ORIENTED BORDISM: CALCULATION AND APPLICATION

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Abstract. In these expanded notes for a talk in Stanford’s Kiddie seminar, we give a quick introduction to oriented bordism. We give an elementary calculation of the first few oriented bordism groups and then sketch Thom’s rational calculation. After that we prove the Hirzebruch signature theorem and sketch several other applications.

These are the expanded notes for a talk in Stanford’s Kiddie seminar. Without assuming many prerequisites apart from basic manifold theory, we introduce the oriented bordism groups, give a straightforward calculation of the first four oriented bordism groups (i.e. up to dimension 3) and show that the bordism group of four-dimensional manifolds is non-trivial using the signature. After that we sketch Thom’s calculation of the rational oriented bordism groups and apply it to prove the Hirzebruch signature theorem. We end with a list of some other applications of oriented bordism.

Convention 0.1. In these notes a manifold will mean a smooth compact manifold, possibly with boundary.

Good references are Weston’s notes [Wes], Miller’s notes [Mil01] or Freed’s notes [Fre12]. Alternatively, one can look at Stong’s book [Sto68].

1. The definition of oriented bordism

Classifying manifolds is a very hard problem and indeed it is known that we could never produce a list of them or give an algorithm to decide whether two manifolds are diffeomorphic. One can see this using the fact that the group isomorphism problem, i.e. telling whether two finitely presented groups are isomorphic, is undecidable and that manifolds of dimension $\geq 4$ can have any finitely presented group as fundamental group.

One can either restrict to particular situations, e.g. three-dimensional manifolds or $(n-1)$-connected $2n$-dimensional manifolds for $n \geq 3$, and solve the problem there. Alternatively – and this is the method that we’ll pursue – one can try to classify manifolds up to a coarser equivalence relation than diffeomorphism. The main difficulty is finding one that is computable, while still being interesting.

In the 50’s, Thom came up with an entire family of equivalence relations that are both interesting and computable [Tho54]. We’ll look at oriented bordism, a typical one, but one that is most easily visualized and relatively easily computed. I hope the reader will trust me for the moment that this will be an interesting and computable notion and will let me get to the definition.

Definition 1.1. Let $M_1$ and $M_2$ be two $d$-dimensional oriented manifolds. We say that $M_1$ and $M_2$ are bordant if there exists a $(d+1)$-dimensional oriented manifold $W$ with boundary such that $\partial W$ is diffeomorphic as an oriented $d$-dimensional manifold to $M_1 \sqcup M_2$.

Here $\bar{M}_2$ is the manifold $M_2$ with opposite orientation. To make sense of this we need to say what the orientation on $\partial W$ is. An ordered basis $(v_1, \ldots, v_d)$ of $T_p \partial W$ is oriented if $(v_1, \ldots, v_d, \nu)$ is an oriented basis of $T_p W$, where $\nu$ is the outward pointed normal vector.

A manifold $W$ with boundary like the one that appeared in the previous definition we call a cobordism between $M_1$ and $M_2$. See figure [1] for an example.

Lemma 1.2. Bordism is an equivalence relation, i.e. it has the following properties:

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Figure 1. A two-dimensional cobordism between two one-dimensional manifolds.

Figure 2. The figures (a), (b) and (c) demonstrate the identity, symmetry and transitivity of the bordism relation.

(i) Identity: every $d$-dimensional oriented manifold $M$ is bordant to itself.
(ii) Symmetry: if $M_1$ is bordant to $M_2$, then $M_2$ is bordant to $M_1$.
(iii) Transitivity: if $M_1$ is bordant to $M_2$ and $M_2$ is bordant to $M_3$, then $M_1$ is bordant to $M_3$.

Proof. See figure 2. For (i) we note that $M \times I$ is a cobordism from $M$ to $M$. For (ii) remark that if $W$ is a cobordism between $M_1$ and $M_2$, then $\overline{W}$ is a cobordism between $M_2$ and $M_1$. Finally, for (iii) we note that if $W_1$ is a cobordism between $M_1$ and $M_2$ and $W_2$ is a cobordism between $M_2$ and $M_3$, then $W_1 \cup_{M_2} W_2$ is a cobordism between $M_1$ and $M_3$. \qed

Definition 1.3. We define $\Omega^SO_d$ to be the set of $d$-dimensional oriented manifolds modulo bordism. These are called the oriented bordism groups.

Natural operations on manifolds should give natural operations on the bordism groups. Indeed, disjoint union makes each $\Omega^SO_d$ into an abelian group and later we will see that Cartesian product makes the graded abelian group $\Omega^SO_*$ into a graded ring.

Lemma 1.4. Disjoint union gives $\Omega^SO_*$ the structure of an abelian group.

Proof. It suffices to check that if $M_1$ is bordant to $M'_1$ and $M_2$ is bordant to $M'_2$, then $M_1 \sqcup M_2$ is bordant to $M'_1 \sqcup M'_2$. To see this just takes disjoint union of the cobordisms. \qed
2. Calculating oriented bordism groups for dimension \( \leq 4 \)

Let’s figure out some of the groups \( \Omega^d_{SO} \) defined in the previous section. We’ll do \( d = 0, 1, 2 \) with relative ease and \( d = 3 \) with slightly more trouble. These early results seem to indicate that the oriented bordism groups are trivial, so we end with showing that is not true. In fact, we’ll show that there is a surjective homomorphism \( \sigma : \Omega^4_{SO} \to \mathbb{Z} \).

2.1. The low-dimensional groups \( \Omega^0_{SO}, \Omega^1_{SO} \) and \( \Omega^2_{SO} \). We will start with \( \Omega^1_{SO} \) and \( \Omega^2_{SO} \), concluding that they are trivial. The real reason this proof will work is that we know what all one- and two-dimensional manifolds are.

Let us think about what that means: all one- or two-dimensional oriented manifolds are bordant to each other, or equivalently to the empty manifold. But that just means they bound a \((d+1)\)-dimensional manifold.

