical competition between producers. In the former, producers sell substitute goods that are graphically represented by parallel edges. In the latter, producers' goods are complements that are represented by consecutive edges on a path: capacity on one edge can be used only if it is also bought on all the others in the path. One of our main contributions is to show how this structure affects economic efficiency.

2 Results

Our main result is to show that the price of anarchy is bounded by a constant, as long as there are at least three producers for every good. Our result applies to markets over series-parallel graphs, and under an additional technical assumption on user demand, to markets over arbitrary graphs.

The precise value the price of anarchy takes depends heavily on the level of horizontal and vertical competition among producers, on their cost functions and on the elasticity of demand. Within series-parallel graphs, our techniques yield precise expressions for the efficiency as a function of these characteristics. Most notably, we show that the price of anarchy equals 2/3 in a perfectly competitive market, 3/4 in a monopsony, and $2\epsilon(2-\epsilon)/(4-\epsilon)$ in a monopoly where consumer valuation functions have a constant elasticity of ϵ.

More generally, this paper makes the following contributions:

- *We analyze forms of pricing used in practice.* Mechanisms in which producers naturally announce prices and consumers specify quantities have been proposed (Valancius et al., 2008; Esquivel et al., 2009) and deployed numerous times, most notably in commercial networking solutions (see $[1, 3, 2]$ and the references in citevalancius08,esquivel09). Unfortunately, most of these mechanisms are provably inefficient (Roughgarden and Chawla, 2008). Our mechanism is both efficient and is a slight variation to ones currently in use. It represents an example of how current systems can be improved and of how new ones might be designed.
- *We show that simple Bertrand and Cournot mechanisms admit good price of anarchy guarantees in two-sided markets.* Bertrand and Cournot mechanisms have been extensively studied (Johari and Tsitsiklis, 2005; Acemoglu and Ozdaglar, 2007; Correa et al., 2010), and yet almost all results hold only when either supply or demand is fixed. We prove they can be nearly efficient, and in doing so we extend several existing results to the general market setting. Moreover, our paper is one of the few to study general markets and offers an example of how this difficult setting can be approached.
- *We analyze how market structure affects economic efficiency.* This paper is the first to show how horizontal and vertical competition affect economic efficiency in a two-sided market.

3 Related work

Pricing methods have been studied within electrical engineering and computer science as a way of allocating bandwidth between self-interested users. One of the main results is the proportional allocation mechanism (PAM) (Kelly, 1997) for distributing a fixed supply among consumers. Johari and Tsitsiklis (2005) show that the PAM has a price of anarchy of 3/4 in fixed-supply markets; Kuleshov and Vetta (2010) extend this result to two-sided markets. In both settings, the PAM admits the best price of anarchy guarantee within a large class of mechanisms.

Although the PAM is provably optimal, more natural mechanisms — especially ones that are Cournot or Bertrand — have also received significant attention because such mechanisms are more likely to be deployed in practice. In the fixed-supply Cournot setting, the price of anarchy varies between 0 and 2/3, depending on how resources are priced (Harks and Miller, 2009). The Bertrand approach achieves a price of anarchy of 5/6 in the fixed-demand single-resource setting , but is very inefficient in multi-resource markets (Acemoglu and Ozdaglar, 2007).

Correa et al. (2010) propose an alternative Bertrand pricing scheme for fixed-demand markets that accepts from producers linear pricing functions instead of scalar prices. They establish constant price of anarchy bounds in the multi-resource setting and also show how market structure affects efficiency.

Pricing mechanisms for markets containing *both* consumers and producers have been recently studied by Roughgarden and Chawla (2008). Their system operates as the Cournot mechanism of Johari and Tsitsiklis (2005) on the demand side and as the Bertrand mechanism of Acemoglu and Ozdaglar (2007) on the supply side. Although that mechanism is very simple and natural, its price of anarchy is zero in most settings.

Here, we present a mechanism that is not only natural, but also efficient. It combines the demand side of Johari and Tsitsiklis (2005) and the supply side of Correa et al. (2010).

4 Definitions and assumptions

We first refer the reader to the introduction for a high-level definition of the mechanism. As we mentioned, in its most general form, the mechanism is defined over a multigraph $G = (V, E)$. We use P to denote the set of paths in G. We call a set of parallel edges between two vertices a *link*; the set of all links is denoted by L. Intuitively, links correspond to goods. We call a path in the induced graph (V, L) a *route*; the set of all routes is denoted by T. Two sets of users operate on the multigraph: consumers Q and producers R. Consumer $q \in Q$ owns a source and a sink $s_q, t_q \in V$; producer $r \in R$ operates on some edge $e \in E$. The strategy of consumer q is a positive vector $\boldsymbol{d}_q = (d_{qp})_{p \in P_q}$, specifying a flow on each (s_q, t_q)-path in the set $P_q \subset P$; the strategy of producer r is a scalar $\gamma_r > 0$, specifying a linear pricing function $p_r(f) = \gamma_r f$. We will discuss in the paper several alternative, equivalent ways to collect flow information from consumers; here, we use one that is standard in the literature (Johari and Tsitsklis, 2004). We also assume there is a one-to-one relationship between edges and producers and throughout the paper we use both r and e to index providers.

We make the following assumptions on the utilities of the players:

Assumption 1. The utility of consumer q for sending a flow of size $d_q = \sum_{p \in P_q} d_{qp}$ is

$$U_q(d_q) = V_q(d_q) - \sum_{p \in P_q} d_{qp} \sum_{e \in p} p_e(f_e),$$

where $V_q(d_q)$ is q's *valuation function*. The valuation functions $V_q(d_q) : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous, increasing, concave, and differentiable for all $q \in Q$.

Assumption 2. The utility of producer r for supplying f_r units of capacity on its edge is

$$U_r(f_r) = p_r(f_r) f_r - C_r(f_r),$$

where $C_r(f_r)$ is r's *cost function*. For all $r \in R$, the cost function $C_r(f) : \mathbb{R}_+ \to \mathbb{R}_+$ is of the form

$$C_r(f) = \int_0^f c_r(x)dx$$

where $c_r(x) : \mathbb{R}_+ \to \mathbb{R}_+$ is the *marginal cost* function. It is continuous, strictly increasing, convex, and $c_r(0) = 0$.

Both assumptions are standard in the literature, see for example Johari, Tsitsiklis and Mannor (2006), Harks and Miller (2009), Correa et al. (2010). Assumption 1 is relatively modest. Requiring convex marginal costs in Assumption 2 is much stronger; yet it is necessary to obtain good efficiency guarantees. Together, the above utilities define the welfare:

Definition 1. *The* social welfare *within the mechanism equals*

$$\sum_{q\in Q} V_q(d_q) - \sum_{r\in R} C_r(f_r)$$

The demand f_r faced by producer r is a function of all the edge prices in the mechanism (which we denote by the vector $\boldsymbol{\gamma} = (\gamma_e)_{e\in E}$) when consumers are at a Nash equilibrium in the Cournot game with edge prices $\boldsymbol{\gamma}$. It is well-defined because Nash equilibria in Cournot mechanisms always exist and are unique (Johari and Tsitsiklis, 2005).

Definition 2. *A* Nash equilibrium *of the Bertrand-Cournot mechanism is a set of strategies* $\{\boldsymbol{d}_q, \gamma_r \,|\, q \in Q, r \in R\}$ *such that*

1. *The* $\boldsymbol{d}_q$ *form a* demand-side equilibrium *given prices* $\boldsymbol{\gamma}$ *and utilities* U_q*. That is, for all* $q \in Q$*,* $\boldsymbol{d}_q = \arg\max_{\boldsymbol{d}} U_q(\boldsymbol{d}, \boldsymbol{d}_{-q}, \boldsymbol{\gamma})$*, where* $\boldsymbol{d}_{-q}$ *are the strategies of all consumers except* q*.*
2. *The prices* $\boldsymbol{\gamma}$ *form a* supply-side equilibrium *given demand functions* f_r *and utilities* U_r*. That is, for all* $r \in R$*,*

$$\gamma_r = \arg\max_{\gamma} \left(\gamma f_r(\gamma, \boldsymbol{\gamma}_{-r}) - C_r(f_r(\gamma, \boldsymbol{\gamma}_{-r}))\right),$$

where $\boldsymbol{\gamma}_{-r}$ *are the strategies of all providers except* r*. Thus* γ_r *is the best response to the other prices when* r *anticipates consumers' equilibrium demand.*

We measure economic efficiency using the concept of price of anarchy.

Definition 3. *The* price of anarchy *is defined as the smallest* welfare ratio

$$\frac{\sum_{q\in Q} V_q(\boldsymbol{d}_q^{NE}) - \sum_{r\in R} C_r(f_r(\gamma^{NE}))}{\sup_{\boldsymbol{d}_q, f_r} \sum_{q\in Q} V_q(\boldsymbol{d}_q) - \sum_{r\in R} C_r(f_r)},$$

where the $\boldsymbol{d}_q^{NE}, \gamma^{NE}$ *form Nash equilibrium.*

Our goal is to lower-bound the price of anarchy across all instances of the mechanism.

The producers' strategies in the mechanism are primarily determined by consumers' responses to price, which can be simply described by the *elasticity* of demand with respect to price.

Definition 4. *Let $f(x) : \mathbb{R} \to \mathbb{R}$ be a differentiable function. The elasticity of f with respect to x is a function $\epsilon_x f(y) : \mathbb{R} \to \mathbb{R}$ defined as*

$$\epsilon_x f(y) = \frac{df}{dx}(y)\frac{y}{f(y)}$$

We mainly work with $\epsilon_{\gamma_r} f_r$, the elasticity of demand to producer r. When referring to $\epsilon_{\gamma_e} f(\gamma_e)$, we often drop the γ_e subscript and simply write $\epsilon_e f_e$ or ϵf_e. We also use properties of elasticity that are summarized in Appendix A.

5 Markets with a single good

We begin our analysis with a market in which there is only a single resource. In this setting, the multigraph G consists of two nodes, s and t, and one link; every producer r offers to carry flow from s to t over an edge on the link.

Our main result is to show that the price of anarchy is bounded by a constant as long as there are at least 3 producers in the market. We give specific bounds for several settings, including monopolies, monopsonies, and perfectly competitive markets.

To establish these results, we analyze the demand and supply sides of the market separately. On the demand side, welfare is usually lost because consumers that value the resource less end up receiving goods that should go to the consumers that value the resource the most. We call that *demand-side inefficiency*. On the supply side, welfare is lost because producers charge consumers higher than at their marginal costs (marginal cost pricing can be shown to be socially optimal). We call that *supply-side inefficiency*. We adopt the following three-step procedure to measure these two inefficiencies:

1. **Defining a simplified version of the mechanism.** We define an equivalent mechanism in which we ask consumers for the size of the flow they want to send across the link, and have the mechanism split it across providers automatically. The per-unit flow price is set using a single linear pricing function $P(f) = \Gamma f$, where Γ is a function of the prices γ_e.
2. **Measuring inefficiency on the demand side.**
   (a) We first show that the worst price of anarchy occurs in a game where valuations are linear and costs are quadratic.
   (b) We then formulate the price of anarchy as the minimum of an optimization problem that minimizes the welfare ratio over all possible linear valuations and marginal costs and over all relevant strategy profiles.

   (c) We analytically solve this problem and find that the price of anarchy equals $2(\rho(2-\rho)/(4-\rho)$, where $0 \le \rho \le 1$ is a parameter measuring supply-side inefficiency.
3. **Measuring inefficiency on the supply side.** We bound ρ when there are at least three producers in the market, and we show how it varies with the number of producers and with the elasticity of consumer demand.

In later sections, we use the same approach to analyze the price of anarchy in more complex markets.

5.1 Defining a simplified version of the mechanism

Observe that from a consumer's perspective, there is only a single resource in the market: (s,t)-flow. It turns out that we can define a single price for that flow that aggregates producers' different prices and then split demand across providers the way it would have been split at equilibrium.

Definition 5. *In the* simplified *single-link Bertrand-Cournot mechanism,*

1. *Producers submit linear pricing functions as in the regular mechanism. The aggregate pricing function is set to $P(f) = \Gamma f$, where*

$$\Gamma = \frac{1}{\sum_e 1/\gamma_e}.$$

2. *Consumer q submits an (s,t)-flow d_q and pays for it $\Gamma f d_q$, where $f = \sum_{q \in Q} d_q$.*
3. *The mechanism sends*

$$f_e = \frac{1/\gamma_e}{\sum_{e' \in E} 1/\gamma_{e'}} f$$

   over edge e and pays the producer $\gamma_e f_e^2$.

This new mechanism is easier to analyze, easier to use, and from the point of view of a consumer, its communication complexity no longer depends on the number of producers. It is also equivalent to the original mechanism.

Theorem 1. *The Nash equilibria of the standard and simplified mechanisms are identical, and at equilibrium, the utilities of each player are the same.* □

Interestingly, when the producers' costs are quadratic, we can also aggregate their cost functions $\frac{\beta_r}{2} f_r^2$ into a single cost function $\frac{B}{2} f^2$ that specifies the smallest cost for sending a total flow of f across all the edges. We can then use this cost to compute the size of the socially optimal flow without worrying how it is distributed among producers.

Definition 6. *When the marginal costs at the edges are of the form $c_e(f) = \beta_e f$, the slope of the* aggregate cost *function of the link is defined to be*

$$B = \frac{1}{\sum_e 1/\beta_e}.$$

Theorem 2. *When producers' marginal costs are linear, a cost-minimizing allocation f costs the producers a total of $\frac{B}{2}f^2$.* □

The proofs of these theorems can be found in Appendix C.

5.2 Measuring inefficiency on the demand side

We can formulate the price of anarchy as the solution to the following optimization problem, taken over all possible functions $(V_q)_{q\in Q}$, $(C_r)_{r\in R}$ and over scalars $d_q, d_q^*, \gamma_r, f_r^*$ (the $f_r(\boldsymbol{\gamma})$ are implicitly defined by the V_q):

$$\begin{aligned} \min \quad & \frac{\sum_{q\in Q} V_q(d_q) - \sum_{r\in R} C_r(f_r(\boldsymbol{\gamma}))}{\sum_{q\in Q} V_q(d_q^*) - \sum_{r\in R} C_r(f_r^*)} \\ \text{s.t.} \quad & \text{The } V_q \text{ are valuations satisfying Assumption 1 for all } q. \\ & \text{The } C_r \text{ are costs satisfying Assumption 2 for all } r. \\ & \text{The } d_q \text{ and the } \gamma_r \text{ are equilibrium strategies given } (V_q)_{q\in Q},\ (C_r)_{r\in R}. \\ & \text{The } d_q^* \text{ and the } f_r^* \text{ are optimum allocations given } (V_q)_{q\in Q},\ (C_r)_{r\in R}. \end{aligned}$$

This problem is not very practical because it is infinite-dimensional. However, it turns out that we can restrict our attention to settings where the valuations and marginal costs are all linear (therefore the costs themselves are quadratic), which yields a simpler, finite-dimensional problem. More formally, we have the following:

Lemma 1. *Given any game instance $\mathcal{G}$, one can construct a new game instance $\mathcal{G}'$ where:*

1. *Consumers have linear valuations and producers have quadratic costs.*
2. *Producers set prices as if the demand functions f_r they were facing were the ones in $\mathcal{G}$.*

The price of anarchy of $\mathcal{G}'$ is a lower bound on that of $\mathcal{G}$. □

See Section D.1 in the appendix for a more formal version of this lemma and a proof.

Therefore, when the demand functions $f_r(\boldsymbol{\gamma})$ are fixed, the above infinite-dimensional problem can be replaced by the following problem that has only $2(Q+R)$ scalar variables $\bar{d}_q$, $\bar{\gamma}_r$, α_q and β_r and 3 "helper" variables f, Γ, B:

Lemma 2. *The price of anarchy is lower-bounded by the solution to:*

$$\begin{aligned} \min_{d_q,\gamma_r,\alpha_q,\beta_r,f,\Gamma,B} \quad & \frac{\sum_{q=1}^{Q} \alpha_q d_q - \frac{B}{2} f^2}{\max_{\bar{f}} (\max_{q\in Q} \alpha_q \bar{f} - \frac{B}{2}\bar{f}^2)} \\ s.t. \quad & \alpha_q = \Gamma f + \Gamma d_q \text{ for all } q \in Q \\ & \beta_r = \gamma_r \left(2 - \frac{1}{\epsilon f_r}\right) \text{ for all } r \in R \\ & \sum_{q\in Q} d_q = f \\ & 0 \le \alpha_q, d_q, \Gamma, B \end{aligned}$$

When valuation functions are linear, this bound is tight. □

The first two constraints are necessary and sufficient conditions for an allocation to be a Nash equilibrium. The third constraint ensures that supply equals demand. Variables Γ, B are the aggregate prices that we defined in Section 5.1, and ϵf_r denotes the elasticity of the demand function f_r faced by provider r (which we take as fixed).