We’ll check this in the case \( d = 1 \). Every oriented one-dimensional manifold is a disjoint union of circles. Since disjoint union gives the abelian group structure, it suffices prove that circle is bordant to the empty set. But the circle \( S^1 \) is naturally the boundary of the oriented disk \( D^2 \), which can be considered as cobordism from \( S^1 \) to \( \emptyset \). See part (a) of figure 3. We conclude the following:

**Proposition 2.1.** \( \Omega^1_{SO} = 0 \).

The case \( d = 2 \) is slightly more difficult, mainly because there are more two-dimensional oriented manifolds. In particular, the connected ones are classified by their genus \( g \geq 0 \): for genus 0 we have the sphere \( S^2 \), for genus 1 the torus \( T^2 \) and for genus \( g \geq 2 \) the hyperbolic surfaces \( \Sigma_g \). But all of these are bordant to the empty set again. Indeed, they are all the boundaries of solid handlebodies. For the sphere this is the disk \( D^3 \), for the torus the solid torus \( S^1 \times D^2 \) and for \( \Sigma_g \) the solid handlebody \( H_g = \#_g(S^1 \times D^2) \) (where \( \# \) denotes connected sum). See part (b) of figure 3. We again conclude:

**Proposition 2.2.** \( \Omega^2_{SO} = 0 \).

Let us now do the case \( d = 0 \). Here we have to admit being slightly sloppy before. The actual structure that we care about is not an orientation but an orientation of the stable normal bundle, whatever that is. Using this there are in fact two zero-dimensional oriented manifolds, the positively oriented point \( *_+ \) and the negatively oriented one \( *_- \). By taking disjoint unions of these, we see that \( \Omega^0_{SO} \) is a quotient of \( \mathbb{Z}^2 \). It is not hard to see that the oriented interval can be considered as a cobordism from \( *_+ \sqcup *_- \) to the empty set. Hence \( *_+ \) is identified with \( -*_- \). All other cobordisms from some oriented 0-dimensional manifold to the empty set are disjoint unions of such intervals, so we conclude that there are no more relations coming from bordism and thus:

**Proposition 2.3.** \( \Omega^0_{SO} = \mathbb{Z} \).
2. $\Omega_{3}^{SO}$. All of the previous calculations were straightforward, so what about higher dimensions? It turns out that one can still do the case $d = 3$ in a similar geometric fashion and it will turn out to vanish as well. This proof was first given by Rourke [Rou85]. To start this proof we will need surgery decompositions of oriented three-dimensional manifolds.

2.2. Surgery decompositions. Every smooth manifold admits a triangulation, a fact one can prove using Whitney’s embedding theorem. This means we obtain any 3-manifold $M$ by glueing together solid tetrahedra along their faces. We’ll use this to write our manifold as follows (for some $g \geq 0$):

$$M = D^3 \cup g \bigcup_{i=1}^g (D^1 \times D^2) \cup g \bigcup_{i=1}^g (D^2 \times D^1) \cup D^3$$

Let’s make this decomposition more precise. It means that $M$ is built as follows:

(i) We start with a disk $D^3$.

(ii) We glue on $g$ copies of $D^1 \times D^2$ along embeddings of $S^0 \times D^2$ into the boundary of $D^3$, which is a sphere $S^2$. The result is a solid handlebody $H_g$ of genus $g$.

(iii) We glue on $g$ copies of $D^2 \times D^1$ along embeddings of $S^1 \times D^1$ into the boundary of $H_g$, which is a genus $g$ surface $\Sigma_g$.

(iv) Finally one can check that the remaining boundary is a sphere $S^2$, and we glue in a disk $D^3$.

So how do we get such a decomposition of our manifold from the triangulation? See figure 4 for a picture to keep in mind.

(i) Consider the graph obtained by glueing the 1-skeletons of the tetrahedra together and pick a maximal tree $T$ in it. A thickened neighborhood of it is homeomorphic to a disk $D^3$: $D^3_T$.

(ii) If we add the thickened remaining edges $\{e\}$, we see that each such edge $e$ contributes a glued on copy of $D^1 \times D^2$: $(D^1 \times D^2)_e$. The union of $D^3$ with $\bigcup_e (D^1 \times D^2)_e$ is our $H_g$.

(iii) Now consider the dual graph obtained by taking a vertex for each tetrahedron and an edge for each face of a tetrahedron. Pick a maximal tree $T'$ and then the remaining edges $\{f\}$ correspond to a maximal set $\{f\}$ of faces such that the complement of their union with $H_g$ is a disk. Thickening these gives the glued on copies of $D^2 \times D^1$: $(D^2 \times D^1)_f$.

(iv) Finally the interiors of the tetrahedra with the remaining faces form a disk $D^3$ by construction. This follows because they correspond to the maximal tree $T'$: $D^3_{T'}$.

So why are the number of edges and faces used above the same? If we do glueing in a slightly different order: (i) and (ii) first, then (iii) and (iv) separately, and then glueing the two results together. We see that at the final step we glued a handlebody of genus $\#{\{e\}}$ to a handlebody of genus $\#{\{f\}}$ along their boundaries. This is only possible if these boundaries are the same, i.e. both have boundary $\Sigma_g$. 

**Figure 4.** This figure demonstrates locally the idea that a triangulation of a connected 3-manifold gives a decomposition into a pair of $D^3$’s, some tubes and some thickened $D^2$’s.
In fact, it’s useful to recast our construction in terms of the boundary $\Sigma_g$. There are two collections of $g$ curves on this surface: the $\alpha_i$ are the circles $\left(\frac{1}{2}\right) \times S^1$ in the thickened edges, the $\beta_i$ are the circles $(S^1 \times \{\frac{1}{2}\})_f$ in the thickened disks. Note that all of the $\alpha_i$ are disjointly embedded, as are the $\beta_i$, and both the $\alpha$’s and the $\beta$’s cut $\Sigma_g$ into a disk.

We can then alternatively think of our construction as starting with $\Sigma_g$ (maybe even slightly thickened to $\Sigma_g \times I$), gluing a disk $D^2$ to the $\alpha_i$ (on the $\Sigma_g \times \{0\}$ in the thickened version) and $\beta_i$ (on the $\Sigma_g \times \{1\}$ in the thickened version), and filling the two remaining $S^2$-boundaries with a disk $D^3$ (it’s clearer that there are two $S^2$ boundary components in the thickened version).

**Definition 2.4.** A surgery decomposition of $M$ is a pair $\alpha, \beta$ of collections of $g$ disjointly embedded curves on $\Sigma_g$ which cut $\Sigma_g$ into a disk, such that gluing a disk $D^2$ to each of the $\alpha_i$ and $\beta_i$ and then filling the two remaining $S^2$-boundaries with a disk $D^3$ gives us $M$. In this case we say $(g, \alpha, \beta)$ is a surgery decomposition of genus $g$. We write $M = M(g, \alpha, \beta)$ if we want to think of $M$ as built from the surgery decomposition.

**Remark 2.5.** A surgery decomposition is closely related to a Heegaard decomposition. In fact they are equivalent and differ only in the way they are presented. The data for a Heegaard decomposition is just a single diffeomorphism $\Sigma_g \rightarrow \Sigma_g$ and we can create a 3-manifold out of this by taking two copies of a handlebody $H_g$ and glueing their boundaries together using the diffeomorphism. Since all diffeomorphisms of $D^2$ relative to its boundary are isotopic, it turns out that the diffeomorphism is uniquely determined up to isotopy by where it sends a collection of $g$ disjointly embedded curves that cut $\Sigma_g$ into a disk. Hence from $(g, \alpha, \beta)$ we can construct a unique diffeomorphism $\Sigma_g \rightarrow \Sigma_g$ up to isotopy by pretending that it mapped the $\alpha_i$ to the $\beta_i$.

2.2.2. **Simplifying surgery decompositions.** Clearly a surgery decomposition is not canonical and depends on many choices. We take advantage of this by simplifying surgery decompositions. The following cancellation lemma is a special case of a very general cancellation lemma in surgery theory.

**Lemma 2.6.** Let $(g, \alpha, \beta)$ be a surgery decomposition of $M$ of genus $g$. If $\alpha_1$ and $\beta_1$ meet transversely in a single point, then we can find surgery decomposition of $M$ of genus $g - 1$: $(g - 1, \alpha \setminus \{\alpha_1\}, \beta \setminus \{\beta_1\})$.

**Proof.** Glue in the disks $D^2$ to the $\alpha_i$’s and the $D^3$ corresponding boundary $S^2$. The situation is then as in figure [5]. We see that if we glue in a thickened disk $D^2 \times D^1$ to a neighborhood of $\beta_1$, it cancels the handle that $\alpha_1$ is on: isotoping away the disk $D^3$ that’s indicated in the figure we see that the handlebody is now of genus $g - 1$. From this handlebody with the $\alpha_i$ and $\beta_i$ for $i \geq 2$ in its boundary, we get a surgery decomposition of genus $g - 1$. Indeed, isotoping away the disk doesn’t influence the other $\alpha_i$ or $\beta_i$ as they don’t intersect $\alpha_1$ and $\beta_1$ respectively. \linebreak

So if we’re lucky, we can cancel all of the $\alpha$’s and $\beta$’s against each other and see that our $M$ is obtained from glueing a $D^3$ to a $D^3$ along $S^2$, i.e. a three-dimensional sphere $S^3$. This bounds a $D^3$ and hence is bordant to the empty set. In that case we are done.

However there is no reason for us to be this lucky. If we’re not we’ll just make ourselves lucky, by showing using an induction that our $M$ is bordant to oriented 3-manifold that allows for cancellation.

2.2.3. **Surgery modifications.** There is a very easy way to construct manifolds $\tilde{M}$ bordant to $M$. We start with an identity cobordism $M \times I$ and glue on a $D^2 \times D^2$ to the boundary $M \times \{1\}$ along a $S^1 \times D^2$. This keeps the incoming boundary $M \times \{0\}$ the same, but turns the outgoing boundary into a 3-manifold $\tilde{M} = (M \setminus S^1 \times D^2) \cup S^3 \times S^1 (D^2 \times S^1)$.

As an example let’s consider two-dimensional manifolds. Taking a torus $T^2$, taking the identity cobordism $T^2 \times I$ and glueing on a tube $(D^2 \times D^2)$ along $S^0 \times D^2$ in $T^2 \times \{1\}$, makes the outgoing boundary into $\tilde{(T^2 \setminus S^0 \times D^2)} \cup (S^0 \times S^3) (D^3 \times S^3)$. If we look at the surface of genus 2 we’ll consider a special case of this construction, based on another collection $\gamma$ of $g$ disjointly embedded curves in $\Sigma_g$ such that $\Sigma_g$ cut along the $\gamma$ is a disk. Let’s rethink our reconstruction of
If we attach disks to two curves that intersect once transversally, they cancel out, i.e. just form a disk that can be isotoped away.

$M$ from a surgery decomposition $(g, \alpha, \beta)$. In the thickened version there’s a $\Sigma_g \times I$ in the middle. We think of the $\gamma$ as lying in $\Sigma_g \times \{\frac{1}{2}\}$ and thicken them to a collection of $g$ disjoint copies of $S^1 \times D^2$ in $\Sigma_g \times (0, 1) \subset M$. If we now start with an identity cobordism $M \times I$ and do the previous construction $g$ times for all the thickened $\gamma_i$ in $M \times \{\frac{1}{2}\}$, we get a cobordism from $M$ to

$$\tilde{M} = \left(M \bigcup_{i=1}^{g} (S^1 \times D^2)_{\gamma_i}\right) \cup \left(\bigcup_{i=1}^{g} (D^2 \times S^1)_{\gamma_i}\right)$$

**Proposition 2.7.** We have that

$$\tilde{M} = M(g, \alpha, \gamma) \# M(g, \gamma, \beta)$$

**Proof.** We start by cutting $M \setminus \bigcup_{i=1}^{g} (S^1 \times D^2)_{\gamma_i}$ along the level surface $\Sigma \times \{\frac{1}{2}\}$: we get two components, let’s denote by $M_1$ the one which contains all the disks glued to the $\alpha_i$ and by $M_2$ the one which contains all the disks glued to the $\beta_i$. Similarly it divides each boundary component $(S^1 \times S^1)_{\gamma_i}$ into a copy of $(S^1 \times D^1)_{\gamma_i}$ in both $M_1$ and $M_2$, where in both cases the image of $S^1$ is the curve $\gamma$. If to these we glue $(D^2 \times D^1)_{\gamma_i}$ we obtain from $M_1$ the manifold $M(g, \alpha, \gamma)$ with one of the disks $D^3$ missing, leaving an $S^2$ boundary, and from $M_2$ the manifold $M(g, \gamma, \beta)$ similarly with one of the disks $D^3$ missing. If we now glue both guys together along their common boundary, we exactly obtain $M$. But if one has two three-dimensional manifolds, removes a disk $D^3$ to both and glues the resulting $S^2$ boundaries together, then this is by definition the connected sum. □

We can finish our proof that every three-dimensional manifold is bordant to $S^3$, which implies it’s bordant to the empty set.

**Theorem 2.8.** Every connected (oriented) 3-manifold $M$ is bordant to $S^3$.

**Proof.** Pick a surgery decomposition $M(g, \alpha, \beta)$ and set $r = \min_{i,j} |\alpha_i \cap \beta_j|$ (without loss of generality they all intersect transversally). We will do an induction over $g$ and $r$, i.e. assume that the statement is true for (i) $g' < g$, and (ii) $g' = g$ and $r' < r$. The case $g = 0$ must have $r = 0$ and then we already know we have a sphere.

- If $r = 1$, then by lemma 2.6 we have that $M$ can be described by a surgery decomposition of genus $g - 1$ and hence by induction we are done.
- If $r = 0$, we’ll find a collection of curves $\gamma$ such that $|\alpha_1 \cap \gamma_1| = 1$ and $|\beta_1 \cap \gamma| = 1$. Then $M$ is bordant to $M(g, \alpha, \gamma) \# M(g, \gamma, \beta)$, both of which can be given by a surgery
decomposition of genus \( g - 1 \) and by induction are bordant to spheres. By taking connected sum of cobordisms, it is not hard to see that then \( M(g, \alpha, \gamma) \# M(g, \gamma, \beta) \) is bordant to \( S^3 \# S^3 = S^3 \) and we are done.