In Appendix D, we analytically solve the above problem using techniques developed by Johari and Tsitsiklis (2004) and Acemoglu and Ozdaglar (2007) . As a result, we obtain the following lemma.

Lemma 3. *The price of anarchy in a single-good market is bounded by* $2\rho(2-\rho)/(4-\rho)$*, where* $0 \le \rho \le 1$ *is an overcharging parameter that equals* B/Γ*. At equilibrium, this ratio is constrained by the equation*

$$\beta_r = \gamma_r \left(2 - \frac{1}{\epsilon f_r}\right) \text{ for all } r. \quad (1)$$

When valuations are linear, this bound is tight. □

Lemma 3 basically indicates that the price of anarchy has two distinct components: one arising from demand-side inefficiency and another from supply-side inefficiency. Demand-side inefficiency has been accounted for by the minimization problem in Lemma 2. All that remains is to combine that analysis with a measure of supply-side inefficiency ρ. Note this parameter corresponds to the ratio of true producer costs over the prices that they charge the users, which was how we defined supply-side inefficiency.

5.3 Measuring inefficiency on the supply side

The simplest setting we consider is one with only one monopolist producer in the market, so that $\Gamma = \gamma_e$.

Theorem 3. *Suppose the market is a monopoly. Suppose users have monomial valuation functions* $V_q(d_q) = \alpha_q d_q^x$*, where* $0 < x \le 1$ *and* $\alpha_q > 0$*. The price of anarchy is bounded by* $(2x(2-x)/(4-x)$*. When valuations are linear, the bound equals* $2/3$ *and is tight.* □

Notice that when $x \to 0$, the elasticity of demand decreases and the bound tends to zero. It can be shown that as $x \to 0$, this is actually tight. This observation is not surprising at all: if consumer demand changes very little with price, there is nothing to stop the monopolist from substantially overcharging its customers.

It can be shown this kind of overcharging can happen even when there are two producers (the demand functions have to be highly inelastic). However, it turns out that with *three* competitors in the market, the price of anarchy can be bounded by a constant. This is our main result for the single-resource case.

Theorem 4. *Suppose there are at least 3 producers. Suppose there is a* $0 < \Delta \leq 1$ *such that* $\min_e \beta_e / \max_e \beta_e \geq \Delta$*. Then the price of anarchy is bounded by a constant.*

Proof. In the simplified mechanism, the flow f_e over edge e equals $S_e f$, where f is the total flow across the link, and S_e denotes the fraction that is routed through edge e. Using properties of elasticity, we obtain

$$\begin{aligned}\epsilon_e f_e &= \epsilon_e S_e + \epsilon_e f = \epsilon_e S_e + \epsilon_\Gamma f \, \epsilon_e \Gamma \\ &= \epsilon_e \frac{1/\gamma_e}{\sum_{e' \in E} 1/\gamma_{e'}} + \epsilon_\Gamma f \, \epsilon_e \frac{1}{\sum_{e'} 1/\gamma_{e'}} \\ &= -\frac{\sum_{e' \neq e} 1/\gamma_{e'}}{\sum_{e'} 1/\gamma_{e'}} + \epsilon_\Gamma f \, \frac{1/\gamma_e}{\sum_{e'} 1/\gamma_{e'}} \end{aligned} \tag{2}$$

Now suppose for a contradiction that $\rho(\boldsymbol{\beta}_n, \boldsymbol{\gamma}_n) \to 0$ for some sequence $(\boldsymbol{\beta}_n, \boldsymbol{\gamma}_n)_{n=1}^{\infty}$ (where $\boldsymbol{\beta}_n$ and $\boldsymbol{\gamma}_n$ are vectors of costs and prices, indexed by edges). We claim that this implies $\beta_{en} = o(\gamma_{en})$ as $n \to \infty$ for all $e \in E$. If $\beta_{en} \neq o(\gamma_{en})$ for some e, then

$$\rho(\boldsymbol{\beta}_n, \boldsymbol{\gamma}_n) = \frac{\sum_{e'} 1/\gamma_{e'n}}{\sum_{e'} 1/\beta_{e'n}} \geq \left(\sum_{e'} 1/\gamma_{e'n} \right) \Delta \beta_{en} \geq \Delta \frac{\beta_{en}}{\gamma_{en}} \nrightarrow 0$$

where n is arbitrary, thus contradicting $\rho \to 0$.

Since $\beta_{en} = o(\gamma_{en})$ for all e, by equation (1) we have $\epsilon_e f_e(\boldsymbol{\beta}_n, \boldsymbol{\gamma}_n) \to -1/2$ for all e as $n \to \infty$. In particular, there must exist $(\boldsymbol{\beta}_N, \boldsymbol{\gamma}_N)$ for some $N \geq 0$ sufficiently high at which $\epsilon_e f_e(\boldsymbol{\beta}_N, \boldsymbol{\gamma}_N) \geq -1/2 - \epsilon/|E|$ for $\epsilon > 0$ and for all $e \in E$.

Inserting expression (44) into $\epsilon f_e \geq -1/2 - \epsilon/|E|$ and summing the result over all e, we obtain

$$\epsilon_\Gamma f - (|E| - 1) \geq -\frac{|E|}{2} - \epsilon,$$

which cannot be achieved for small values of ϵ when $|E| \geq 3$, because $\epsilon_\Gamma f$ cannot be positive. Thus we get a contradiction. □

The requirement that $\min_e \beta_e / \max_e \beta_e \geq \Delta$ for some $0 < \Delta \leq 1$ ensures that producers are able to compete with each other[3]. If one producer had much higher

[3] In the general setting, this condition could be formulated as $\min_r c_r(f_r^{\text{NE}}) / \max_r c_r(f_r^{\text{NE}}) \geq \Delta$

costs than the others, it could not profitably charge lower than its competitors, and would have no effect on the market.

The precise constant that bounds the price of anarchy depends on both Δ and the number of producers. It can be computed numerically by formulating ρ as the minimum of an optimization problem. Interestingly, it tends to $2/3$ as $|R| \to \infty$, which is the same as when demand has an elasticity of 1. Thus market competition may entirely offset the effects of inelastic demand.

Theorem 5. *Consider a single-resource market. Suppose there is a* $0 < \Delta \leq 1$ *such that* $\min_{e \in E} \beta_e / \max_{e \in E} \beta_e \geq \Delta$ *for any number of edges* $|E| > 0$*. Then as* $|E| \to \infty$*, the price of anarchy goes to* $2/3$*.* □

The best possible price of anarchy guarantee is achieved when there is no competition among consumers.

Corollary 1. When there is only one user and an infinite number of producers, the price of anarchy equals $3/4$. □

Proofs of both theorems can be found in Apprendix E.

6 Multi-resource markets over series-parallel graphs

In the setting where the market contains multiple resources, the efficiency becomes dependent on whether producers compete horizontally or vertically with each other. Recall that in the former, producers sell substitute goods that are graphically represented by parallel edges; in the latter, producers' goods are complements that are represented by consecutive edges on a path.

The effects of horizontal and vertical competition are most easily understood by looking at *series-parallel* graphs. Informally, a series-parallel graph is built recursively by connecting smaller series-parallel graphs in parallel or in series, starting from edges. See Correa et al. (2010) for a full definition. For our purposes, it will be enough to look at the restricted class of *two-level* series-parallel graphs.

Definition 7. *A* two-level series-parallel graph G *consists of a set of* T *disjoint parallel routes that connect two special nodes: a source* s *and a target* t*.*

Such graphs are useful for studying the effects of horizontal and vertical competition; however, our results also carry over to arbitrary series-parallel graphs (usually by an induction argument). We also assume in this section that consumers have linear valuations $\{\alpha_q d_q \mid q \in Q\}$ and that providers have quadratic costs $\{\frac{\beta_r}{2} f_r^2 \mid r \in R\}$; in the next section, we formally establish that this is the worst-case setting.

Our analysis follows the same blueprint as in the single-resource case. First, we show that the mechanism can be simplified on the consumer side like in Section 5.1. Then, using the same argument as in the single-resource setting, we derive a price of anarchy bound that is a function of supply-side inefficiency (an analogue of Lemma 3). Finally, we bound the supply-side inefficiency and obtain the full price of anarchy. See Section 5 for more details on this procedure.

6.1 Defining a simplified version of the mechanism

Let G be a two-level series-parallel graph with a common source-target pair (s, t) shared by all consumers. We can define like in Section 5.1 a pricing function $P(f) = \Gamma f$ for (s, t)-flow f and show that charging consumers for their total flow using P results in a game that is equivalent to the original. More formally, given the graph G, we definte Γ to be

$$\Gamma = \frac{1}{\sum_{t \in T} 1/\Gamma_t}, \text{ where } \Gamma_t = \sum_{l \in t} \Gamma_l \text{ and } \Gamma_l = 1/\sum_{e \in l} 1/\gamma_e.$$

The intuition here is that the price of a route is a sum of the prices of its links, and parallel routes with prices Γ_t can be aggregated like edges in a link.

Definition 8. *In the* simplified *Bertrand-Cournot mechanism for a two-level series-parallel graph G,*

1. *Each producer r submits a linear pricing functions with slope γ_r like in the regular mechanism, and the aggregate pricing function is set to $P(f) = \Gamma f$.*
2. *Each consumer q chooses to send d_q units of (s, t)-flow and pays $\Gamma f d_q$, where $f = \sum_{q \in Q} d_q$.*
3. *The mechanism divides payments and flow proportionally to the producers' contribution to Γ. The producer on edge e on link l on route t receives the following fraction of the payments:*

$$\frac{1/\Gamma_t}{\sum_{t' \in T} 1/\Gamma_{t'}} \frac{\Gamma_l}{\sum_{l' \in t} \Gamma_{l'}} \frac{1/\gamma_e}{\sum_{e' \in l} 1/\gamma_{e'}}.$$

Theorem 6. *The Nash equilibria of the standard and simplified mechanisms are identical, and at equilibrium, the utilities of each player are the same.* □

See Appendix F.1 for a proof. We can also define a true cost B is the same way as we defined Γ:

$$B = \frac{1}{\sum_{t \in T} 1/B_t}, \text{ where } B_t = \sum_{l \in t} B_l \text{ and } B_l = 1/\sum_{e \in l} 1/\beta_e.$$

6.2 Measuring inefficiency on the demand side

Observe that from a consumer's perspective, there is again only a single resource in the simplified mechanism: (s, t)-flow. This provides an intuition for why graph structure turns out not to affect consumer behavior. In fact the same argument as in the single-resource setting establishes the following bound on the price of anarchy (an analogue of Lemma 3), which is independent of graph structure.

Lemma 4. *The price of anarchy in a single-good market is bounded by $2\rho(2-\rho)/(4-\rho)$, where $0 \leq \rho \leq 1$ is an overcharging parameter that equals B/Γ. At equilibrium, this ratio is constrained by the equation*

$$\beta_r = \gamma_r \left(2 - \frac{1}{\epsilon f_r}\right) \text{ for all } r.$$

□

The parameter ρ can be seen as the ratio of the true cost of (s,t)-flow over the price that the users are charged. See Appendix F.2 for a proof.

6.3 Measuring inefficiency on the supply side

Unlike consumer behavior, the behavior of producers depends heavily on the topology of the graph G. In particular, when producers are located on parallel edges, competition tends to drive down the price, whereas when producers are on edges connected in series, the opposite happens.

In this section, we look at how the supply-side efficiency measured by ρ varies with the structure of G and identify the lowest value it can attain. Although the formula for market efficiency can be somewhat complicated, we can precisely describe how horizontal and vertical competition affect the elasticity of the flow at an edge. Analogue formulas for the price of anarchy can then be obtained by plugging the expression for $\epsilon_e f_e$ into ρ.

Theorem 7. *In a two-level series-parallel graph G, let e be an edge located on link l on route t. The elasticity of the (s,t)-flow f_e at e with respect to γ_e equals*

$$\begin{aligned}\epsilon_e f_e = &- \frac{\sum_{e' \in l; e' \neq e} 1/\gamma_{e'}}{\sum_{e' \in l} 1/\gamma_{e'}} - \frac{\sum_{t \in T; t' \neq t} 1/\Gamma_{t'}}{\sum_{t' \in T} 1/\Gamma_{t'}} \frac{\Gamma_l}{\sum_{l' \in t} \Gamma_{l'}} \frac{1/\gamma_e}{\sum_{e' \in l} 1/\gamma_{e'}} \\ &+ \frac{1/\Gamma_t}{\sum_{t' \in T} 1/\Gamma_{t'}} \frac{\Gamma_l}{\sum_{l' \in t} \Gamma_{l'}} \frac{1/\gamma_e}{\sum_{e' \in l} 1/\gamma_{e'}} \epsilon_\Gamma f\end{aligned}$$

□

This somewhat complicated-looking expression actually has three distinct terms. The first term approaches -1 (its best possible value) as horizontal competition at the link containing e increases. Similarly, the other two terms drive the elasticity down when the number of parallel routes increases. On the contrary, when the number of serial links increases, the last two terms increase towards zero, and the elasticity worsens. Thus horizontal competition leads to higher efficiency, while vertical competition drives efficiency down.

The theorems below formalize this claim. Our first results pertain to *route graphs* — graphs containing exactly one route of L serial links — which turn out to admit the worst price of anarchy of all series-parallel graphs. We assume there are at least two edges in every link; otherwise there is no equilibrium in the market. As mentioned earlier, we also assume that costs are of the form $\frac{\beta_r}{2} f_r^2$.

Theorem 8. *Let G be a route graph with two edges per link. Suppose that $\min_{e \in E} \beta_e / \max_{e \in E} \beta_e \geq \Delta$ for some $0 < \Delta \leq 1$. For any fixed L, the price of anarchy is bounded by a constant. As $L \to \infty$, the price of anarchy goes to zero.* □

But just as in a single-resource market, having only three competitors per link guarantees a constant price of anarchy bound.

Theorem 9. *Let G be a route graph with three producers per link and suppose that for all $l \in L$ $\min_e \beta_e / \max_e \beta_e \geq \Delta_l$ for all $e \in l$ and for some $0 < \Delta_l \leq 1$. The price of anarchy is bounded by a constant.* □

The core of our argument is the same as in the proof of Theorem 4. Similar techniques also yield the following result.

Theorem 10. *Let $G = (V, E)$ be a route graph with m producers per link and suppose that there exists a $0 < \Delta \leq 1$ such that $\min_{e \in E} \beta_e / \max_{e \in E} \beta_e \geq \Delta$ for all m. As m goes to infinity, ρ goes to one.* □

In general series-parallel graphs, there is more competition among producers, since consumers are offered alternative routes. The following theorems show that this indeed improves the price of anarchy.

Theorem 11. *Let G be a two-level series-parallel graph with at least three providers on every link and suppose that and suppose that there exists a $0 < \Delta \leq 1$ such that $\min_{e \in E} \beta_e / \max_{e \in E} \beta_e \geq \Delta$. When the number of parallel routes of G goes to infinity, the elasticity at each edge tends to that obtained from linear valuations.* □

Moreover, for any demand elasticity, the route graph exhibits the worst price of anarchy.

Theorem 12. *Let G be a two-level series-parallel graph. The price of anarchy of G is lower-bounded by that of a route series-parallel graph.* □

Proofs of all the above theorems are found in Appendix F.3.

7 Multi-resource markets over arbitrary graphs

Consider the model we defined at the beginning of the paper. Let G be an arbitrary graph and let us return to our initial assumptions on user utilities (Assumptions 1 and 2). Although we can no longer precisely describe how the structure of an arbitrary graph affects efficiency, our two most important results carry over to this general setting.