The collection \( \gamma \) is obtained by finding \( \gamma_1 \) and then randomly picking other disjointly embedded \( \gamma_i \) to make the \( \gamma \)'s cut \( \Sigma_g \) into a disk. We cut \( \Sigma_g \) along \( \alpha_1 \) and glue in disks to get \( \Sigma \). If \( \beta_1 \) cuts \( \Sigma \) into two components, then the disks must lie on opposite sides of \( \beta_1 \) (or \( \beta_1 \) would have cut \( \Sigma_g \) into two pieces) and we can connect them by an arc that intersects \( \beta_1 \) once. This arc corresponds to an embedded curve \( \gamma_1 \) in \( \Sigma_g \) that intersects \( \beta_1 \) and \( \alpha_1 \) once. If \( \beta_1 \) doesn’t cut \( \Sigma \) into two components, then cut \( \Sigma \) along \( \beta_1 \) and glue in disks to obtain a surface \( \Sigma' \) with four disks on it. Connect pair each disk from \( \alpha_1 \) with one from \( \beta_1 \) and connect both of these pairs with an arc. These two arcs corresponds to on \( \Sigma_g \) to an embedded curve \( \gamma_1 \) that intersects \( \beta_1 \) and \( \gamma_1 \) once.

- Finally, if \( r > 1 \), then we can assume without loss of generality that \( |\alpha_1 \cap \beta_1| = r \). We’ll find a collection of curves \( \gamma \) such that \( |\alpha_1 \cap \gamma_1| < r \), \( |\beta_1 \cap \gamma_1| < r \). Then \( M \) is bordant to \( M(g, \alpha, \gamma) \# M(g, \gamma, \beta) \), both of which have smaller \( r \) and by induction are bordant to spheres.

To find the collection \( \gamma \), we again pick \( \gamma_1 \) and complete the collection with random \( \gamma_i \)'s.

To pick \( \gamma_1 \) we note that we can find two adjacent intersection points along \( \beta_1 \) with \( \alpha_1 \). Take the arc \( x \) on \( \beta_1 \) between these. Then one of the two arcs \( y, y' \) on \( \alpha_1 \) between the intersection points has the property that \( x \cup y \) or \( x \cup y' \) does not cut the surface into two pieces, without loss of generality it is \( y \). The curve \( \gamma \) will be obtained by pushing \( x \) and \( y \) a bit to the side and connecting them. See figure 6.

\[ \square \]

This concludes our calculation of \( \Omega_3^{SO} \). The reader may want to think about the unoriented case.

2.3. Signature and the non-triviality of \( \Omega_4^{SO} \). Until now oriented bordism groups have turned out to be quite boring, so we’ll give an example in dimension \( d \) of non-trivial geometric data that it detects. To do this, we will define a surjective homomorphism \( \sigma : \Omega_4^{SO} \to \mathbb{Z} \) called the signature. It is actually an isomorphism, something that we’ll see rationally in the next section.

To define the signature, we need to know something about cup products in cohomology. There are several ways to think about these. Most geometrically one can think of cohomology as de Rham cohomology, i.e. closed forms modulo exact forms, and then we define the cup product \( a \cup b \) of \( a = [\alpha] \) and \( b = [\beta] \) as the cohomology class \( [\alpha \wedge \beta] \), where \( \wedge \) is the exterior product of forms. More algebraically one can think of cohomology as coming from maps from the singular...
simplices to $\mathbb{Z}$. Then $a \cup b$ for $a = [f]$ and $b = [g]$ of degree $k_1$ and $k_2$ respectively is given by $(f \sim g)(\gamma : \Delta^k \to M) = f(\gamma|_{\Delta^1})g(\gamma|_{\Delta^1})$ where the first restriction is to the first face of dimension $k_1$ and the second restriction is to the last face of dimension $k_2$.

**Theorem 2.9** (Poincaré duality in middle dimension). Let $M$ be a connected oriented manifold of even dimension $2d$. Then the cup product induces a non-degenerate bilinear form

$$H^d(M; \mathbb{R}) \otimes H^d(M; \mathbb{R}) \to H^{2d}(M; \mathbb{R}) \cong \mathbb{R}$$

which is symmetric if $d$ is even and skew-symmetric if $d$ is odd.

If the case that $d$ is even, i.e. the dimension of the manifold is divisible by 4, the symmetry tells us that the eigenvalues of the matrix representing the bilinear form are real. Non-degeneracy tells us they are non-zero. The signature $\sigma(M)$ of $M$ is the signature of this bilinear form: the number of positive eigenvalues minus the number of negative eigenvalues.

Let us now specialize to the case of 4-dimensional manifolds. The signature extends easily to the case of $d$-dimensional manifolds, then $\sigma(M)$ is the signature of this bilinear form: the number of positive eigenvalues minus the number of negative eigenvalues.

By $(\sim)$ we see that ker$(f^* \circ f_*)$ is isomorphic to ker$(f_*)$ and by considering the following commutative duality diagram coming from the universal coefficient theorem

$$H^2(W; \mathbb{R}) \xrightarrow{f^*} H^2(M; \mathbb{R})$$

$$\cong \quad \cong$$

$$H_2(W; \mathbb{R}) \xleftarrow{f_*} H_2(M; \mathbb{R})$$

we see that ker$(f_*)$ is isomorphic to im$(f^*)$.

Now the proof is straightforward. Diagonalize the matrix for the bilinear form to get a decomposition $H^2(M; \mathbb{R}) = P \oplus N$, where $P$ is the subspace on which the bilinear form is positive definite and $N$ is the subspace on which it is negative definite. We claim these have the same dimension. For suppose that say $P$ has dimension $\geq \frac{1}{2} \dim H^2(M; \mathbb{R}) + 1 = \dim f^*(H^2(W; \mathbb{R})) + 1$, then $f^*(H^2(W; \mathbb{R}))$ and $P$ must intersect in a subspace of dimension at least $1$ and hence the...
bilinear form can’t vanish on \( f^*(H^2(W; \mathbb{R})) \). This gives a contradiction and thus \( P \) and \( N \) have the same dimension, which implies that the signature of \( M \) is zero.

So to check that \( \sigma \) is surjective, we just need to find a 4-dimensional oriented manifold with signature 1.

**Lemma 2.11.** With its standard orientation, \( \sigma(\mathbb{C}P^2) = 1 \).

**Proof.** To see it’s non-zero, one simply needs to compute that \( H^2(\mathbb{C}P^2) = \mathbb{Z} \), for example using the standard cell decomposition with a single cell in degrees 0, 2, 4. Let \( x \) be the generator in degree 2, then by Poincaré duality we can’t have \( x \cap x = 0 \) and hence the bilinear form used in the signature is non-zero. To see that the signature is 1 instead of \(-1\), one need to compute that the cohomology ring is in fact isomorphic to \( \mathbb{Z}[x]/(x^3) \).