Theorem 13. *Let G be an arbitrary graph with three producers per link and suppose that $\min_{e \in E} \beta_e / \max_{e \in E} \beta_e \geq \Delta$ for some $0 < \Delta \leq 1$. The price of anarchy is bounded by a constant.*

Theorem 14. *Let G be an arbitrary graph with m producers per link and suppose that $\min_{e \in E} \beta_e / \max_{e \in E} \beta_e \geq \Delta$ for some $0 < \Delta \leq 1$. As m goes to infinity, ρ goes to one and the price of anarchy goes to $2/3$.*

These results are established using the same three-step process that was used in previous sections. See Appendix G for more details.

8 Existence of Nash equilibria

With the techniques we developed in earlier sections, we can establish the following result, which is as strong as the best equilibrium result for Bertrand pricing mechanisms over graphs (Correa et al. 2010).

Theorem 15. *Let G be a series-parallel graph with at least two producers per link. Suppose producers' costs are quadratic and that consumers' valuations are linear. Then there exists a Nash equilibrium and best-responses converge on both the demand and the supply side.*

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A Properties of elasticity

Definition 1. *Let $f(x) : \mathbb{R} \to \mathbb{R}$ be a differentiable function. The elasticity of f with respect to x is a function $\epsilon_x f(y) : \mathbb{R} \to \mathbb{R}$ defined as*

$$\epsilon_x \, f(y) = \frac{df(y)}{dy} \frac{y}{f(y)}$$

We use the following properties of elasticity in most of our proofs.

Lemma A1. Let $f(x), g(x) : \mathbb{R} \to \mathbb{R}$ be differentiable functions. Let $a \in \mathbb{R}$. Then the following holds:

1. $\epsilon_x \, af = \epsilon_x \, f$ for $a > 0$.
2. $\epsilon_x \, f^a = a \, \epsilon_x \, f$
3. $\epsilon_x \, f(g(\cdot)) = \epsilon_g \, f \; \epsilon_x \, g$
4. $\epsilon_x \, (f+g) = \frac{f}{f+g} \epsilon_x \, f + \frac{g}{f+g} \epsilon_x \, g$
5. A function has constant point elasticity if and only if it is a monomial.

Proof. Follows from Definition 1. □

B Necessary and sufficient equilibrium conditions

Here, we characterize equilibria of the mechanism as solutions to a set of inequalities that arise from the optimization problem of each user. Recall the definition $d_q = \sum_{p \in P_q} d_{qp}$.

Lemma B2. A set of strategies $\{\boldsymbol{d}_q, \gamma_r \,|\, q \in Q, r \in R\}$ is a Nash equilibrium of the Bertrand-Cournot mechanism if and only if the following holds:

$$V_q'(d_q) \geq \sum_{p'} \sum_{e \in p \cap p'} \gamma_e d_{qp'} + \sum_{e \in p} \gamma_e f_e \text{ for all } q \in Q,\, p \in P_q \tag{3}$$

$$V_q'(d_q) \leq \sum_{p'} \sum_{e \in p \cap p'} \gamma_e d_{qp'} + \sum_{e \in p} \gamma_e f_e \text{ for all } q \in Q,\, p \in P_q \text{ s.t. } d_{qp} > 0 \tag{4}$$

$$c_r(f_r) \leq \gamma_r f_r \left(2 - \frac{1}{\epsilon f_r}\right) \text{ for all } r \in R \tag{5}$$

$$c_r(f_r) \geq \gamma_r f_r \left(2 - \frac{1}{\epsilon f_r}\right) \text{ for all } r \in R \text{ s.t. } f_r > 0 \tag{6}$$

Proof. Recall that the users' utility functions are

$$U_q(\boldsymbol{d_q}, \boldsymbol{d}_{-q}, \boldsymbol{\gamma}) = V_q(\sum_{p \in P_q} d_{qp}) - \sum_{p \in P_q} d_{qp} \left(\sum_{e \in p} \gamma_e f_e\right)$$
$$U_r(\gamma_r, \boldsymbol{\gamma}_{-r}) = \gamma_r f_r^2 - C_r(f_r)$$

Observe that for each consumer q and each vector of bids $(\boldsymbol{d}_{-q}, \boldsymbol{\gamma})$, U_q is continuous and differentiable in d_{qp} for any p over $[0, \infty)$. Moreover, U_q is concave over $[0, \infty)^P$ in $\boldsymbol{d}_q$. Thus, at $\boldsymbol{d}_q \in [0, \infty)^P \backslash \{0\}$ a consumer q would not want to unilaterally deviate from equilibrium if and only if for any $p \in P_q$:

$$\frac{\partial U_q}{\partial d_{qp}}(\boldsymbol{d}_q) = V_q'(d_q) - \sum_{p'} \sum_{e \in p \cap p'} \gamma_e d_{p'} - \sum_{e \in p} \gamma_e f_e = 0.$$

At $\boldsymbol{d}_q = 0$, a consumer would not want to unilaterally deviate if and only if for any $p \in P_q$:

$$\frac{\partial U_q}{\partial d_{qp}}(\boldsymbol{d}_q) = V_q'(d_q) - \sum_{p'} \sum_{e \in p \cap p'} \gamma_e d_{p'} - \sum_{e \in p} \gamma_e f_e \leq 0.$$

Together, these two conditions give us (4), (3).

When $f_r > 0$, the producer r's utility function is differentiable and the producer plays a best response if and only if

$$\frac{\partial U_r}{\partial \gamma}(f_r) = f_r^2 + 2\gamma_r f_r \frac{\partial f}{\partial \gamma} - c_r(f_r) \frac{\partial f_r}{\partial \gamma_r} = 0.$$

which translates to

$$\begin{aligned} c_r(f) &= \frac{f_r^2 + 2\gamma f_r \frac{\partial f_r}{\partial \gamma_r}}{\frac{\partial f_r}{\partial \gamma_r}} \\ &= \frac{2\gamma f_r |\frac{\partial f_r}{\partial \gamma_r}| - f_r^2}{|\frac{\partial f_r}{\partial \gamma_r}|} \\ &= \gamma f_r \left(2 - \frac{f_r}{\gamma |\frac{\partial f_r}{\partial \gamma_r}|}\right) \end{aligned}$$

When $f = 0$ the above equation becomes an inequality, and combining the two cases $f > 0$ and $f = 0$ we obtain conditions (6) and (5). □

C Equivalence of the simplified single-link mechanism

Here, we show that the simplified version of the Bertrand-Cournot mechanism at a single link is equivalent to the original version.

Recall the definition of the simplified mechanism.

Definition 2. *In the* simplified *single-link Bertrand-Cournot mechanism,*

1. *Producers submit linear pricing functions as in the regular mechanism. The aggregate pricing function is set to $P(f) = \Gamma f$, where*

$$\Gamma = \frac{1}{\sum_e 1/\gamma_e}.$$

2. *Consumer q submits an (s,t)-flow d_q and pays for it $\Gamma f d_q$, where $f = \sum_{q \in Q} d_q$.*
3. *The mechanism sends*
$$f_e = \frac{1/\gamma_e}{\sum_{e' \in E} 1/\gamma_{e'}} f$$
over edge e and pays the producer $\gamma_e f_e^2$.

Theorem 1. *The Nash equilibria of the standard and simplified mechanisms are identical, and at equilibrium, the utilities of each player are the same.*

Proof. Applied to the single-link game, the necessary and sufficient conditions of Lemma B2 state that:

$$V_q'(d_q) \leq \gamma_e(d_{qe} + f_e) \text{ for all } e \text{ s.t. } d_{qe} > 0 \tag{7}$$
$$V_q'(d_q) \geq \gamma_e(d_{qe} + f_e) \text{ for all } e \tag{8}$$
$$C_r'(f_r) \geq \gamma_r f_r \left(2 - \frac{1}{\epsilon f_r}\right) \text{ if } f > 0 \tag{9}$$
$$C_r'(f_r) \leq \gamma_r f_r \left(2 - \frac{1}{\epsilon f_r}\right) \tag{10}$$

It is not hard to show that the equilibrium conditions for the aggregate game are:

$$V_q'(d_q) \leq \Gamma(d_q + f) \text{ for all } q \text{ s.t. } d_q > 0 \tag{11}$$
$$V_q'(d_q) \geq \Gamma(d_q + f) \text{ for all } q \tag{12}$$
$$C_r'(f_r) \geq \gamma_r f_r \left(2 - \frac{1}{\epsilon f_r}\right) \text{ if } f_r > 0 \tag{13}$$
$$C_r'(f_r) \leq \gamma_r f_r \left(2 - \frac{1}{\epsilon f_r}\right) \tag{14}$$

Take any demand-side equilibrium $\boldsymbol{d}$ of the aggregate game. We have to show that conditions (7-8) hold for

$$d_{qe} = \frac{1/\gamma_e}{\sum_{e'} 1/\gamma_{e'}} d_q$$

Condition (7) for any edge e follows by algebraic manipulations:

$$V_q'(d_q) \leq \Gamma(d_q + f) = \frac{\gamma_e/\gamma_e}{\sum_{e'} 1/\gamma_{e'}}(d_q + f) = \gamma_e\left(\frac{1/\gamma_e}{\sum_{e'} 1/\gamma_{e'}} d_q + \frac{1/\gamma_e}{\sum_{e'} 1/\gamma_{e'}} f\right) = \gamma_q(d_{qe} + f_e)$$

Condition (8) follows similarly.

Recall that demand-side Cournot games have unique equilibria. Thus both games have the same equilibria, and for any $\boldsymbol{\gamma}$, the demand at each edge remains the same in the aggregate game. Thus the function $f_r(\boldsymbol{\gamma})$ is unchanged, and the equilibrium prices $\boldsymbol{\gamma}$ of the aggregate game also form a supply-side equilibrium in the original game. □

Definition 3. *When the marginal costs at the edges are of the form* $c_e(f) = \beta_e f$, *the slope of the* aggregate cost *function of the link is defined to be*

$$B = \frac{1}{\sum_e 1/\beta_e}.$$

Theorem 2. *When producers' marginal costs are linear, a cost-minimizing allocation* f *costs the producers a total of* $\frac{B}{2}f^2$. □

Proof. When f units of flow are sent in a socially optimal way across the link, the marginal costs $\beta_e f_e$ at each edge e must be equal. From this requirement, it's easy to derive the formula

$$f_e = \frac{1/\beta_e}{\sum_{e' \in l} 1/\beta_{e'}} f$$

and the total cost to society is

$$\sum_{e \in l} \frac{\beta_e}{2} f_e^2 = \frac{1}{2} \sum_{e \in l} \frac{1/\beta_e}{(\sum_{\epsilon' \in l} 1/\beta_{e'})^2} f^2 = \frac{1}{2 \sum_{e' \in l} 1/\beta_{e'}} f^2.$$

□

D Measuring inefficiencies on the demand side for one link.

Here, we analyze the loss of welfare that is due to demand-side inefficiency in a single-link market. Recall that we defined demand-side inefficiency to be the loss of welfare due to low-value consumers selfishly demanding resources that should go to higher-value consumers.

Specifically, the end result of this appendix is a proof of Lemma 3, which established that the full price of anarchy is a function of an overcharging parameter ρ measuring supply-side inefficiency. To establish that lemma, we proceed in steps outlined in the beginning of Section 5.

D.1 Linearizing valuation and marginal cost functions

We start by showing that the price of anarchy in any instance of the mechanism is lower-bounded by that of another instance in which users have linear valuations and linear marginal costs. Specifically, we prove Lemma 1:

Lemma 1. *Given any game instance* $\mathcal{G}$, *one can construct a new game instance* $\mathcal{G}'$ *where:*

1. *Consumers have linear valuations and producers have quadratic costs.*
2. *Producers set prices as if the demand functions* f_r *they were facing were the ones in* $\mathcal{G}$.

The price of anarchy of $\mathcal{G}'$ is a lower bound on that of $\mathcal{G}$. □

The formal statement of this lemma is as follows:

Lemma D3. Suppose agents have valuations $\{V_q \,|\, q \in Q\}$ and costs $\{C_r \,|\, r \in R\}$. Let $\{d_q, \gamma_r \,|\, q \in Q, r \in R\}$ be an equilibrium, let $\{d_q^*, f_r^* \,|\, q \in Q, r \in R\}$ be optimal allocations, and let $f_r(\boldsymbol{\gamma})$ denote the demand functions induced by the V_q.

There exist linear valuations $\{\bar{V}_q(d) = \alpha_q d \,|\, q \in Q\}$ and quadratic costs $\{\bar{C}_r(f) = \frac{\beta_r}{2} f \,|\, r \in R\}$ such that

$$\frac{\sum_{q\in Q} V_q(d_q) - \sum_{r \in R} C_r(f_r(\boldsymbol{\gamma}))}{\sum_{q\in Q} V_q(d_q^*) - \sum_{r \in R} C_r(f_r^*)} \geq \frac{\sum_{q\in Q} \bar{V}_q(d_q) - \sum_{r \in R} \bar{C}_r(f_r(\boldsymbol{\gamma}))}{\sum_{q\in Q} \bar{V}_q(\bar{d}_q^*) - \sum_{r \in R} \bar{C}_r(\bar{f}_r^*)} \tag{15}$$

where

– The γ_r form a supply-side equilibrium given the original demand functions f_r and the new linear costs $\bar{C}_r$.
– The d_q form a demand-side equilibrium given the γ_r and the new linear valuations $\bar{V}_q$.
– The $\bar{d}_q^*$ and the $\bar{f}_r^*$ form an optimal allocation with the new linear valuations and quadratic costs.

When the initial V_q are linear, this bound is tight. □

To do that, we first need to establish several sub-lemmas. Note that the proofs in this subsection assume that consumers play the regular version of the Bertrand-Cournot mechanism, and not the simplified version. This makes our argument more general; one can easily verify that it holds for the simplified version too.

Lemma D4. Let $(\boldsymbol{d}, \boldsymbol{g})$ be any vector of allocations, and let $\boldsymbol{d}^*$, $\boldsymbol{f}^*$ be welfare-maximizing allocations. Then

$$\frac{\sum_{q\in Q} V_q(d_q) - \sum_{r \in R} C_r(f_r)}{\sum_{q\in Q} V_q(d_q^*) - \sum_{r \in R} C_r(f_r^*)} \geq \frac{\sum_{q\in Q} V_q'(d_q) d_q - \sum_{r \in R} C_r(f_r)}{\max_{\bar{\boldsymbol{f}} \geq 0} \left(\left(\max_{q \in Q} V_q'(d_q) \right) \sum_r \bar{f}_r - \sum_{r \in R} C_r(\bar{f}_r) \right)}$$

Proof. By concavity, for any q we have $V_q(d_q^*) \leq V_q(d_q) + V_q'(d_q)(d_q^* - d_q)$. Since $V_q(0) \geq 0$ by assumption, $V_q(d_q) \geq V_q'(d_q) d_q$. Using these two inequalities we obtain

$$\begin{aligned}
\frac{\sum_{q\in Q} V_q(d_q) - \sum_{r \in R} C_r(f_r)}{\sum_{q\in Q} V_q(d_q^*) - \sum_{r \in R} C_r(f_r^*)} &\geq \frac{\sum_{q\in Q} (V_q(d_q) - V_q'(d_q) d_q) + \sum_{q \in Q} V_q'(d_q) d_q - \sum_{r \in R} C_r(f_r)}{\sum_{q\in Q} (V_q(d_q) - V_q'(d_q) d_q) + \sum_{q \in Q} V_q'(d_q) d_q^* - \sum_{r \in R} C_r(f_r^*)} \\
&\geq \frac{\sum_{q\in Q} V_q'(d_q) d_q - \sum_{r \in R} C_r(f_r)}{\sum_{q\in Q} V_q'(d_q) d_q^* - \sum_{r \in R} C_r(f_r^*)} \\
&\geq \frac{\sum_{q\in Q} V_q'(d_q) d_q - \sum_{r \in R} C_r(f_r)}{\max_{\bar{\boldsymbol{f}} \geq 0} \left(\left(\max_{q \in Q} V_q'(d_q) \right) \sum_r \bar{f}_r - \sum_{r \in R} C_r(\bar{f}_r) \right)}.
\end{aligned}$$

In the last line, observe that the consumer with the highest marginal utility receives all the supply. □

Lemma D5. Let $(\boldsymbol{d}, \boldsymbol{\gamma})$ be a vector of strategies such that

1. The vector $\boldsymbol{d}$ forms a demand-side equilibrium given pricing functions $\boldsymbol{\gamma}$.
2. The vector $\boldsymbol{\gamma}$ forms a supply-side equilibrium given elasticity functions $\boldsymbol{\epsilon f}$.

Then there exist $\alpha_q > 0$ for all $q \in Q$ such that

$$\frac{\sum_{q\in Q} V_q(d_q) - \sum_{r\in R} C_r(f_r)}{\sum_{q\in Q} V_q(d_q^*) - \sum_{r\in R} C_r(f_r^*)} \geq \frac{\sum_{q\in Q} \alpha_q d_q - \sum_{r\in R} C_r(f_r)}{\max_{\bar{f}_r} \left((\max_{q\in Q} \alpha_q) \sum_r \bar{f}_r - \sum_{r\in R} C_r(\bar{f}_r) \right)} \tag{16}$$

The vector $\boldsymbol{d}$ is still a demand-side equilibrium given the pricing functions $\boldsymbol{\gamma}$ and the new valuation functions and $\boldsymbol{\gamma}$ is still a supply-side equilibrium given the original elasticity functions $\boldsymbol{\epsilon f}$.

Proof. The inequality follows by applying Lemma D4 at $(\boldsymbol{d}, \boldsymbol{\gamma})$ and choosing $\alpha_q = V_q'(d_q)$ for all q.

Observe that when valuations are linear the derivatives at the equilibrium $\boldsymbol{d}$ are the same as with the original valuations. Thus by the first-order conditions of Lemma B2, $\boldsymbol{d}$ remains an equilibrium. The prices $\boldsymbol{\gamma}$ are still an equilibrium, because provider utilities have not changed. □

The above lemma establishes that the price of anarchy is always worse within games where consumers have linear valuations. It now remains to show that we can similarly assume that marginal costs are linear in the worst case. Before we establish that in Lemma D7, we will need to show that the flow at every edge is greater at optimum than at equilibrium.

Lemma D6. Suppose consumers have linear valuations, and let $(\boldsymbol{d}, \boldsymbol{\gamma})$ be a vector of strategies such that

1. The vector $\boldsymbol{d}$ forms a demand-side equilibrium given pricing functions $\boldsymbol{\gamma}$.
2. The vector $\boldsymbol{\gamma}$ forms a supply-side equilibrium given elasticity functions ϵf_e that satisfy $|\epsilon f_e| \leq 1$.

Let $(d_q^*)_{q\in Q}$ be a welfare-maximizing flow allocation and let $f_e = \sum_{q\in Q} d_{qe}$, $f_e^* = \sum_{q\in Q} d_{qe}^*$. Then $f_e \leq f_e^*$ for all e.

Proof. First observe that for any $q \in Q$, $e \in E$,

$$V_q'(d_q) \underbrace{\geq}_{\text{by (3)}} \gamma_e f_e \left(1 + \frac{d_{qe}}{f_e}\right) \underbrace{\geq}_{f, d_q \geq 0} \gamma_e f_e \underbrace{\geq}_{\text{by } |\epsilon f_r| \leq 1} c_e(f_e) \tag{17}$$

Recall that c_e denotes the marginal cost of the producer at edge e.

Now suppose for a contradiction that $f_e > f_e^*$ for some e. Then there exists a q such that $d_{qe} > d_{qe}^*$, and $d_q > 0$. Since we assumed the V_q were linear, $V_q'(d_q^*) \geq V_q'(d_q)$, and by the strict monotonicity of c_e, $c_e(f_e) > c_e(f_e^*)$. Combining this with (17), we obtain $V_q'(d_q^*) > c_e(f_e^*)$. But at optimum, we must have $V_q'(d_q^*) = c_e(f_e^*)$ for all e and all q such that $d_q' > 0$, and thus we arrive at a contradiction. □

The assumption that $|\epsilon f_e| \leq 1$ is not particularly strong; it is satisfied by most interesting valuation functions, including monomial functions $V(d) = cd^x$.

Lemma D7. Let $(\boldsymbol{d}, \boldsymbol{\gamma})$ be such that

1. The demands $\boldsymbol{d}$ form a demand-side equilibrium given $\boldsymbol{\gamma}$.
2. The prices $\boldsymbol{\gamma}$ form a supply-side equilibrium given some vector of elasticity functions $\boldsymbol{\epsilon f}$.

Suppose that consumers have linear valuations with slopes $(\alpha_q)_{q\in Q}$. Then there exist $\beta_r > 0$ for all $r \in R$ such that

$$\frac{\sum_{q\in Q} \alpha_q d_q - \sum_{r\in R} C_r(f_r)}{\sum_{q\in Q} \alpha_q d_q^* - \sum_{r\in R} C_r(f_r^*)} \geq \frac{\sum_{q\in Q} \alpha_q d_q - \sum_{r\in R} \beta_r (f_r)^2}{\sum_{q\in Q} \alpha_q d_q^{**} - \sum_{r\in R} \beta_r (f_r^{**})^2} \tag{18}$$

where d_q^{**} and f_r^{**} are welfare-maximizing allocations when the cost function of every producer r is $C_r(f_r) = \frac{\beta_r}{2} f_r^2$.

Moreover, $\boldsymbol{\gamma}$ is still a supply-side equilibrium given $\boldsymbol{\epsilon f}$ and the new cost functions.

Proof. Given a producer r, let f_r be the equilibrium flow at their edge, and define $\beta_r = c_r(f_r)/f_r$. Recall that c_r denotes the marginal cost of producer r. We will establish the lemma for these β_r in two steps. First, we will define an intermediary cost function $\hat{C}_r(f)$ and show that replacing the C_r by the $\hat{C}_r$ can only reduce the left-hand side in (D.1). Then, we will show that going from $\hat{C}_r$ to $\frac{\beta_r}{2} f^2$ yields the lower bound in the right-hand side of (D.1), thus establishing the claim.

Let $\hat{C}_r(f)$ be the cost function uniquely determined by the marginal cost function

$$\hat{c}_r(f) = \begin{cases} c_r(f) & \text{if } f \leq f_r \\ \beta f & \text{if } f_r \leq f \end{cases}$$

Observe that $\boldsymbol{d}$ and $\boldsymbol{\gamma}$ are still respectively demand-side and supply-side equilibria once we replace the C_r by the $\hat{C}_r$. That is because $c_r(f_r) = \hat{c}_r(f_r)$ at the equilibrium flow f_r, and the necessary and sufficient conditions (4)-(5) still hold at that point.

Moreover, $C(f_r) = \hat{C}(f_r)$, and the social welfare at $(\boldsymbol{d}, \boldsymbol{\gamma})$ is unchanged. The optimal social welfare, on the other hand, can only improve, as

1. For $f_r \leq f$ $\hat{c}_r(f) = \beta f \leq c_r(f)$ by convexity of the marginal cost function
2. For $f \leq f_r$, $\hat{c}_r(f) = c_r(f)$

These two observations imply that $\hat{C}_r(f) \leq C_r(f)$ for all f, r. Thus replacing the C_r by the $\hat{C}_r$ can only reduce the left-hand side in (D.1). Formally, we have shown that

$$\frac{\sum_{q\in Q} \alpha_q d_q - \sum_{r\in R} C_r(f_r)}{\sum_{q\in Q} \alpha_q d_q^* - \sum_{r\in R} C_r(f_r^*)} \geq \frac{\sum_{q\in Q} \alpha_q d_q - \sum_{r\in R} \beta_r (f_r)^2}{\sum_{q\in Q} \alpha_q \hat{d}_q^* - \sum_{r\in R} \hat{C}_r (\hat{f}_r^*)^2}$$

where $\hat{d}_q^*$ and $\hat{f}_r^*$ denote optimal allocation when the cost functions are $\hat{C}_r$.

Now consider what happens when we pass from $\hat{C}_r$ to $\frac{\beta_r}{2} f^2$. Observe that the marginal cost at equilibrium is βf_r for both functions, and therefore $\boldsymbol{d}$ and $\boldsymbol{\gamma}$ remain equilibrium points. Let $\hat{f}^*$ be the flow at optimum when the cost functions are $\hat{C}_r$. Since valuations are linear, by Lemma D6, $f_r \leq \hat{f}^*$. Observe also that

1. When $f_r \leq f$, $\beta f = \hat{c}_r(f)$.
2. When $f \leq f_r$, $\beta f \geq \hat{c}_r(f)$ by convexity of the marginal cost function.

Thus $\frac{\beta}{2} f^2 \geq \hat{C}_r(f)$ for all f.

Let f_r^{**} be the flow at optimum at edge r when the cost functions are $\frac{\beta_r}{2} f^2$. Then $\frac{\beta_r}{2}(f_r^{**})^2 - \hat{C}(\hat{f}_r^*) = \frac{\beta}{2}(f_r)^2 - \hat{C}(f_r) \geq 0$. This value represents the area on the graph between the function $\hat{c}_r(f)$ and $\beta_r f$. Now, the price of anarchy can be bounded as follows:

$$\begin{aligned}
\frac{\sum_{q\in Q} \alpha_q d_q - \sum_{r\in R} \hat{C}_r(f_r)}{\sum_{q\in Q} \alpha_q \hat{d}_q^* - \sum_{r\in R} \hat{C}_r(\hat{f}_r^*)} &\geq \frac{\sum_{q\in Q} \alpha_q d_q - \sum_{r\in R} \hat{C}_r(f_r) - \sum_{r\in R}(\frac{\beta_r}{2}(f_r)^2 - \hat{C}_r(f_r))}{\sum_{q\in Q} \alpha_q \hat{d}_q^* - \sum_{r\in R} \hat{C}(\hat{f}_r^*) - \sum_{r\in R}(\frac{\beta_r}{2}(f_r^{**})^2 - \hat{C}_r(\hat{f}^*))} \\
&= \frac{\sum_{q\in Q} \alpha_q d_q - \sum_{r\in R} \frac{\beta_r}{2}(f_r)^2}{\sum_{q\in Q} \alpha_q \hat{d}_q^* - \sum_{r\in R} \frac{\beta_r}{2}(f^{**})^2} \\
&\geq \frac{\sum_{q\in Q} \alpha_q d_q - \sum_{r\in R} \frac{\beta_r}{2}(f_r)^2}{\sum_{q\in Q} \alpha_q d_q^{**} - \sum_{r\in R} \frac{\beta_r}{2}(f^{**})^2}
\end{aligned}$$

The latter ratio is the right-hand side of (D.1). This completes the proof. □

Combining Lemma D5 and Lemma D7, we obtain Lemma D3.

Lemma D3. Suppose agents have valuations $\{V_q \mid q \in Q\}$ and costs $\{C_r \mid r \in R\}$. Let $\{d_q, \gamma_r \mid q \in Q, r \in R\}$ be an equilibrium, let $\{d_q^*, f_r^* \mid q \in Q, r \in R\}$ be optimal allocations, and let $f_r(\boldsymbol{\gamma})$ denote the demand functions induced by the V_q.

There exist linear valuations $\{\bar{V}_q(d) = \alpha_q d \mid q \in Q\}$ and quadratic costs $\{\bar{C}_r(f) = \frac{\beta_r}{2} f \mid r \in R\}$ such that

$$\frac{\sum_{q\in Q} V_q(d_q) - \sum_{r\in R} C_r(f_r(\boldsymbol{\gamma}))}{\sum_{q\in Q} V_q(d_q^*) - \sum_{r\in R} C_r(f_r^*)} \geq \frac{\sum_{q\in Q} \bar{V}_q(d_q) - \sum_{r\in R} \bar{C}_r(f_r(\boldsymbol{\gamma}))}{\sum_{q\in Q} \bar{V}_q(\bar{d}_q^*) - \sum_{r\in R} \bar{C}_r(\bar{f}_r^*)} \tag{19}$$

where

- The γ_r form a supply-side equilibrium given the original demand functions f_r and the new linear costs $\bar{C}_r$.
- The d_q form a demand-side equilibrium given the γ_r and the new linear valuations $\bar{V}_q$.
- The $\bar{d}_q^*$ and the $\bar{f}_r^*$ form an optimal allocation with the new linear valuations and quadratic costs.

When the initial V_q are linear, this bound is tight.

Proof. Let V_q, $q \in Q$ denote valuation functions satisfying Assumption 1 and let C_r, $r \in R$ denote cost functions satisfying Assumption 2. Let $(\boldsymbol{d}, \boldsymbol{\gamma})$ be an equilibrium of the mechanism. Let $\boldsymbol{\epsilon f}$ be the vector of elasticity functions at every edge when demand functions are V_q. The price of anarchy is the ratio

$$\frac{\sum_{q\in Q} V_q(d_q) - \sum_{r\in R} C_r(f_r)}{\sum_{q\in Q} V_q(d_q^*) - \sum_{r\in R} C_r(f_r^*)}.$$

By Lemma D5, there exist a set of $\alpha_q > 0$ for $q \in Q$ such that the following bound holds

$$\frac{\sum_{q\in Q} V_q(d_q) - \sum_{r\in R} C_r(f_r)}{\sum_{q\in Q} V_q(d_q^*) - \sum_{r\in R} C_r(f_r^*)} \geq \frac{\sum_{q\in Q} \alpha_q d_q - \sum_{r\in R} C_r(f_r)}{\max_{\bar{f}_r} \left((\max_{q\in Q} \alpha_q) \sum_r \bar{f}_r - \sum_{r\in R} C_r(\bar{f}_r)\right)} \tag{20}$$

and $\boldsymbol{d}$ is still a demand-side equilibrium given $\boldsymbol{\gamma}$. The vector $\boldsymbol{\gamma}$ is still a supply-side equilibrium given the original elasticities $\boldsymbol{\epsilon f}$.

By Lemma D7, there exist $\beta_r > 0$ for all $r \in R$ such that

$$\frac{\sum_{q\in Q} \alpha_q d_q - \sum_{r\in R} C_r(f_r)}{\max_{\bar{f}_r} \left((\max_{q\in Q} \alpha_q) \sum_r \bar{f}_r - \sum_{r\in R} C_r(\bar{f}_r)\right)} \geq \frac{\sum_{q\in Q} \alpha_q d_q - \sum_{r\in R} \beta_r (f_r)^2}{\max_{\bar{f}_r} \left((\max_{q\in Q} \alpha_q) \sum_r \bar{f}_r - \sum_{r\in R} \frac{\beta_r}{2} (\bar{f}_r)^2\right)}$$

and $\boldsymbol{\gamma}$ is still a supply-side equilibrium given $\boldsymbol{\epsilon f}$ when producers' cost functions are replaced by $C_r(f) = \frac{\beta_r}{2} f^2$. This establishes the claim. □

D.2 Formulating and solving a minimization problem

The above lemma says that it is enough to search for the worst price of anarchy within games where users have linear valuations and marginal costs. For fixed Q, R, this search space is finite-dimensional, and so we formally describe the price of anarchy as the solution to a minimization problem.

Here, we formulate and solve that problem. Specifically, we prove Lemmas 2 and 3.

Lemma 2. *The price of anarchy is lower-bounded by the solution to:*

$$\begin{aligned} \min \quad & \frac{\sum_{q=1}^{Q} \alpha_q d_q - \frac{B}{2} f^2}{\max_{\bar{f}} (\max_{q\in Q} \alpha_q \bar{f} - \frac{B}{2} \bar{f}^2)} \\ s.t. \quad & \alpha_q = \Gamma f + \Gamma d_q \text{ for all } q \in Q \\ & \beta_r = \gamma_r \left(2 - \frac{1}{\epsilon f_r}\right) \text{ for all } r \in R \\ & \sum_{q\in Q} d_q = f \\ & 0 \leq \alpha_q, d_q, \Gamma, B \end{aligned}$$

When valuation functions are linear, this bound is tight. □

Proof. By Lemma D3, to lower-bound the price of anarchy, it is enough to find a set of linear valuations $V_q(d) = \alpha_q d$, quadratic costs $C_r(f) = \frac{\beta_r}{2} f^2$, and elasticities $\boldsymbol{\epsilon f}$ that result in the worst welfare loss at a point $(\boldsymbol{d}, \boldsymbol{\gamma})$ that satisfies:

1. The vector $\boldsymbol{d}$ is a demand-side equilibrium given $\boldsymbol{\gamma}$ and linear valuations.
2. The vector $\boldsymbol{\gamma}$ is a supply-side equilibrium given $\boldsymbol{\epsilon f}$ and quadratic costs.