**Corollary 2.12.** \( \Omega^4_{SO} \) is not isomorphic to \( \mathbb{Z} \).

### 3. Thom’s Calculation of the Rational Oriented Bordism Groups

So what \( \Omega^*_SO \) in general? The signature turns out to be a very interesting invariant of 4-dimensional manifolds, so this is an interesting question to try and answer. In this section we’ll describe the answer using an additional ring structure on \( \Omega^*SO \) and then sketch its proof using the Pontryagin-Thom construction.

#### 3.1. The oriented bordism ring

What can we do with oriented manifolds? A very basic operation is taking products of these, and such a product has a canonical orientation. We want to import this structure to the oriented bordism groups. If everything works out, a \( k \)-operation is taking products of these, and such a product has a canonical orientation. We want to check is that this is independent of the choice of representatives of the bordism classes.

**Lemma 3.1.** If \( M \) is bordant to \( M' \) and \( N \) is bordant to \( N' \), then \( M \times N \) is bordant to \( M' \times N' \).

**Proof.** If \( N \) is bordant to \( N' \) by \( W \) and \( M \) is bordant to \( M' \) by \( V \), then \( W \times N \) is a cobordism from \( M \times N \) to \( M' \times N \) and \( M' \times V \) is a cobordism from \( M' \times N \) to \( M' \times N' \). Thus \( M \times N \) and \( M' \times N' \) are bordant.

We conclude that hence the product structure is well defined.

Thom calculated the cobordism ring. As is customary, we want to split the data into its free and primary parts by taking the tensor product with \( \mathbb{Q} \) or \( \mathbb{F}_p \) respectively. In this note, we’ll just look at the rational part and that case the result is as follows.

**Theorem 3.2.** We have an isomorphism of rings

\[
\Omega^*_SO \otimes \mathbb{Q} \cong \mathbb{Q}[x_{4i} | i \geq 1]
\]

with \( |x_{4i}| = 4i \). Furthermore \( x_{4i} \) is a non-zero multiple of \( \mathbb{C}P^{2n} \).

**Remark 3.3.** The complex projective spaces \( \mathbb{C}P^{2n} \) do not generate the oriented cobordism ring integrally. However, Wall has constructed more complicated generators for the integral case.

Note that the orientable bordism ring is concentrated in degrees divisible by four, for example for every five-dimensional oriented manifold some finite number of disjoint copies of it bounds a six-dimensional one. The reason for the finite number is that that the five-dimensional oriented manifold might represent a torsion class in the bordism group.

Also note that our calculation of the signature shows that the map \( \sigma : \Omega^4_{SO} \otimes \mathbb{Q} \to \mathbb{Q} \) is an isomorphism. However, it is more natural to use that the Pontryagin numbers as maps \( \Omega^*_{SO} \otimes \mathbb{Q} \to \mathbb{Q} \) which distinguish all bordism classes. These Pontryagin classes are geometrically constructed characteristic classes \( p_i(M) \in H^{4i}(M; \mathbb{Q}) \) and they distinguish all the bordism classes in the sense that if we consider all ordered non-decreasing tuples \( I \) of positive integers with \( \sum_{i \in I} i = n \) given by

\[
\prod_{\{i \in I\}} p_i(M), [M] \in \mathbb{Q}
\]
we get a basis for the dual space to $\Omega_{4n}^{SO}$. Part of this statement is that the Pontryagin numbers are invariant under oriented bordism.

The reader can now continue with a sketch of Thom’s calculation or take the calculation of the oriented bordism ring on faith and skip to the next section about the Hirzebruch signature theorem, which tells us how to compute the signature of a $4n$-dimensional manifold in terms of its Pontryagin numbers.

### 3.2. The Pontryagin-Thom construction

To solve the problem of computing the oriented bordism ring, we do one of the few things an algebraic topologists can do: convert the problem into computing homotopy groups of something. This something will be an object known as a spectrum. One should think of this a space up to suspension or roughly equivalently an infinite loop space:

**Definition 3.4.** A prespectrum $E$ is a sequence $\{E_n\}_{n \geq 0}$ of pointed spaces together with structure maps $\Sigma E_n \to E_{n+1}$ or equivalently $E_n \to \Omega E_{n+1}$.

A map of prespectra $E \to F$ is a sequence of pointed maps $E_n \to F_n$ compatible with the structure maps. A weak equivalence of prespectra is defined as a map inducing an isomorphism on homotopy groups of spectra discussed earlier and they are defined as $\pi_k(E) = \lim_{k \to \infty} \pi_k E_{k+i}$. To explain our previous remark about infinite loop spaces, we note that every prespectrum is weakly equivalent to an $\Omega$-spectrum, i.e. a prespectrum such that all the maps $E_n \to \Omega E_{n+1}$ are weak equivalences, which is the same as an infinite loop space.

Thom’s theorem expresses $\Omega^{SO}_{4n}$ as homotopy groups of a spectrum $MSO$. Before we define that spectrum we discuss some other examples. The simplest one is the suspension spectrum $\Sigma^\infty X$ of a pointed space $X$: $(\Sigma^\infty X)_n = \Sigma^n X$ and the structure maps $\Sigma \Sigma^n X \to \Sigma^{n+1} X$ are the identity. The most basic of these is the sphere spectrum $S$, which is the suspension spectrum of $S^0$. Its homotopy groups are the stable homotopy groups of spheres.

To define the Thom spectrum we need to introduce the Thom space construction for a (compact) space $B$ with a (finite-dimensional oriented real) vector bundle $\xi$ over it. It is simply the one-point compactification of the total space of $\xi$:

$$\text{Thom}(\xi) = \xi \cup \{\infty\}$$

If $B$ is not compact, one one-point compactifies each fiber of $\xi$ and identifies all the points at infinity to a single point. The classifying space $BSO(n)$ for $n$-dimensional oriented real vector bundles is the universal example of a space with a $n$-dimensional oriented real vector bundle over it: the universal bundle $\xi_u$. Every other $n$-dimensional real vector bundle over some space $B$ is obtained by pullback along some map $f : B \to BSO(n)$. It is not surprising that if Thom spaces are interesting at all, Thom$(\xi_u)$ is the most interesting of all. Note that there is a natural map Thom($f^*\xi_u$) $\to$ Thom($\xi_{u+1}$).

**Definition 3.5.** The Thom spectrum $MSO$ is given by $MSO_n = \text{Thom}(\xi_u)$. The structure maps are induced by the inclusion $BSO(n) \to BSO(n+1)$, since the pullback of $\xi_{n+1}$ is $\xi_n \oplus \mathbb{R}$ and Thom($\xi_u \oplus \mathbb{R}$) $\cong$ Thom($\xi_{u+1}$).