Formally, the price of anarchy is lower-bounded by the optimum of the following optimization problem:

$$\begin{aligned}
\min \quad & \frac{\sum_{q=1}^{Q} \alpha_q d_q - \frac{B}{2} f^2}{\max_{\bar{f}}(\max_{q\in Q} \alpha_q \bar{f} - \frac{B}{2}\bar{f}^2)} \\
\text{s.t.} \quad & \alpha_q \le \mathit{\Gamma} f + \mathit{\Gamma} d_q \text{ if } d_q > 0, \forall q \in Q \\
& \alpha_q \ge \mathit{\Gamma} f + \mathit{\Gamma} d_q, \forall q \in Q \\
& \beta_r \ge \gamma_r \left(2 - \frac{1}{\epsilon f_r}\right) \text{ if } f_r > 0, \forall r \in R \\
& \beta_r \le \gamma_r \left(2 - \frac{1}{\epsilon f_r}\right), \forall r \in R \\
& \sum_{q\in Q} d_q = f \\
& 0 < \alpha_q \le 1, \forall q \in Q \\
& 0 \le d_q, \mathit{\Gamma}, B
\end{aligned}$$

The objective function is precisely the bound of Lemma D7; it is minimized over all possible linear valuations with slopes α_q, all possible quadratic costs with parameters β_r, and all possible strategies $\boldsymbol{d}$, $\boldsymbol{\gamma}$ under the constraints we informally defined above.

□

Lemma 3. *The price of anarchy in a single-good market is bounded by*

$$\frac{2\rho(2-\rho)}{4-\rho}$$

where $0 \le \rho \le 1$ is an overcharging parameter that equals $B/\mathit{\Gamma}$. At equilibrium, this ratio is constrained by the supply-side condition

$$\beta_r = \gamma_r \left(2 - \frac{1}{\epsilon f_r}\right) \textit{ for all } r.$$

When valuations are linear, this bound is tight. □

Proof. The optimal allocation when users have linear valuations and linear marginal costs can be calculated using Theorem 2. In particular, the least expensive way to send a flow of f costs the producers $\frac{B}{2} f^2$. The highest-value consumer derives a utility of of $\max_q \alpha_q f$ from that flow. Thus, the optimal welfare is the maximum of

$$\max_q \alpha_q f - \frac{B}{2} f^2,$$

which is

$$\frac{\max_q \alpha_q^2}{2B}.$$

Observe also that in the minimization program, we can assume that the total flow f in the graph equals 1 and that

$$1 = \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_Q > 0$$

This poses no loss of generality because we can always normalize the numerator and the denominator of the objective function so that these assumptions hold.

Without loss of generality, we can also assume that the necessary and sufficient equilibrium conditions hold with equality. Indeed, if $d_q = 0$ for some q, we can reduce α_q until equality holds without affecting the price of anarchy. If $\Gamma = 0$, then $\alpha_q = 0$ for all q, and the theorem holds trivially. This yields the form of the minimization problem we were seeking.

Thus we have to solve the following program:

$$\begin{aligned}
\min \quad & \frac{d_1 + \sum_{q=2}^{Q} \alpha_q d_q - B/2}{1/2B} && (21)\\
\text{s.t.} \quad & 1 = \Gamma + \Gamma d_1, && (22)\\
& \alpha_q = \Gamma + \Gamma d_q, \text{ for} q = 2, ..., Q && (23)\\
& \beta_r = \gamma_r \left(2 - \frac{1}{\epsilon f_r}\right), \forall r \in R && (24)\\
& \sum_{q \in Q} d_q = 1 && (25)\\
& 0 < \alpha_q \leq 1, \forall q \in Q && (26)\\
& 0 \leq d_q, \Gamma, B && (27)
\end{aligned}$$

Plugging in (22), (23) and (24) into (21) we obtain

$$\begin{aligned}
\min \quad & \frac{\frac{1-\Gamma}{\Gamma} + \sum_{q=2}^{Q} (\Gamma + \Gamma d_q) d_q - B/2}{1/2B} && (28)\\
\text{s.t.} \quad & \sum_{q=2}^{Q} d_q = 1 - \frac{1-\Gamma}{\Gamma} && (29)\\
& \Gamma + \Gamma d_q \leq 1, \forall q \in Q && (30)\\
& \beta_r = \gamma_r \left(2 - \frac{1}{\epsilon f_r}\right), \forall r \in R && (31)\\
& 0 < \Gamma \leq 1 && (32)\\
& 0 \leq d_q && (33)
\end{aligned}$$

Observe that (29) is symmetric and convex in the variables d_q. Since a convex function admits a unique minimum, all d_q must be equal at that point. Thus, we must have $d_q = \frac{1-(1-\Gamma)/\Gamma}{Q-1}$. Observe that that solution is feasible if and only if

$$\frac{1}{Q} \leq \frac{1-\Gamma}{\Gamma} \leq 1$$

The optimization problem becomes:

$$\begin{aligned} \min \quad & \frac{\frac{1-\Gamma}{\Gamma} + (\Gamma + \Gamma\frac{1-(1-\Gamma)/\Gamma}{Q-1})(1-(1-\Gamma)/\Gamma) - B/2}{1/2B} \\ \text{s.t.} \quad & \frac{1}{Q} \leq \frac{1-\Gamma}{\Gamma} \leq 1 \\ & \beta_r = \gamma_r\left(2 - \frac{1}{\epsilon f_r}\right), \forall r \in R \\ & 0 < \Gamma \leq 1 \end{aligned}$$

Observe that the function is decreasing in Q. Thus the minimum occurs when we let $Q \to \infty$. This yields the program

$$\min \quad \frac{\frac{1-\Gamma}{\Gamma} + \Gamma(1-(1-\Gamma)/\Gamma) - B/2}{1/2B} \tag{34}$$

$$\text{s.t.} \quad 0 \leq \frac{1-\Gamma}{\Gamma} \leq 1 \tag{35}$$

$$\beta_r = \gamma_r\left(2 - \frac{1}{\epsilon f_r}\right), \forall r \in R \tag{36}$$

To get rid of B, we introduce the ratio $\rho = B/\Gamma$.

$$\min \quad \frac{\frac{1-\Gamma}{\Gamma} + \Gamma(1-(1-\Gamma)/\Gamma) - \rho\Gamma/2}{1/2\rho\Gamma} \tag{37}$$

$$\text{s.t.} \quad 0 \leq \frac{1-\Gamma}{\Gamma} \leq 1 \tag{38}$$

$$\rho = B/\Gamma \tag{39}$$

$$\beta_r = \gamma_r\left(2 - \frac{1}{\epsilon f_r}\right), \forall r \in R \tag{40}$$

This program finally evaluates to

$$\min \quad \frac{2\rho(2-\rho)}{4-\rho} \tag{41}$$

$$\text{s.t.} \quad \rho = B/\Gamma \tag{42}$$

$$\beta_r = \gamma_r\left(2 - \frac{1}{\epsilon f_r}\right), \forall r \in R \tag{43}$$

□

E Measuring inefficiencies on the supply side for one link

Here we prove lower bounds on ρ in the single-link setting. First, we need to establish a technical lemma.

E.1 Reducing the number of users

The coefficient ρ is a function of the elasticity of demand faced by each producer, which we took as fixed when solving our optimization problem. Here, we will show that the elasticity does not depend on the number of users when they have monomial valuations.

Formally, suppose that the Q consumers have valuation functions $V_q(d_q) = \alpha_q d_q^x$, where $0 < x \leq 1$ and $\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_Q > 0$. Let $f = \sum_{q \in Q} d_q$. Let Γ be the aggregate price for the entire graph. If there was only one consumer, the flow would be the maximizer of that conumer's utility function:

$$U(f) = \alpha f^x - \Gamma f^2.$$

We can check that

$$f = \left(\frac{x\alpha}{2\Gamma}\right)^{1/(2-x)}$$

maximizes $U()$. Consequently, the elasticity of demand with respect to Γ equals

$$\epsilon_\Gamma f = -\frac{1}{2-x}.$$

The goal of this subsection is to show that $\epsilon_\Gamma f = -1/(2-x)$ no matter how many consumers there are. To establish that, we will use the following technical lemma:

Lemma E8 (Matrix determinant). Suppose A is an invertible $n \times n$ matrix, and U, V are $m \times n$ matrices. Then

$$\det(A + UV^T) = \det(I + V^T A^{-1} U) \det(A)$$

Lemma E9. Suppose there are Q consumers with valuation functions $V_q(d_q) = \alpha_q d_q^x$, where $0 < x \leq 1$ and $\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_Q > 0$. Let $f = \sum_{q \in Q} d_q$. Let Γ be the aggregate price for the graph. Then

$$\epsilon_\Gamma f = -\frac{1}{2-x}$$

Proof. At equilibrium, for any q,

$$\alpha_q x d_q^{x-1} = \Gamma(f + d_q)$$

Taking the elasticity on both sides, we obtain

$$(x-1)\epsilon_\Gamma d_q = 1 + \sum_{r \neq q} \frac{d_r}{\sum_{r \neq q} d_r + 2d_q} \epsilon_\Gamma d_r + \frac{2d_q}{\sum_{r \neq q} d_r + 2d_q} \epsilon_\Gamma d_q$$

The function $\epsilon_\Gamma d_q$ must satisfy the above equation. It is easy to check that $\epsilon_\Gamma d_q = -1/(2-x)$ for all q (the elasticity in the single-consumer case) is a solution to this linear system of equations. It remains to verify that there are no other solutions to the system.

Observe that we can rewrite this system using matrices as

$$-1 = ((1-x)I + ABC)\epsilon$$

where ϵ is a vector of variables, $A = diag(1/(\sum_{r\neq 1} d_r + 2d_1), ..., 1/(\sum_{r\neq Q} d_r + 2d_Q))$, $C = diag(d_1, ..., d_Q)$, and B has the form

$$B = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

Observe that we can rewrite the determinant of $(1-x)I + ABC$ as $(1-x)^Q \det(I + \frac{1}{1-x}ABC)$. To evaluate this determinant, we will use the matrix determinant lemma.

Applying the lemma to our determinant, we obtain

$$(1-x)^Q \det(I + \frac{1}{1-x}ABC) = (1-x)^Q \det(B^{-1} + \frac{1}{1-x}AC)\det(B^{-1})$$

It can be verified that the eigenvalues of B are $n+1, 1$, where n is the dimension of B. Thus B and B^{-1} are positive definite. Clearly, AC is also positive definite. Thus $B^{-1} + \frac{1}{1-x}AC$ is positive definite, and hence invertible. It follows that the determinant of $(1-x)I + ABC$ is non-zero and that the solution is unique.

From Lemma A1, we obtain

$$\epsilon_\Gamma f = \epsilon_\Gamma \sum_q d_q = \sum_q \frac{d_q}{\sum_q d_q} \epsilon_\Gamma d_q$$

and since we have just established that $\epsilon_\Gamma d_q = -1/(2-x)$, it follows that $\epsilon_\Gamma f = -1/(2-x)$. □

Thus the elasticity of the flow in the multi-consumer case is identical to the single-consumer case.

E.2 Bounding ρ

Theorem 3. *Suppose the market is a monopoly. Suppose users have monomial valuation functions $V_q(d_q) = \alpha_q d_q^x$, where $0 < x \leq 1$ and $\alpha_q > 0$. The price of anarchy is bounded by*

$$\frac{2x(2-x)}{4-x}$$

When valuations are linear, the bound equals $2/3$ and is tight. □

Proof. By the results of the previous section, the elasticity faced by the provider is $-1/(2-x)$. The theorem then follows directly from Lemma 3.

Lemma E10. As $x \to 0$, the price of anarchy goes to zero. That is, the lower bound of the previous theorem is also tight in the limit.

Proof. Suppose there is only one user with a valuation function $V(f) = f^x$ and one producer with a cost of f^2. As $x \to 0$, the price γ goes to infinity, and the equilibrium welfare goes to zero. □

Theorem 4. *Suppose there are at least 3 producers. Suppose there is a* $0 < \Delta \leq 1$ *such that* $\min_e \beta_e / \max_e \beta_e \geq \Delta$*. Then the price of anarchy is bounded by a constant.*

Proof. In the simplified mechanism, the flow f_e over edge e equals $S_e f$, where f is the total flow across the link, and S_e denotes the fraction that is routed through edge e. Using properties of elasticity, we obtain

$$\begin{aligned} \epsilon_e f_e &= \epsilon_e S_e + \epsilon_e f = \epsilon_e S_e + \epsilon_\Gamma f \, \epsilon_e \Gamma \\ &= \epsilon_e \frac{1/\gamma_e}{\sum_{e' \in E} 1/\gamma_{e'}} + \epsilon_\Gamma f \, \epsilon_e \frac{1}{\sum_{e'} 1/\gamma_{e'}} \\ &= -\frac{\sum_{e' \neq e} 1/\gamma_{e'}}{\sum_{e'} 1/\gamma_{e'}} + \epsilon_\Gamma f \, \frac{1/\gamma_e}{\sum_{e'} 1/\gamma_{e'}} \end{aligned} \tag{44}$$

Now suppose for a contradiction that $\rho(\boldsymbol{\beta}_n, \boldsymbol{\gamma}_n) \to 0$ for some sequence $(\boldsymbol{\beta}_n, \boldsymbol{\gamma}_n)_{n=1}^{\infty}$ (where $\boldsymbol{\beta}_n$ and $\boldsymbol{\gamma}_n$ are vectors of costs and prices, indexed by edges). We claim that this implies $\beta_{en} = o(\gamma_{en})$ as $n \to \infty$ for all $e \in E$. If $\beta_{en} \neq o(\gamma_{en})$ for some e, then

$$\rho(\boldsymbol{\beta}_n, \boldsymbol{\gamma}_n) = \frac{\sum_{e'} 1/\gamma_{e'n}}{\sum_{e'} 1/\beta_{e'n}} \geq \left(\sum_{e'} 1/\gamma_{e'n} \right) \Delta \beta_{en} \geq \Delta \frac{\beta_{en}}{\gamma_{en}} \nrightarrow 0$$

where n is arbitrary, thus contradicting $\rho \to 0$.

Since $\beta_{en} = o(\gamma_{en})$ for all e, by equation (1) we have $\epsilon_e f_e(\boldsymbol{\beta}_n, \boldsymbol{\gamma}_n) \to -1/2$ for all e as $n \to \infty$. In particular, there must exist $(\boldsymbol{\beta}_N, \boldsymbol{\gamma}_N)$ for some $N \geq 0$ sufficiently high at which $\epsilon_e f_e(\boldsymbol{\beta}_N, \boldsymbol{\gamma}_N) \geq -1/2 - \epsilon/|E|$ for $\epsilon > 0$ and for all $e \in E$.

Inserting expression (44) into $\epsilon f_e \geq -1/2 - \epsilon/|E|$ and summing the result over all e, we obtain

$$\epsilon_\Gamma f - (|E| - 1) \geq -\frac{|E|}{2} - \epsilon,$$

which cannot be achieved for small values of ϵ when $|E| \geq 3$, because $\epsilon_\Gamma f$ cannot be positive. Thus we get a contradiction. □

Theorem 5. *Consider a single-resource market. Suppose there is a* $0 < \Delta \leq 1$ *such that* $\min_{e \in E} \beta_e / \max_{e \in E} \beta_e \geq \Delta$ *for any number of edges* $|E| > 0$*. Then as* $|E| \to \infty$*, the price of anarchy goes to* $2/3$*.*

Proof. Consider an infinite but countable set of producers $R = \{r_1, r_2, ...\}$ with quadratic cost functions $\{\frac{\beta_r}{2} f_r^2 \,|\, r \in R\}$, and let $R_m = \{r_1, ..., r_m\}$ denote the set of the first m producers. Let $\boldsymbol{\gamma}_m$ denote the vector of prices in the game where the set of providers is R_m, and define

$$\rho_m = \frac{B_m}{\Gamma_m} = \frac{\sum_{r \in R_m} 1/\gamma_{mr}}{\sum_{r \in R_m} 1/\beta_r}.$$

We have to show that $\rho_m \to 1$ as $m \to \infty$. For simplicity, assume that $\epsilon_\Gamma f = 0$; it is easy to show that more elastic demand functions always lead to less overcharging and a better supply-side efficiency measure ρ. Note that when $\epsilon_\Gamma f = 0$,

$$\beta_r = \gamma_{mr} \left(1 - \frac{1/\gamma_{mr}}{\sum_{r \in R_m} 1/\gamma_{mr}} \right) \tag{45}$$

for all m, r.

First, we claim that for all $\epsilon > 0$, there is an N such that for all m, the number of players in R_m for which

$$\frac{1/\gamma_{mr}}{\sum_{r \in R_m} 1/\gamma_{mr}} > \epsilon$$

is less than N. If not, then for some ϵ, we can find a set R_M such that the above holds for a set of at least $1/\epsilon$ players R'_M, and so

$$1 = \sum_{r \in R_M} \frac{1/\gamma_{Mr}}{\sum_{r \in R_M} 1/\gamma_{Mr}} \geq \sum_{r \in R'_M} \frac{1/\gamma_{Mr}}{\sum_{r \in R_M} 1/\gamma_{Mr}} > \frac{1}{\epsilon}\epsilon = 1$$

which is a contradiction.

So fix an $\epsilon > 0$ and an m and let R_m^ϵ denote the set of producers in R_m for which

$$\frac{1/\gamma_{mr}}{\sum_{r \in R_m} 1/\gamma_{mr}} < \epsilon.$$

Note that by equation (45), we have for all these producers that

$$\beta_r > \gamma_{mr} \left(1 - \epsilon\right).$$

We can now express the ratio ρ_m as

$$\rho_m = \frac{\sum_{r \in R_m} 1/\gamma_{mr}}{\sum_{r \in R_m} 1/\beta_r} \geq \frac{\sum_{r \in R_m^\epsilon} 1/\gamma_{mr}}{\sum_{r \in R_m} 1/\beta_r} > \frac{(1-\epsilon) \sum_{r \in R_m^\epsilon} 1/\beta_r}{\sum_{r \in R_m^\epsilon} 1/\beta_r + \sum_{r \in R_m \backslash R_m^\epsilon} 1/\beta_r}.$$

Since this holds for all m, since the set $R_m \backslash R_m^e$ is finite, and since there is a $0 < \Delta \leq 1$ such that $\min_{r \in R_m} \beta_r / \max_{r \in R_m} \beta_r \geq \Delta$ for any R_m, we can easily establish by picking m large enough that

$$\liminf_{m \to \infty} \rho_m \geq 1 - \epsilon.$$

But since ϵ was arbitrary, it must follow that $\liminf_{m \to \infty} \rho_m \geq 1$, which is what we wanted to prove. □

Corollary 1. In a monopsony, that is when there is only one user and an infinite number of producers, the price of anarchy equals $3/4$.

Proof. By Lemma D5, the worst price of anarchy occurs when the user's valuation is linear. As $E \to \infty$, $\Gamma \to B$. In a monopsony, there is no demand-side competition, and the socially optimal allocation is the maximizer of

$$f - \frac{B}{2} f^2,$$

which yields an optimum welfare of $f = 1/2B = 1/2\Gamma$. The equilibrium welfare, on the other hand, will equal the maximizer of

$$f - \Gamma f^2,$$

which yields a welfare of $f = 3/8\Gamma$.

The ratio of these two welfares is $3/4$. □

F Measuring inefficiencies on the demand-side for a series-parallel graph

We now restrict our attention to *series-parallel* graphs; they help us understand the effects of horizontal and vertical competition on market efficiency. Informally, a series-parallel graph is built recursively by connecting smaller series-parallel graphs in parallel or in series, starting from edges. See Correa et al. (2010) for a full definition. For our purposes, it will be enough to look at the restricted class of *two-level* series-parallel graphs.

Definition 4. *A* two-level series-parallel graph G *consists of a set of* T *disjoint parallel routes that connect two distinguished nodes: a source* s *and a target* t*.*

Two-level series-parallel graphs capture the essence of horizontal and vertical competition; however, all of our results still carry over to arbitrary series-parallel graphs (usually by an induction argument). We will also assume that consumers have linear valuations and that providers have linear marginal costs. Later, we will formally establish that this is indeed the worst-case setting.

Our analysis will follow the same blueprint as in the single-resource case. First, we show that the mechanism can be simplified on the consumer side. Then we prove a lemma that returns the price of anarchy as a function of the market's supply-side inefficiency. Finally, we bound supply-side inefficiency.

F.1 Defining a simplified mechanism

We first show that just like we did for a link, we can define a single price for sending flow across the entire graph. This makes the mechanism simpler from the consumers' point of view and easier to analyze mathematically.

Definition 5. *In the* simplified *Bertrand-Cournot mechanism for a two-level series-parallel graph G,*

1. *Each producer r submits a linear pricing functions with slope γ_r like in the regular mechanism, and the aggregate pricing function is set to $P(f) = \Gamma f$.*
2. *Each consumer q chooses to send d_q units of (s,t)-flow and pays $\Gamma f d_q$, where $f = \sum_{q \in Q} d_q$.*
3. *The mechanism divides payments and flow proportionally to the producers' contribution to Γ. The producer on edge e on link l on route t receives the following fraction of the payments:*

$$\frac{1/\Gamma_t}{\sum_{t' \in T} 1/\Gamma_{t'}} \frac{\Gamma_l}{\sum_{l' \in t} \Gamma_{l'}} \frac{1/\gamma_e}{\sum_{e' \in l} 1/\gamma_{e'}}.$$

Theorem 6. *The Nash equilibria of the standard and simplified mechanisms are identical, and at equilibrium, the utilities of each player are the same.* □

Proof. Let P denote the set of all paths in G. The necessary and sufficient conditions of Lemma B2 applied to G state that for all q and r:

$$V_q'(d_q) \leq \sum_{p'} \sum_{e \in p \cap p'} \gamma_e d_{p'} + \sum_{e \in p} \gamma_e f_e \text{ for all } p \in P \text{ such that } d_p > 0 \tag{46}$$

$$V_q'(d_q) \geq \sum_{p'} \sum_{e \in p \cap p'} \gamma_e d_{p'} + \sum_{e \in p} \gamma_e f_e \text{ for all } p \in P \tag{47}$$

$$C_r'(f_r) \geq \gamma_r f_r \left(2 - \frac{1}{\epsilon f_r}\right) \text{ if } f > 0 \tag{48}$$

$$C_r'(f_r) \leq \gamma_r f_r \left(2 - \frac{1}{\epsilon f_r}\right) \tag{49}$$

The efficiency conditions for the aggregate game are can be expressed as:

$$V_q'(d_q) \leq \Gamma(d_q + f) \text{ for all } q \text{ s.t. } d_q > 0 \tag{50}$$

$$V_q'(d_q) \geq \Gamma(d_q + f) \text{ for all } q \tag{51}$$

$$C_r'(f_r) \geq \gamma_r f_r \left(2 - \frac{1}{\epsilon f_r}\right) \text{ if } f_r > 0 \tag{52}$$

$$C_r'(f_r) \leq \gamma_r f_r \left(2 - \frac{1}{\epsilon f_r}\right) \tag{53}$$

Take any demand-side equilibrium $\boldsymbol{d}$ of the aggregate game. We have to show that conditions (46-47) hold for the path flows d_{qp} that the aggregate mechanism chooses for consumers. Observe that the d_{qp} are chosen so that the flow at an edge e on link l within path t equals

$$\sum_{p: e \in p} d_{qp} = \frac{1/\Gamma_t}{\sum_{t'} 1/\Gamma_{t'}} \frac{1/\gamma_{tle}}{\sum_{e'} 1/\gamma_{tle'}} d_q.$$

Condition (46) for any path $p \in P$ follows by simple algebra. Path p must coincide with some route $t \in T$ and at every link $l \in t$ take some edge $e_p \in l$. We then derive condition (46) as follows.

$$\begin{aligned} V_q'(d_q) \leq \Gamma(d_q + f) &= \Gamma_t(\frac{1/\Gamma_t}{\sum_{t'} 1/\Gamma_{t'}} d_q + \frac{1/\Gamma_t}{\sum_{t'} 1/\Gamma_{t'}} f) = \Gamma_t(d_{qt} + f_t) \\ &= \sum_{l \in t} \Gamma_{lt}(d_{qt} + f_t) \\ &= \sum_{l \in t} \gamma_{e_p} \frac{1/\gamma_{e_p}}{\sum_{e' \in l} 1/\gamma_{tle'}} (d_{qt} + f_t) \\ &= \sum_{e \in p} \gamma_e \sum_{p' \in P : e \in p'} d_{p'} + \sum_{e \in p} \gamma_e f_e \\ &= \sum_{p' \in P} \sum_{e \in p' \cap p} \gamma_e d_{p'} + \sum_{e \in p} \gamma_e f_e \end{aligned}$$

Condition (47) follows similarly.

This shows that for any $\boldsymbol{\gamma}$, the demand at each edge remains the same in the aggregate game. Thus the function $f_r(\boldsymbol{\gamma})$ is unchanged, and the equilibrium prices $\boldsymbol{\gamma}$ of the aggregate game also form a supply-side equilibrium in the original game. □

F.2 Measuring inefficiency on the demand side

From a consumer's perspective, the market contains a single resource: (s,t)-flow. This suggests that graph structure does not affect consumer behavior, and in fact the same procedure as in the single-resource setting can be used to compute the demand-side inefficiency of our market.

Lemma F11. The price of anarchy is lower-bounded by the solution to:

$$\begin{aligned} \min \quad & \frac{\sum_{q=1}^{Q} \alpha_q d_q - \frac{B}{2} f^2}{\max_{\bar{f}} (\max_{q \in Q} \alpha_q \bar{f} - \frac{B}{2} \bar{f}^2)} \\ \text{s.t.} \quad & \alpha_q = \Gamma f + \Gamma d_q \text{ for all } q \in Q \\ & \beta_r = \gamma_r \left(2 - \frac{1}{\epsilon f_r}\right) \text{ for all } r \in R \\ & \sum_{q \in Q} d_q = f \\ & 0 \leq \alpha_q, d_q, \Gamma, B \end{aligned}$$

□

Proof. Since we assume that valuations and marginal costs are linear, to lower-bound the price of anarchy, it is enough to find a set of valuations $V_q(d) = \alpha_q d$, quadratic costs $C_r(f) = \frac{\beta_r}{2} f^2$, and elasticities $\boldsymbol{\epsilon f}$ that result in the worst welfare loss at a point $(\boldsymbol{d}, \boldsymbol{\gamma})$ that satisfies:

1. The vector $\boldsymbol{d}$ is a demand-side equilibrium given $\boldsymbol{\gamma}$ and linear valuations.
2. The vector $\boldsymbol{\gamma}$ is a supply-side equilibrium given $\boldsymbol{\epsilon f}$ and quadratic costs.

Formally, the price of anarchy is lower-bounded by the optimum of the following optimization problem:

$$\begin{aligned}
\min \quad & \frac{\sum_{q=1}^{Q} \alpha_q d_q - \frac{B}{2} f^2}{\max_{\bar{f}} (\max_{q \in Q} \alpha_q \bar{f} - \frac{B}{2} \bar{f}^2)} \\
\text{s.t.} \quad & \alpha_q \le \Gamma f + \Gamma d_q \text{ if } d_q > 0, \forall q \in Q \\
& \alpha_q \ge \Gamma f + \Gamma d_q, \forall q \in Q \\
& \beta_r \ge \gamma_r \left(2 - \frac{1}{\epsilon f_r}\right) \text{ if } f_r > 0, \forall r \in R \\
& \beta_r \le \gamma_r \left(2 - \frac{1}{\epsilon f_r}\right), \forall r \in R \\
& \sum_{q \in Q} d_q = f \\
& 0 < \alpha_q \le 1, \forall q \in Q \\
& 0 \le d_q, \Gamma, B
\end{aligned}$$

The objective function is precisely the welfare ratio at $(\boldsymbol{d}, \boldsymbol{\gamma})$; the ratio minimized over all possible linear valuations with slopes α_q, all possible quadratic costs with parameters β_r, and all possible strategies $\boldsymbol{d}$, $\boldsymbol{\gamma}$ under the constraints we informally defined above. □

Lemma 4. *The price of anarchy in a single-good market is bounded by*

$$\frac{2\rho(2-\rho)}{4-\rho}$$

where $0 \le \rho \le 1$ is an overcharging parameter that equals B/Γ. At equilibrium, this ratio is constrained by the supply-side condition

$$\beta_r = \gamma_r \left(2 - \frac{1}{\epsilon f_r}\right) \text{ for all } r. \tag{54}$$

Proof. Follows directly from Lemma 3. □

F.3 Measuring inefficiency on the supply side

Theorem 7. *In a two-level series-parallel graph G, let e be an edge located on link l on route t. The elasticity of the (s,t)-flow f_e at e with respect to γ_e equals*

$$\begin{aligned}
\epsilon_e f_e = & -\frac{\sum_{e' \in l; e' \ne e} 1/\gamma_{e'}}{\sum_{e' \in l} 1/\gamma_{e'}} - \frac{\sum_{t \in T; t' \ne t} 1/\Gamma_{t'}}{\sum_{t' \in T} 1/\Gamma_{t'}} \frac{\Gamma_l}{\sum_{l' \in t} \Gamma_{l'}} \frac{1/\gamma_e}{\sum_{e' \in l} 1/\gamma_{e'}} \\
& + \frac{1/\Gamma_t}{\sum_{t' \in T} 1/\Gamma_{t'}} \frac{\Gamma_l}{\sum_{l' \in t} \Gamma_{l'}} \frac{1/\gamma_e}{\sum_{e' \in l} 1/\gamma_{e'}} \epsilon_\Gamma f
\end{aligned}$$

□

Proof. By the definition of the mechanism,

$$f_e = \frac{1/\gamma_e}{\sum_{e' \in l} 1/\gamma_{e'}} \frac{1/\Gamma_t}{\sum_{t' \in T} 1/\Gamma_{t'}} f,$$

where f is the size of the total (s,t)-flow. By the properties of elasticity, it follows that

$$\begin{aligned} \epsilon_e f_e &= \epsilon_e \frac{1/\gamma_e}{\sum_{e' \in l} 1/\gamma_{e'}} + \epsilon_e \frac{1/\Gamma_t}{\sum_{t' \in T} 1/\Gamma_{t'}} + \epsilon_e f \\ &= \epsilon_e \frac{1/\gamma_e}{\sum_{e' \in l} 1/\gamma_{e'}} + \epsilon_{\Gamma_t} \frac{1/\Gamma_t}{\sum_{t' \in T} 1/\Gamma_{t'}} \epsilon_e \Gamma_t + \epsilon_e \Gamma_t \epsilon_{\Gamma_t} \Gamma \epsilon_\Gamma f. \end{aligned} \tag{55}$$

By directly applying the properties of elasticity, we find that the first term equals

$$\epsilon_e \frac{1/\gamma_e}{\sum_{e' \in l} 1/\gamma_{e'}} = -\frac{\sum_{e' \in l; e' \neq e} 1/\gamma_{e'}}{\sum_{e'} 1/\gamma_{e'}}$$

and the exact same manipulations can also establish that

$$\epsilon_{\Gamma_t} \frac{1/\Gamma_t}{\sum_{t' \in T} 1/\Gamma_{t'}} = -\frac{\sum_{t \in T; t' \neq t} 1/\Gamma_{t'}}{\sum_{t' \in T} 1/\Gamma_{t'}}.$$

Next, observe that

$$\begin{aligned} \epsilon_e \Gamma_t &= -\epsilon_e \sum_{l' \in t} \Gamma_l = -\frac{\Gamma_l}{\sum_{l' \in t} \Gamma_{l'}} \epsilon_e \Gamma_l \\ &= -\frac{\Gamma_l}{\sum_{l' \in t} \Gamma_{l'}} \epsilon_e \frac{1}{\sum_{e' \in l} 1/\gamma_{e'}} \\ &= \frac{\Gamma_l}{\sum_{l' \in t} \Gamma_{l'}} \frac{1/\gamma_e}{\sum_{e' \in l} 1/\gamma_{e'}} \end{aligned}$$

Finally, observe that

$$\epsilon_{\Gamma_t} \Gamma = \epsilon_{\Gamma_t} \frac{1}{\sum_{t' \in T} 1/\Gamma_{t'}} = \frac{1/\Gamma_t}{\sum_{t' \in T} 1/\Gamma_{t'}}$$

Taking all these observations, and combining them with (55), we obtain our claim. □

Theorem 8. *Let G be a route graph with two edges per link. Suppose that $\min_{e \in E} \beta_e / \max_{e \in E} \beta_e \geq \Delta$ for some $0 < \Delta \leq 1$. For any fixed L the price of anarchy is bounded by a constant.* □

Proof. When there is only one route in the graph and demand is linear, the elasticity equals

$$\epsilon_e f_e = -\frac{\sum_{e' \in l; e' \neq e} 1/\gamma_{e'}}{\sum_{e' \in l} 1/\gamma_{e'}} - \frac{\Gamma_l}{\sum_{l' \in t} \Gamma_{l'}} \frac{1/\gamma_e}{\sum_{e' \in l} 1/\gamma e'} \tag{56}$$

When there are two producers per link, $\rho = B/\Gamma = (\sum_{l=1}^{L} \frac{1}{1/\beta_{l1}+1\beta_{l2}})/(\sum_{l=1}^{L} \frac{1}{1/\gamma_{l1}+1/\gamma_{l2}})$ Inserting (56) into the above and changing variables to $g_{le} = 1/\gamma_{le}$, we obtain

$$\frac{\sum_{l=1}^{L} \dfrac{1}{\dfrac{1}{\dfrac{1}{g_{l1}} - \dfrac{1}{g_{l2} + 1/\sum_{l' \neq l} \Gamma_{l'}}} + \dfrac{1}{\dfrac{1}{g_{l2}} - \dfrac{1}{g_{l1} + 1/\sum_{l' \neq l} \Gamma_{l'}}}}}{\sum_{l=1}^{L} 1/(g_{l1} + g_{l2})}. \tag{57}$$

Without loss of generality, we can restrict our domain to the set of all g_{le} such that $g_{l1} + g_{l2} \geq 1$. That is because we can normalize any feasible solution such that the inequality holds and the original objective function value is unchanged.