We can now state Thom’s theorem.

**Theorem 3.6 (Thom).** We have an isomorphism of rings

$$\Omega^{SO}_* \cong \pi_*(MSO)$$

**Proof.** We will just describe maps $\Omega^{SO}_n \to \pi_n(MSO)$ and $\pi_n(MSO) \to \Omega^{SO}_n$ and leave it to the reader or the references to check that these are mutually inverse and ring homomorphisms. These maps are called the Pontryagin-Thom construction.

Let’s start with the map $\Omega^{SO}_n \to \pi_n(MSO)$. The star here is the normal bundle. If we take an oriented $n$-dimensional manifold $M$, we can embed it into some $\mathbb{R}^{n+k}$ and it has a $k$-dimensional normal bundle $\nu$ there. The tubular neighborhood theorem gives an embedding $\nu \hookrightarrow \mathbb{R}^{n+k}$ such that the restriction to the zero-section is the embedding of $M$ into $\mathbb{R}^{n+k}$. Note that if we collapse the complement of the image of $\nu$ to a point, we get something that is homeomorphic to the Thom
space of \( \nu \) and we can extend this collapse map to the point at \( \infty \) in \( \mathbb{R}^{n+k} \) by sending it to \( \{\infty\} \) in \( \text{Thom}(\nu) \). Now consider the following sequence of maps

\[
S^{n+k} = \mathbb{R}^{n+k} \cup \{\infty\} \to \mathbb{R}^{n+k}/(\mathbb{R}^{n+k}\setminus \text{im}(\nu)) = \text{Thom}(\nu) \to \text{Thom}(\xi_k)
\]

where only the last map remains to be explained: it is the map induced by a classifying map \( M \to BSO(k) \) for \( \nu \). Passing to homotopy classes gives us an element in \( \pi_{n+k}(MSO_k) \) and hence in \( \pi_n(MSO) \).

We must check that is independent of the choice of embedding, tubular neighborhood and classifying map. The latter two are easy: every two tubular neighborhood are isotopic and every two classifying maps are homotopic, hence the two induced maps \( S^{n+k} \to \text{Thom}(\xi_k) \) are homotopic. To check that our construction doesn’t depend on the choice of embedding, one notes that if we use the standard embedding of \( \mathbb{R}^{n+k} \) into \( \mathbb{R}^{n+k+1} \), the same construction gives an element of \( \pi_{n+k+1}(MSO_{k+1}) \) that equal to the image of our original element of \( \pi_{n+k}(MSO_k) \) under suspension. Hence we can assume that the two embeddings are into the same \( \mathbb{R}^{n+k} \) and that \( k \geq n + 1 \). In that case any two embeddings are isotopic and the induced maps homotopic. Applying the same construction to a cobordism between two manifolds gives a homotopy between their corresponding elements of \( \pi_n(MSO) \), so this construction factors over \( \Omega^n_{SO} \).

For the map \( \pi_n(MSO) \to \Omega^n_{SO} \) we will use transversality. Any element of \( \pi_n(MSO) \) is represented by some map \( S^{n+k} \to MSO_k \) for \( k \geq n + 1 \). By compactness, the latter factors through \( \text{Thom}(\xi_{k,l}) \) for some sufficiently large \( l \), where \( \xi_{k,l} \) is the canonical \( k \)-dimensional vector bundle over the Grassmannian of oriented \( k \)-planes in \( \mathbb{R}^l \). Because \( k \geq n + 1 \), we can assume the map from \( S^{n+k} \) is smooth and by transversality we can assume it transverse to the zero-section and only \( \infty \in S^{n+k} = \mathbb{R}^{n+k} \cup \{\infty\} \) hits the point at infinity in Thom space. Thus we obtain a smooth map from \( \mathbb{R}^{n+k} \) to the total space the total space of the canonical vector bundle, both manifolds, that intersects transversally the zero section, a codimension \( k \) submanifold. The inverse image of the zero-section is a \( n \)-dimensional manifold \( M \subset \mathbb{R}^{n+k} \), which can be shown to be canonically oriented.

This is independent of the choices made: the representative map \( S^{n+k} \to MSO_k \) and the perturbation of this. Increasing \( k \) just increases the dimension of the Euclidean space \( M \) is embedded in. Picking two different homotopic representatives gives by a similar transversality argument a cobordism between the two manifolds, as does a different perturbation of the map.

\[\square\]

3.3. The rational homotopy groups of \( MSO \). The computation of the rational homotopy groups of \( MSO \) is suprisingly easy, given several standard results in algebraic topology.

**Proposition 3.7.** We have that

\[
\pi_*(MSO) \otimes \mathbb{Q} = H_*(MSO; \mathbb{Q}) = H_*(BSO; \mathbb{Q}) = \mathbb{Q}[p_i | i \geq 1]
\]

where the \( p_i \) of degree \( 4i \) are the Pontryagin classes mentioned before.

One of the terms in this sequence of equalities hasn’t been defined yet: the stable rational homotopy group \( H_*(MSO; \mathbb{Q}) := \text{colim}_{k \to \infty} H_{i+k}(MSO(k); \mathbb{Q}) \).

**Proof.** We’ll start by proving that \( \text{colim}_{k \to \infty} \pi_{i+k}(MSO(k)) \otimes \mathbb{Q} = \text{colim}_{k \to \infty} H_{i+k}(MSO(k); \mathbb{Q}) \).

To prove this, it suffices to note that the natural map \( S^n_{SO} \to K(\mathbb{Q}, n) \) is either a weak equivalence or \((2n - 2)\)-connected. We conclude that that if \( X \) is \( k \)-connected, then \( \pi_i(X) \otimes \mathbb{Q} \to H_i(X; \mathbb{Q}) \) is an isomorphism in degrees \( 0 \leq i \leq 2k - 2 \). We remarked before \( MSO(k) \) is \( k \)-connected, so \( \pi_{i+k}(MSO(k)) \otimes \mathbb{Q} = H_{i+k}(MSO(k); \mathbb{Q}) \) for \( 0 \leq i \leq k - 2 \). As \( k \to \infty \), we get the first isomorphism in the statement of the theorem.