Observe now that at any $\boldsymbol{\gamma}$, $(\sum_l B_l(\boldsymbol{\gamma}))/(\sum_l \Gamma_l(\boldsymbol{\gamma})) \geq \min_l (B_l(\boldsymbol{\gamma}))/(\Gamma_l(\boldsymbol{\gamma}))$. Since the functions B_l and Γ_l are symmetric in the g_{le}, we can, by relabeling the variables if necessary, assume that $B_1/\Gamma_1 = \min_l B_l/\Gamma_l$ at the minimizer of ρ:

$$\rho \geq \frac{\dfrac{1}{\dfrac{1}{\dfrac{1}{g_{11}} - \dfrac{1}{g_{12} + 1/\sum_{l' \neq l} \Gamma_{l'}}} + \dfrac{1}{\dfrac{1}{g_{12}} - \dfrac{1}{g_{11} + 1/\sum_{l' \neq l} \Gamma_{l'}}}}}{1/(g_{11} + g_{12})}. \tag{58}$$

Thus it is enough to minimize B_1/Γ_1. Observe that (58) is increasing in $1/\sum_{l' \neq l} \Gamma_{l'}$. The term $1/\sum_{l' \neq l} \Gamma_{l'}$ attains its smallest value when $g_{l1} + g_{l2} = 1$ for $l \neq 1$, and so we can assume that that equality holds:

$$\rho = \frac{\dfrac{1}{\dfrac{1}{\dfrac{1}{g_{11}} - \dfrac{1}{g_{12} + 1/(L-1)}} + \dfrac{1}{\dfrac{1}{g_{12}} - \dfrac{1}{g_{11} + 1/(L-1)}}}}{1/(g_{11} + g_{12})}. \tag{59}$$

Since $g_{l1} + g_{l2} \geq 1$, (59) only goes to zero when β_{11} or β_{12} tend to zero. But it's easy to see that they can't both go to zero, and when $\beta_{11} \to 0$, β_{12} goes to some constant, violating the $\beta_{11}/\beta_{12} \geq \Delta$ condition. Thus, ρ is lower-bounded by a constant which can be computed numerically for different values of Δ. □

Unfortunately, elasticity decreases as $L \to \infty$, and in the limit this results in an unbounded price of anarchy.

Corollary 2. Let G be a route graph with two edges per link. As $L \to \infty$, the price of anarchy goes to zero.

Proof. Observe that (59) is attained at any point for which $\Gamma_l = 1$ for all l. Then for any value of g_{l1}, g_{l2}, the ratio goes to zero as $\Delta \to \infty$. □

But just as in a single-resource market, having only three competitors per link guarantees a constant price of anarchy bound.

Theorem 9. *Let G be a route graph with three producers per link and suppose that for all $l \in L$ $\min_e \beta_e / \max_e \beta_e \geq \Delta_l$ for all $e \in l$ and for some $0 < \Delta_l \leq 1$. The price of anarchy is bounded by a constant.*

Proof. Suppose for a contradiction that $\rho(\boldsymbol{\beta}_n, \boldsymbol{\gamma}_n) \to 0$ for some sequence $(\boldsymbol{\beta}_n, \boldsymbol{\gamma}_n)_{n=1}^{\infty}$ (where $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ are vectors of costs and prices, indexed by edges). We claim that there exists an $\bar{l} \in L$ such that $\liminf_{n\to\infty} \beta_{en}/\gamma_{en} = 0$ for all $e \in \bar{l}$ as $n \to \infty$. If not, then taking $l(n) = \arg\min_{l' \in L} B_{l'n}/\Gamma_{l'n}$, and taking an $e \in l(n)$ such that $\liminf_{n\to\infty} \beta_{en}/\gamma_{en} > 0$ we find

$$\begin{aligned}\rho(\boldsymbol{\beta}_n, \boldsymbol{\gamma}_n) &= \frac{\sum_l B_l}{\sum_l \Gamma_l} \geq \frac{B_{l(n)}}{\Gamma_{l(n)}} = \frac{\sum_{e' \in l(n)} 1/\gamma_{e'n}}{\sum_{e' \in l(n)} 1/\beta_{e'n}} \\ &\geq \left(\sum_{e'} 1/\gamma_{e'n}\right) \frac{\Delta_{l(n)}}{|E_{l(n)}|} \beta_{en} \geq \frac{\Delta_{l(n)}}{|E_{l(n)}|} \frac{\beta_{en}}{\gamma_{en}} \geq \delta > 0\end{aligned}$$

for some $\delta > 0$. Since n is arbitrary, this contradicts $\rho \to 0$.

Since $|E|$ and $|L|$ are finite, without loss of generality we can pick a sequence $(\boldsymbol{\beta}_n, \boldsymbol{\gamma}_n)_{n=1}^{\infty}$ so that $\lim_{n\to\infty} \beta_{en}/\gamma_{en} = 0$ for all $e \in \bar{l}$.

Now let

$$S_{le} = \frac{1/\gamma_e}{\sum_{e' \in E_l} 1/\gamma_{e'}}.$$

and observe that $f_e = S_{le} f$ for any $l \in L$, $e \in l$. Then just like in the proof of Theorem 4, we can show that

$$\epsilon_e f_e = -\frac{\sum_{e' \neq e, e' \in l} 1/\gamma_{e'}}{\sum_{e' \in l} 1/\gamma_{e'}} + \epsilon_{\Gamma_l} f \, \frac{1/\gamma_e}{\sum_{e' \in l} 1/\gamma_{e'}}$$

Since $\lim_{n\to\infty} \beta_{en}/\gamma_{en} = 0$ for all $e \in \bar{l}$, by equation (54) we have $\epsilon_e f_e(\boldsymbol{\beta}_n, \boldsymbol{\gamma}_n) \to -1/2$ for all $e \in \bar{l}$ as $n \to \infty$. In particular, there must exist $(\boldsymbol{\beta}_N, \boldsymbol{\gamma}_N)$ for some $N \geq 0$ sufficiently high at which $\epsilon_e f_e(\boldsymbol{\beta}_N, \boldsymbol{\gamma}_N) \geq -1/2 - \epsilon/|E|$ for $\epsilon > 0$ and for all $e \in \bar{l}$.

Inserting expression (44) into $\epsilon f_e \geq -1/2 - \epsilon/|E_l|$ and summing the result over all $e \in \bar{l}$, we obtain

$$\epsilon_{\Gamma_l} f - (|E_l| - 1) \geq -\frac{|E_l|}{2} - \epsilon,$$

which cannot be achieved for small values of ϵ when $|E_l| \geq 3$, because $\epsilon_{\Gamma_l} f$ cannot be positive. Thus we get a contradiction. □

Theorem 10. *Let $G = (V, E)$ be a route graph with m producers per link and suppose that there exists a $0 < \Delta \leq 1$ such that $\min_{e \in E} \beta_e / \max_{e \in E} \beta_e \geq \Delta$ for all m. As m goes to infinity, ρ goes to one.*

Proof. First observe that

$$\rho = \frac{\sum_{l' \in L} B_{l'}}{\sum_{l' \in L} \Gamma_{l'}} \geq \min_{l \in L} \frac{B_l}{\Gamma_l}.$$

Since the proof of Theorem 5 holds for any form of consumer demand (including one arising from links placed series), $B_l/\Gamma_l \to 1$ for all l as $m \to \infty$, thus establishing our claim. □

Theorem 11. *Let G be a two-level series-parallel graph with at least three providers on every link and suppose that and suppose that there exists a $0 < \Delta \le 1$ such that $\min_{e\in E} \beta_e / \max_{e\in E} \beta_e \ge \Delta$. When the number of parallel routes of G goes to infinity, the elasticity at each edge tends to that obtained from linear valuations.*

Proof. First observe that the existence of a $0 < \Delta \le 1$ such that $\min_{e\in E} \beta_e / \max_{e\in E} \beta_e \ge \Delta$ implies that $\min_{t\in T} B_t / \max_{t\in T} B_t \ge \Delta$.

Next recall that in Theorem **??** we established that the elasticity of flow at edge e equals:

$$\begin{aligned}\epsilon_e f_e = &- \frac{\sum_{e'\in l; e'\neq e} 1/\gamma_{e'}}{\sum_{e'\in l} 1/\gamma_{e'}} - \frac{\sum_{t\in T; t'\neq t} 1/\Gamma_{t'}}{\sum_{t'\in T} 1/\Gamma_{t'}} \frac{\Gamma_l}{\sum_{l'\in t} \Gamma_{l'}} \frac{1/\gamma_e}{\sum_{e'\in l} 1/\gamma_{e'}} \\ &+ \frac{1/\Gamma_t}{\sum_{t'\in T} 1/\Gamma_{t'}} \frac{\Gamma_l}{\sum_{l'\in t} \Gamma_{l'}} \frac{1/\gamma_e}{\sum_{e'\in l} 1/\gamma_{e'}} \epsilon_\Gamma f\end{aligned}$$

To establish our claim, it is enough to show that

$$\frac{\sum_{t\in T; t'\neq t} 1/\Gamma_{t'}}{\sum_{t'\in T} 1/\Gamma_{t'}} \to 1$$

as $|T| \to \infty$, for all routes t and any set of cost provider cost functions $(\beta_r)_{r\in R}$ and equilibrium strategies $(\gamma_r)_{r\in R}$. Observe that this is equivalent to

$$\frac{1/\Gamma_t}{\sum_{t'\in T} 1/\Gamma_{t'}} \to 0.$$

Suppose for a contradiction that

$$\frac{1/\Gamma_t}{\sum_{t'\in T} 1/\Gamma_{t'}} \nrightarrow 0$$

for some t and some set of cost and pricing functions. Then it must be that $\Gamma_t/\Gamma_t' \to \infty$ for some two routes t, t'.

But then

$$\frac{\Gamma_t}{\Gamma_{t'}} = \frac{\Gamma_t}{B_t} \frac{B_t}{\Gamma_{t'}} \le \frac{\Gamma_t}{B_t} \frac{B_{t'}/\Delta}{\Gamma_{t'}} \le \frac{\Gamma_t}{\Delta B_t}$$

and so $\Gamma_t/B_t \to \infty$. But we have established in Theorem 9 that the ratio ρ cannot go to ∞ in a route graph under our assumptions (3 providers per link, and the existence of a Δ). Therefore we get a contradiction. □

Theorem 12. *Let G be a two-level series-parallel graph. The price of anarchy of G is lower-bounded by that of a route series-parallel graph.*

Proof. The efficiency ratio ρ can be bounded by that of the worst parallel route:

$$\frac{B}{\Gamma} = \frac{\frac{1}{\sum_i 1/B_i}}{\frac{1}{\sum_i 1/G_i}} \geq \min_i \frac{1/\Gamma_i}{1/B_i}$$

Then this argument can be applied inductively to the serial elements of every top-level route i that also contain sub-routes. Eventually, we will get that

$$\frac{B}{\Gamma} = \frac{\frac{1}{\sum_i 1/B_i}}{\frac{1}{\sum_i 1/G_i}} \geq \frac{1/\Gamma_t}{1/B_t}$$

for some route t.

Finally observe that Theorem 7 implies that the function $\epsilon_e f_e$ is pointwise smaller at every pricing vector γ when there is only a single route in the graph. Thus, passing to route elasticities cannot improve the ratio

$$\frac{1/\Gamma_t}{1/B_t}.$$

□

G Analysis of markets over arbitrary graphs

Here we analyze the loss of welfare arising from demand-side inefficiencies when the market is represented by an arbitrary graph $G = (V, E)$. In this section, we return to our initial assumptions on utilities: Assumptions 1 and 2.

G.1 Measuring inefficiency on the demand side

Before we start our demand-side analysis, we will show that our mechanism is equivalent to one in which consumers buy capacities on every link using the single-link mechanism and send a max-flow on the resulting capacitated graph (V, L). This game is easier to analyze and may be more convenient to use in consumers. We show this by first reducing the path-based mechanism to a similar one where consumers choose flows on routes instead of paths.

Definition G1. In the *route-based* mechanism,

1. Producers submit linear pricing functions $\gamma_r f_r$.
2. Consumer q selects a flow d_{qt} on every (s_q, t_q)-route t.
3. The flow on every path is defined by splitting the route flows at every link proportionally to $1/\gamma_r$, like in the single-resource mechanism.

Lemma 5. *The Nash equilibria and the equilibrium utilities of the two games are identical are identical.*

Proof. The proof is very similar to that of Lemma **??**. Let P denote the set of all paths in G. The necessary and sufficient conditions of Lemma B2 applied to G state that for all q and r:

$$V_q'(d_q) \leq \sum_{p'} \sum_{e \in p \cap p'} \gamma_e d_{qp'} + \sum_{e \in p} \gamma_e f_e \text{ for all } p \in P \text{ such that } d_{qp} > 0 \tag{60}$$

$$V_q'(d_q) \geq \sum_{p'} \sum_{e \in p \cap p'} \gamma_e d_{qp'} + \sum_{e \in p} \gamma_e f_e \text{ for all } p \in P \tag{61}$$

$$C_r'(f_r) \geq \gamma_r f_r \left(2 - \frac{1}{\epsilon f_r}\right) \text{ if } f > 0 \tag{62}$$

$$C_r'(f_r) \leq \gamma_r f_r \left(2 - \frac{1}{\epsilon f_r}\right) \tag{63}$$

For any q, the efficiency conditions for the aggregate game are can be expressed as:

$$V_q'(d_q) \leq \sum_{t'} \sum_{l \in t \cap t'} \mathit{\Gamma}_l d_{qt'} + \sum_{l \in t} \mathit{\Gamma}_l f_l \text{ for all } t \in T \text{ such that } d_{qt} > 0 \tag{64}$$

$$V_q'(d_q) \geq \sum_{t'} \sum_{l \in t \cap t'} \mathit{\Gamma}_l d_{qt'} + \sum_{l \in t} \mathit{\Gamma}_l f_l \text{ for all } t \in T \tag{65}$$

$$C_r'(f_r) \geq \gamma_r f_r \left(2 - \frac{1}{\epsilon f_r}\right) \text{ if } f_r > 0 \tag{66}$$

$$C_r'(f_r) \leq \gamma_r f_r \left(2 - \frac{1}{\epsilon f_r}\right) \tag{67}$$

Consider a vector strategies $(d_{qt})_{t \in T}$ that satisfies conditions (64)-(67) and a route t. The aggregate mechanism will allocate rates d_{qp} on paths $p \in t$ that coincide with t according to the same rules as for series-parallel graphs: proportionally to the reciprocals of the slopes of the pricing functions. We have to show that conditions (60)-(61) hold for these d_{qp}.