Next we remark that \( H_{i+k}(MSO(k); \mathbb{Q}) = H_{i}(BSO(k); \mathbb{Q}) \) by the Thom isomorphism. Since the inclusion \( BSO(k) \to BSO \) is \( k \)-connected, we have that \( H_{i}(BSO(k); \mathbb{Q}) = H_{i}(BSO; \mathbb{Q}) \) for \( k \) sufficiently large, proving the second isomorphism. The final isomorphism is a standard computation using the Serre spectral sequence. \[\square\]
4. The Hirzebruch signature theorem

We will now explain a classical application of oriented bordism: a formula of the signature of a 4n-dimensional manifold in terms of its Pontryagin numbers. This is a result by Hirzebruch, explained nicely in [Hir66]. The set-up is straightforward: any map of abelian groups

$$\Omega^{SO}_{4n} \otimes \mathbb{Q} \to \mathbb{Q}$$

can be written as a linear combination of Pontryagin numbers, as they are the dual space for $$\Omega^{SO}_{4n} \otimes \mathbb{Q}$$, and we can find the coefficients by evaluating our map on products of $$CP^{2n}$$'s. If the map actually comes from a ring homomorphism

$$\Omega^{SO}_{\ast} \otimes \mathbb{Q} \to \mathbb{Q}$$

we have that its value is actually determined in terms of its value on the $$CP^{2n}$$'s.

The application we have in mind is for the signature. We only defined it for 4-dimensional manifolds, but the exact same construction gives a map $$\sigma : \Omega^{SO}_{4n} \to \mathbb{Z}$$. Furthermore by the Künneth formula $$\sigma(M \times N) = \sigma(M)\sigma(N)$$, so the signature is a ring homomorphism. We will find the coefficients for its expression in terms of Pontryagin numbers in this section, but only after defining genera as a useful context for studying these types of questions.

4.1. Genera and the L-genus as an example. We start with the definition of a genus, directly copying the properties of the signature.

**Definition 4.1.** A genus $$\phi$$ with values in a ring $$R$$ is a homomorphism

$$\phi : \Omega^{SO}_{\ast} \otimes R \to R$$

In the case where 2 is invertible in $$R$$, we have that $$\Omega^{SO}_{2} \otimes R = R[p_i | i \geq 1]$$ (this follows from the remark that the only torsion is $$H_*(BSO)$$ is 2-torsion). In this conditions we will give very general construction of a genus from a power series $$Q(t) = R[[t]]$$ with leading coefficient 1.

Let’s first think about how we would go about getting a genus from a formal power series. Our goal will be to find some polynomial expression $$P_n^Q(z_1, \ldots, z_n)$$ obtained from the power series such that if we substitute Pontryagin numbers of $$M$$ in $$z_i = (p_i(M), [M])$$ we get the value of our genus on $$M$$. Since powers in Pontryagin numbers are additive in disjoint union, this is an additive homomorphism.

Thus the main restriction on the $$P_n^Q$$’s is that they have to be compatible with cartesian product. For this we need to use the product formula, which says that (if 2 is invertible) $$p(E \oplus F) = p(E) \cup p(F)$$ for the full Pontryagin class $$p(E) = 1 + \sum_{i=1}^{\infty} p_i(E)$$. This tells us that $$p(M \times N) = p(M) \cup p(N)$$. It turns out to be useful to package the $$P_n^Q$$ into a single expression $$1 + \sum_{i=1}^{\infty} P_i^Q$$. In that conclusion that our $$P_n^Q$$ must satisfy that if 1 + $$\sum_{i=1}^{\infty} z_i t^i = (1 + \sum_{i=1}^{\infty} x_i t^i)(1 + \sum_{i=1}^{\infty} w_i t^i)$$ then $$P^Q(z) = P^Q(x)P^Q(w)$$.

Let us now give the definition of $$P_n^Q$$ in a way that forces $$P^Q(z) = P^Q(x)P^Q(w)$$. We write the coefficients of $$Q$$ by $$q_i$$. We define for a sequence $$I$$ of numbers $$i_1 \leq \ldots \leq i_k$$ with $$\sum_j i_j = n$$ the element $$Q(I)$$ to be $$\prod_{j=1}^{k} q_{i_j}$$. Furthermore we define a polynomial $$s_I(z_1, \ldots, z_n)$$ to be the unique polynomial such that $$s_I(\sigma_1(t), \ldots, \sigma_n(t)) = \sum t^I$$, where the $$\sigma_i$$ are the elementary symmetric polynomials and the sum is over distinct permutations of the indices of the $$t_i$$. These polynomials satisfy $$\sum_I s_I(z) = \sum_{I \cup J = I} s_I(z) s_J(w)$$. We then define

$$P_n^Q = \sum_I Q(I) s_I(z_1, \ldots, z_n)$$

It is now easy to check that $$P^Q(z) = P^Q(x)P^Q(w)$$ and we have thus defined a genus which we call $$\phi_Q$$ by defining for a 4n-dimensional manifold $$M$$

$$\phi_Q(M) = P_n^Q(\langle p_1(M), [M] \rangle, \ldots, \langle p_n(M), [M] \rangle) = \langle P_n^Q(p_1(M), \ldots, p_n(M)), [M] \rangle$$

In fact there is an inverse to this construction, giving a power series $$Q_\phi$$ for each genus $$\phi$$ such that $$\phi_Q = \phi$$ and $$Q_{\phi_Q} = Q$$. It involves the log-series of a formal power series with leading coefficient 1.
Remark 4.2. It is not hard to see that the coefficient of \(z^n\) in \(P_n^Q\) is exactly \(q_n\).

Example 4.3. A very simple power series over \(\mathbb{Q}\) is \(1 + t\). To figure out what the corresponding genus is, we note that only \(I = (1, \ldots, 1)\) has a non-zero coefficient and \(s_1, \ldots, 1\) exactly is the elementary symmetric polynomial \(s_n\). So the corresponding genus is given on a 4n-dimensional manifold by the \(n\)th Pontryagin number \((p_n(M), [M])\).

Example 4.4. Another nice example of a genus is the Todd genus \(\text{Td}\), which plays an important role in Atiyah-Singer and Hirzebruch-Riemann-Roch. It is the unique genus with values in \(\mathbb{Q}\) such that \(\text{Td}(CP^{2n}) = 1\) for all \(n\). The corresponding power series is the expansion of \(\frac{t}{1 - \exp(-t)}\).

4.2. The \(L\)-genus gives signature. We will be concerned with the genus coming from the power series \(Q(t)\) obtained by expanding \(\frac{t}{\tanh(\sqrt{x})}\). We call it the \(L\)-genus.

Theorem 4.5 (Hirzebruch signature theorem). For all 4n-dimensional manifolds we have

\[
\sigma(M) = \phi_L(M)
\]

Proof. It suffices to prove the equality on the \(CP^{2n}\). The signature is always 0 on those, so all the difficulty will be in proving the same for \(\phi_L\).

We now need that \(p(CP^{2n}) = (1 + x^2)^{2n+1}\), if we use \(x\) to denote the generator of the cohomology ring of \(CP^{2n}\). Due to multiplicativity of the \(P^L\), we have that \(P^L((1 + x^2)^{2n+1}) = P^L(1 + x^2)^{2n+1}\).

So what is \(P^L(1 + x^2)\)? The only relevant term is \(x^2\), so it is \(\sum P^L_n(x^2, 0, \ldots)\). We know the coefficient of \(z^n\) is exactly the \(n\)th Taylor coefficient of \(\frac{x}{\tanh(\sqrt{x})}\), so we get \(P^L(1 + x^2) = \frac{\sqrt{2\pi}}{\tanh(\sqrt{x})}\).

We conclude that

\[
\phi_L(CP^{2n}) = \left\langle \left(\frac{x}{\tanh(\sqrt{x})}\right)^{2n+1}, [CP^{2n}] \right\rangle
\]

so that our goal is to prove that the coefficient \(a_{2n}\) of \(x^{2n}\) in \(\left(\frac{x}{\tanh(\sqrt{x})}\right)^{2n+1}\) is 1. We can do this using complex analysis:

\[
a_{2n} = \frac{1}{2\pi i} \oint \frac{1}{\tanh(z)^{2n+1}} \, dz
\]

This can be computed using the substitution \(u = \tanh(z)\), which has \(du = (1 - \tanh(z)^2)dz = (1 - u^2)dz\):

\[
a_{2n} = \frac{1}{2\pi i} \oint \frac{du}{(1 - u^2)u^{2n+1}} = \frac{1}{2\pi i} \oint \frac{1 + u^2 + \ldots + u^{2n} + \ldots}{u^{2n+1}} \, du = 1
\]

5. More applications

Finally, we sketch some other applications of oriented bordism. The first two are related to particular genera and their properties, and the last one was the actual motivation for Thom to introduce bordism: figuring out which homology classes are represented by the image of the fundamental class of a manifold mapping into your space.

5.1. Positive scalar curvature and the \(A\)-genus. Recall that the scalar curvature of a manifold with metric is the full trace of the Riemann curvature tensor. Geometrically it is related to the difference between the volumes of small spheres in the manifold and Euclidean volume of spheres. The result is a real-valued function on \(M\) and the properties of this function is closely related to the geometry of \(M\).

Related to this is another example of a genus: the genus coming from the rational power series \(Q(x) = \frac{x^2/2}{\sinh(\sqrt{x}/2)}\). This is called that \(A\)-genus. The exact expression is not important (yet), but the following result is:

Theorem 5.1 (Lichnerowicz). Let \(M\) be a spin-manifold (i.e. the classifying map \(M \to BSO(n)\) of the tangent bundle lifts to \(BSpin(n)\)). Then \(M\) can only have a positive scalar curvature metric if \(\hat{A}(M) = 0\).
In other words we’ve found a topological invariant giving an obstruction to \( M \) possessing a metric with scalar curvature being pointwise positive. This opens up a very interesting direction for studying the question whether a manifold admits a positive scalar curvature metric. Completely opposite from this obstruction, there is a positive result concerning bordism and positive scalar curvature: if \( M \) has a positive scalar curvature metric and there is a cobordism \( W \) from \( M \) to \( M' \) which can be made by handle attachments of codimension \( \geq 3 \), then \( M' \) has a metric of positive scalar curvature as well.

Using the explicit determination of the oriented and spin cobordism rings, Gromov-Lawson and Stolz completely answered the question which simply-connected manifolds admit a positive scalar curvature metric.

**Theorem 5.2.** If \( M \) is a simply-connected non-spin manifold of dimension \( \geq 3 \), then \( M \) admits a metric of positive scalar curvature. If \( M \) is a simply-connected spin-manifold of dimension \( \geq 3 \), then \( M \) admits positive scalar curvature metric if and only if an invariant \( \alpha(M) \) (essentially a refined \( \hat{A} \)-genus admitting torsion) vanishes.

A good survey about these and related results is [RS01].

### 5.2. Elliptic genera.

Both the \( L \)-genus and \( \hat{A} \)-genus are special cases of the elliptic genus. It is a genus

\[
\text{Ell}: \Omega^\text{SO}_\ast \to \mathbb{Z}[1/2, \delta, \epsilon]
\]

defined by \( Q(z) = \frac{\sqrt{z}}{f(\sqrt{z})} \) for \( f \) given by

\[
f(z) = \int_0^z \frac{du}{\sqrt{1 - \delta u^2 + \epsilon u^4}}
\]

Though the motivation for this is clearer if one looks at complex bordism (closely related to oriented bordism by complexification), the following examples should show this notion is interesting: if we set \( \delta = \epsilon = 1 \), we get \( f(z) = \int_0^z \frac{du}{1-u^2} = \tanh(z) \), and if we set \( \delta = -\frac{1}{8} \) and \( \epsilon = 0 \), we get \( f(z) = \int_0^z \frac{du}{\sqrt{1 + \frac{1}{8} u^2}} = 2 \sinh\left(\frac{z}{2}\right) \).

**Definition 5.3.** An elliptic genus is a genus obtained from \( \text{Ell} \) by specifying \( \delta \) and \( \epsilon \).

So in particular the \( L \)- and \( \hat{A} \)-genus are elliptic. There is an interesting geometric characterisation of elliptic genera due to Ochanine.

**Theorem 5.4** (Ochanine). A genus with values \( \mathbb{Z}[1/2] \) is elliptic if and only if it vanishes on projectivizations of complex vector bundles.

Pursuing the connection between elliptic curves and topology further leads to elliptic cohomology theories [Tho99] and eventually TMF [Lur09]. It also led Witten to define the Witten genus and conjecture relations to the scalar curvature of loop spaces [Wit87].

### 5.3. Geometric representatives for homology classes.

Finally, we go back all the way to the conception of oriented bordism and look at the question that Thom wanted it to answer. If one has a (not necessarily oriented) manifold \( M \) and a map \( f \) from an oriented manifold \( N \) into it, we can consider the image \( f_\ast([N]) \) of the fundamental class of \( M \). Especially in earlier times when the definition of homology was still in flux, people were interested in answering the question which homology classes of \( M \) were realized by such maps.

Thom ended up solving this question by proving it is always possible with \( F_2 \)-coefficients, not always with \( F_p \)-coefficients for \( p \geq 3 \) prime and up to an odd multiple with integers coefficients. See [Sul04] for a survey.

### 6. Other things I could have talked about

For the reader that wants to learn more, here are some suggestions of topics.

- As a spectrum oriented bordism actually defines a generalized (co)homology theory. One useful computational tool is the Atiyah-Hirzebruch spectral sequence.
• Bordism groups for other tangential structures exist, e.g. unoriented bordism $MO$ or complex bordism $MU$.
• Complex bordism $MU$ is closely related to one-dimensional formal groups laws, leading nicely into the story about elliptic cohomology (via Landweber exactness) and TMF.
• Bordism theories with cone-like singularities, known as Baas-Sullivan theories, are a way to increase the scope of the (co)homology theories obtained from bordism. For example, one can define ordinary cohomology with integral coefficients in terms of a bordism theory.
• On a related note, Conner-Floyd tells us how to define K-theory in terms of bordism.

References