It can be verified that if we take any link $l \in t$ and any edge $e \in l$, the sum of the flows on paths $p \in t$ that pass by e equals

$$\sum_{p \in t, e \in p} d_{qp} = \frac{1/\gamma_e}{\sum_{e' \in l} 1/\gamma_{e'}} d_{qt}.$$

We use this observation to derive condition (60) for any path $p \in P$. Path p must coincide with a unique route $t \in T$ and at every link $l \in t$ take some edge

$e_l \in l$. We derive condition (60) as follows.

$$\begin{aligned}
V_q'(d_q) &\leq \sum_{t'} \sum_{l \in t \cap t'} \Gamma_l d_{qt'} + \sum_{l \in t} \Gamma_l f_l \\
&= \sum_{t'} \sum_{l \in t \cap t'} \gamma_{e_l} \frac{1/\gamma_{e_l}}{\sum_{e' \in l} 1/\gamma_{e'}} d_{qt'} + \sum_{l \in t} \gamma_{e_l} \frac{1/\gamma_{e_l}}{\sum_{e' \in l} 1/\gamma_{e'}} f_l \\
&= \sum_{t'} \sum_{l \in t \cap t'} \gamma_{e_l} \sum_{p' \in t', e_l \in p'} d_{qp'} + \sum_{l \in t} \gamma_{e_l} f_{e_l} \\
&= \sum_{t'} \sum_{p' \in t'} \sum_{e \in p \cap p'} \gamma_e d_{qp'} + \sum_{e \in p} \gamma_e f_e \\
&= \sum_{p'} \sum_{e \in p \cap p'} \gamma_e d_{qp'} + \sum_{e \in p} \gamma_e f_e
\end{aligned}$$

Condition (61) follows similarly.

This shows that for any $\boldsymbol{\gamma}$, the demand at each edge remains the same in the aggregate game. Thus the function $f_r(\boldsymbol{\gamma})$ is unchanged, and the equilibrium prices $\boldsymbol{\gamma}$ of the aggregate game also form a supply-side equilibrium in the original game. □

This formulation of the mechanism may be useful by itself. Here, we use it to reduce the path-based mechanism to a link-based mechanism.

Definition G2. In the *link-based* mechanism,

1. Producers submit linear pricing functions $\gamma_r f_r$.
2. Consumer q buys a vector of link capacities $\boldsymbol{x}_q = (x_{ql}, l \in L)$ at aggregate prices Γ_l and sends a max (s_q, t_q)-flow of size $d_q(\boldsymbol{x}_q)$ in the $\boldsymbol{x}_q$-capacitated graph $G_L = (V, L)$.
3. Money and flow is distributed among providers on a link proportionally to $1/\gamma_e$, just as in the single-link mechanism.

The proof of the following lemma uses techniques developed by Johari and Tsitsiklis (2005).

Lemma G12. At equilibrium, the equilibria in the path, route and link based mechanism are identical and the utilities of every player are the same.

Proof. Let the pricing functions at every edge be fixed. It is enough to show that the route-based game is equivalent to the link-based game. Observe that from the consumers perspective, they are choosing flows on every path in the graph $G_L = (V, L)$.

Thus, by Theorem 3.23 in Johari (2004), if $(d_{qt})_{q \in Q, t \in T}$ is a demand-side equilibrium in the game where users submit rates on routes, then the vector of link rates $x_{ql} = \sum_{t \in T_q} \sum_{l \in t} d_{qt}$ forms a demand-side equilibrium in the game where users submit rates on links.

Conversely, if $\boldsymbol{x}$ is a demand-side equilibrium of the bid-per-link game, then the $d_{qt}(x_q)$ form a demand-side equilibrium of the bid-per-route game by the same theorem.

Since we fixed pricing functions at every edge at arbitrary values, the above holds for any producer strategy, and since both games distribute link flows between edges in the same way, the demand function at every edge remains the same. Thus the demand elasticity faced by any producer remains the same, and the supply-side strategies of both games are also identical, and this completes the proof of the theorem.. □

Using this lemma, we can prove that valuation and marginal cost functions can be linearized just like in the single-resource case. Moreover, the price of anarchy in the full game is lower bounded by that the price of anarchy at the worst link.

Lemma 6. *Suppose agents have valuations $(V_q)_{q\in Q}$ and costs $(C_r)_{r\in R}$. Let $(\boldsymbol{d},\boldsymbol{\gamma})$ be an equilibrium of the link-based game, let $(d_q^*)_{q\in Q}$ and $(f_r^*)_{r\in R}$ be an optimal allocation, and let $f_r(\boldsymbol{\gamma})$ denote the demand functions induced by the V_q.*

There exist linear valuations $(\bar{V}_{ql}(d) = \alpha_{ql} d)_{q\in Q, l\in L}$ and linear marginal costs $(\bar{c}_e(f) = \beta_e f)_{e\in E}$ such that

$$\frac{\sum_{q\in Q} V_q(d_q) - \sum_{r\in R} C_r(f_r(\boldsymbol{\gamma}))}{\sum_{q\in Q} V_q(d_q^*) - \sum_{r\in R} C_r(f_r^*)} \geq \frac{\sum_{l\in E}\sum_{q\in Q} \bar{V}_{ql}(\bar{d}_{ql}) - \sum_{e\in E} \bar{C}_e(f_e(\boldsymbol{\gamma}))}{\sum_{e\in E}\sum_{q\in Q} \bar{V}_{ql}(\bar{d}_{ql}^*) - \sum_{e\in E} \bar{C}_e(\bar{f}_e^*)}$$

$$\geq \min_{l\in L} \frac{\sum_{q\in Q} \bar{V}_{ql}(\bar{d}_{ql}) - \sum_{e\in l} \bar{C}_e(f_e(\boldsymbol{\gamma}))}{\sum_{q\in Q} \bar{V}_{ql}(\bar{d}_{ql}^*) - \sum_{e\in l} \bar{C}_e(\bar{f}_e^*)} \quad (68)$$

where

- *The $(\gamma_r)_{r\in R}$ form a supply-side equilibrium given the original demand functions $(f_r)_{r\in R}$ and the new costs $(\bar{C}_e)_{e\in E}$.*
- *At every link l, the $(\bar{d}_{ql})_{q\in Q}$ form a demand-side equilibrium given $\bar{\gamma}_e$ and the new linear valuations $(\bar{V}_{ql})_{q\in Q}$ and $\sum_q d_{ql} = f_l(\boldsymbol{\gamma})$.*
- *The $(\bar{d}_{ql}^*)_{q\in Q}$ and the $(\bar{f}_e^*)_{e\in l}$ form an optimal allocation with the new linear valuations and marginal costs at every link $l \in L$.*

□

Proof (Sketch). Let the suppliers strategies γ_e be fixed. For simplicity, we will denote the link pricing functions by $p_l(\boldsymbol{x}_q, \boldsymbol{x}_{-q})$.

Let $\boldsymbol{x}$ be a Nash equilibrium of the per-link bid game. By definition, for all q, $\boldsymbol{x}_q$ maximizes $\bar{U}_q$:

$$\boldsymbol{x}_q \in \arg\max_{\bar{\boldsymbol{x}}} \left(V_q(d_q(\bar{\boldsymbol{x}})) - \sum_{e\in E} \bar{x}_e p_e(\bar{x}_e, \boldsymbol{x}_{-q}) \right)$$

As usual, $d_q(\bar{\boldsymbol{x}})$ is the size of the maximum (s_q, t_q)-flow in G when link capacities equal $\bar{\boldsymbol{x}}$.

The function $-\bar{U}_q$ is proper and convex; therefore the subdifferential $\partial(-\bar{U}_q)$ is non-empty at $\boldsymbol{x}_q$. In particular, since $\boldsymbol{x}_q$ maximizes $\bar{U}_q$, $\mathbf{0}$ is a subgradient of $-\bar{U}_q$.

It can be established using a theorem in convex analysis (specifically, Theorem 23.8 in Rockafellar, 1970) that

$$\partial(-\bar{U}_q(\bar{x})) = \partial(-(V_q(d_q(\bar{x})) + \partial(\sum_{e\in E} \bar{x}_e p_e(\bar{x}_e, \boldsymbol{x}_{-q}))$$

where $+$ denotes summation of sets: $A + B = \{a + b | a \in A, b \in B\}$.

Thus there exist $\boldsymbol{\alpha}_q \in -\partial(-(V_q(d_q(\bar{x})))$ and $\boldsymbol{\beta}_q \in -\partial(\sum_{l\in L} \bar{x}_l p_l(\bar{x}_l, \boldsymbol{x}_{-q}))$ such that $\boldsymbol{\alpha}_q = -\boldsymbol{\beta}_q$. Since V_q is non-decreasing in $\bar{\boldsymbol{x}}$, $\boldsymbol{\alpha}_q \geq 0$.

But then by the same theorem, $\mathbf{0}$ is also going to be a subgradient of $\boldsymbol{\alpha}_q^\top \bar{\boldsymbol{x}} - \sum_{l\in L} \bar{x}_l p_l(\bar{x}_l, \boldsymbol{x}_{-q})$ at the Nash equilibrium $\boldsymbol{x}_q$. In particular, we will have:

$$\boldsymbol{x}_q \in \arg\max_{\bar{\boldsymbol{x}}} \left(\boldsymbol{\alpha}_q^\top \bar{\boldsymbol{x}} - \sum_{l\in L} \bar{x}_l p_e(\bar{l}_e, \boldsymbol{x}_{-q}) \right)$$

which implies that for all l:

$$\boldsymbol{x}_{ql} \in \arg\max_{\bar{x}} \left(\alpha_l \bar{x} - \bar{x}_l p_l(\bar{x}_l, \boldsymbol{x}_{-q}) \right)$$

Thus the bids received at link l form a demand-side Nash equilibrium of a single-link game at l in which consumers have linear valuations with slopes α_{ql}.

To prove the theorem, we need to lower-bound the price of anarchy of the network game by that of the worst link. Let $\boldsymbol{x}^*$ be the set of per-link bids that maximizes social welfare in the network game. Observe that by the definition of a subderivative, we have

$$V_q(\boldsymbol{x}_q^*) \leq V_q(\boldsymbol{x}_q) + \boldsymbol{\alpha}_q^\top (\boldsymbol{x}_q^* - \boldsymbol{x}_q).$$

Also, it is clear that

$$\sum_{q\in Q} \boldsymbol{\alpha}_q^\top \boldsymbol{x}_q^* - \sum_{e\in E} C_e(f_e^*) = \sum_{l\in L} \left(\sum_{q\in Q} \alpha_{qe} x_{qe}^* - \sum_{e\in l} C_e(f_e^*) \right) \leq \sum_{l\in L} \max_{\bar{f}} \left(\max_q \alpha_{qe} \bar{f} - \sum_{e\in l} C_e(\bar{f}) \right).$$

Finally, by the definition of the subderivative and the fact that $V_q(d_q(\mathbf{0}) \geq 0$, it follows that

$$V_q(\boldsymbol{x}_q) - \boldsymbol{\alpha}_q^\top \boldsymbol{x}_q \geq 0.$$

Applying the first, the second, and then the third inequality, we obtain

$$\begin{aligned}
\frac{\sum_{q\in Q} V_q(\boldsymbol{x}_q) - \sum_{e\in E} C_e(f_e)}{\sum_{q\in Q} V_q(\boldsymbol{x}_q^*) - \sum_{e\in E} C_e(f_e^*)} &\geq \frac{\sum_{q\in Q}\left(V_q(\boldsymbol{x}_q) + \boldsymbol{\alpha}_q^\top \boldsymbol{x}_q - \boldsymbol{\alpha}_q^\top \boldsymbol{x}_q\right) - \sum_{e\in E} C_e(f_e)}{\sum_{q\in Q}\left(V_q(\boldsymbol{x}_q) + \boldsymbol{\alpha}_q^\top(\boldsymbol{x}_q^* - \boldsymbol{x}_q)\right) - \sum_{e\in E} C_e(f_e^*)} \\
&= \frac{\sum_{q\in Q}\left(V_q(\boldsymbol{x}_q) - \boldsymbol{\alpha}_q^\top \boldsymbol{x}_q\right) + \sum_{q\in Q} \boldsymbol{\alpha}_q^\top \boldsymbol{x}_q - \sum_{e\in E} C_e(f_e)}{\sum_{q\in Q}\left(V_q(\boldsymbol{x}_q) - \boldsymbol{\alpha}_q^\top \boldsymbol{x}_q\right) + \sum_{q\in Q} \boldsymbol{\alpha}_q^\top \boldsymbol{x}_q^* - \sum_{e\in E} C_e(f_e^*)} \\
&\geq \frac{\sum_{q\in Q}\left(V_q(\boldsymbol{x}_q) - \boldsymbol{\alpha}_q^\top \boldsymbol{x}_q\right) + \sum_{l\in L}\left(\sum_{q\in Q}\alpha_{ql}x_{ql} - \sum_{e\in l} C_e(f_e)\right)}{\sum_{q\in Q}\left(V_q(\boldsymbol{x}_q) - \boldsymbol{\alpha}_q^\top \boldsymbol{x}_q\right) + \sum_{l\in L}\max_{\bar f}\left(\max_q \alpha_{ql}\bar f - \sum_{e\in l} C_e(\bar f_e)\right)} \\
&\geq \frac{\sum_{l\in L}\left(\sum_{q\in Q}\alpha_{ql}x_{ql} - \sum_{e\in l} C_e(f_e)\right)}{\sum_{l\in L}\max_{\bar f}\left(\max_q \alpha_{ql}\bar f - \sum_{e\in l} C_e(\bar f_e)\right)} \\
&\geq \min_{l\in L} \frac{\sum_{q\in Q}\alpha_{ql}x_{ql} - \sum_{e\in l} C_e(f_e)}{\max_{\bar f}\max_q \alpha_{ql}\bar f - \sum_{e\in l} C_e(\bar f).}
\end{aligned}$$

Recall that at the beginning of the proof we fixed the producers' strategies. Thus for any set of pricing functions, the demand at a Nash equilibrium will be the same as in the original mechanism.

Finally, to obtain equation (68), we apply Lemma D6 and Lemma D7 at every link $l \in L$. □

Lemma 7. *The price of anarchy of the two-sided mechanism for a single link is*

$$\frac{2\rho(2-\rho)}{4-\rho}$$

where

$$\rho = \min_{l\in L} \frac{B_l}{\Gamma_l}.$$

Proof. Follows by applying Lemma 2 and Lemma 3 at every link $l \in L$. □

G.2 Measuring inefficiency on the supply side

Theorem 13. *Let G be an arbitrary graph with three producers per link and suppose that $\min_{e\in E}\beta_e / \max_{e\in E}\beta_e \geq \Delta$ for some $0 < \Delta \leq 1$. The price of anarchy is bounded by a constant.*

Proof (Sketch). As in the single-resource analysis, one can show that at any link l,

$$\epsilon_e f_e = -\frac{\sum_{e'\in l; e'\neq e} 1/\gamma_{e'}}{\sum_{e'\in l} 1/\gamma_{e'}} + \epsilon_{\Gamma_l} f_l \frac{1/\gamma_e}{\sum_{e'\in l} 1/\gamma_{e'}}.$$

If $B_l/\Gamma_l \to 0$ at some l, then the same argument as in Theorem 4 establishes a contradiction by deriving that $\epsilon_{\Gamma_l} f_l > 0$. □

Theorem 14. *Let G be an arbitrary graph with m producers per link and suppose that there exists a $0 < \Delta \leq 1$ such that $\min_{e \in E} \beta_e / \max_{e \in E} \beta_e \geq \Delta$ for all m. As m goes to infinity, ρ goes to one and the price of anarchy goes to $2/3$.*

Proof. Since the proof of Theorem 5 holds for any form of consumer demand (including one arising from links placed series), $B_l / \Gamma_l \to 1$ for all l as $m \to \infty$, thus establishing our claim.

Theorem 15. *Let G be a series-parallel graph with at least two producers per link. Suppose producers' costs are quadratic and that consumers' valuations are linear. Then there exists a Nash equilibrium and best-responses converge on both the demand and the supply side.*

Proof. Sketch One can view the equilibrium equation in Theorem 7 that relates a producer's cost, price and demand elasticity as a best-response function that specifies the producer's action given the other's prices and its cost. The equilibrium existence result follows by applying Brower's fixed point theorem to this best-response function within the compact set $[0, \max_r \beta_r]^R$.