THE ALGEBRAIC K-THEORY OF ALGEBRAICALLY CLOSED FIELDS

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Abstract. We sketch the proof of Suslin’s theorem that an inclusion of algebraically closed induces an isomorphism on algebraic $K$-theory.

These are the notes for a talk in the course Math 395. The references are [Sus83] and [Wei13]. We mostly follow the latter, section VI.1.

1. Suslin’s theorem

This note is about determining the algebraic $K$-theory of fields. We have a straightforward description of these groups in low degree:

$$K_n(F) = \begin{cases} 
\mathbb{Z} & \text{if } n = 0 \\
F^\times & \text{if } n = 1 \\
(F^\times \otimes F^\times) / \langle x \otimes (1-x) = 0 \text{ for all } x \neq 0, 1 \rangle & \text{if } n = 2 \\
? & \text{otherwise}
\end{cases}$$

Our goal is to determine all these groups for algebraically closed fields $F$, at least with finite coefficients. What are these? The $K$-theory groups are the homotopy groups of some spectrum $K(F)$: $K_n(F) = \pi_{n+1}(K(F))$. This leads to the definition of algebraic $K$-theory of finite coefficients:

$$K_n(F; \mathbb{Z}/m\mathbb{Z}) = \pi_{n+1}(K(F) \wedge M(\mathbb{Z}/m\mathbb{Z}))$$

where $M(\mathbb{Z}/m\mathbb{Z})$ is the mod $m$ Moore spectrum. This is related to ordinary algebraic K-theory by the following short exact sequence

$$0 \to K_n(F) \otimes \mathbb{Z}/m\mathbb{Z} \to K_n(F; \mathbb{Z}/m\mathbb{Z}) \to \ker(\times m : K_n(F) \to K_n(F)) \to 0$$

The slogan for Suslin’s theorem is “the algebraic $K$-theory of an algebraically closed field only depends on the characteristic”. More precisely, the theorem is as follows:

Theorem 1.1. If $i : k \hookrightarrow F$ is an inclusion of algebraically closed fields, then $i_* : K_{n-1}(k; \mathbb{Z}/m\mathbb{Z}) \to K_{n-1}(F; \mathbb{Z}/m\mathbb{Z})$ for all $m \in \mathbb{N}$.

Before we start with our first technical tool, the localization theorem, we recall a subset of the various ways to define algebraic $K$-theory of rings. Indeed, we can define algebraic $K$-theory in a varying degrees of generality, e.g. for rings, symmetric monoidal categories, exact categories, abelian categories and Waldhausen categories. More precisely, for a ring $R$, we can define the ordinary $K$-theory groups $K_n(R)$ in almost all of these ways:

(i) As the homotopy groups of $\Omega BG\mathcal{L}_\infty(R)^+$, where $+$ is the Quillen plus construction.
(ii) As the homotopy groups of $BS^{-1}SP(R)^\oplus$, where $SS^{-1}$ is the $S^{-1}S$-construction for symmetric monoidal categories, here applied to the symmetric monoidal category $P(R)^\oplus$ of finitely generated projective $R$-modules under direct sum.
(iii) As the homotopy groups of the space $\Omega BQ\mathcal{P}(R)$, where $Q$ is Quilen’s $Q$-construction, here applied to the exact category of finitely generated projective $R$-modules $\mathcal{P}(R)$.
(iv) As the homotopy groups of $\Omega \mathcal{S}_*\tilde{P}(R)$, which is the Waldhausen $\mathcal{S}_*$-construction applied to the Waldhausen category of finitely generated projective $R$-modules $\tilde{P}(R)$.
In this talk only the first and third are relevant. The first is used only to see the following: if \( R = \text{colim} R_\alpha \) then we have

\[
K_n(R) = \text{colim}_\alpha K_n(R_\alpha)
\]

We will apply this several times, for example if \( k \subset F \) to all finitely-generated subfields of \( F \) containing \( k \) or all finitely generated \( k \)-subalgebras contained in \( F \).

2. The localization theorem (and secretly resolution and devissage as well)

Our first technical tool critically uses the third definition of algebraic K-theory, as a construction of a spectrum out of an exact category.

The definition of an exact category was inspired by that of its more well-behaved cousin, the abelian category. One can do more construction with abelian categories and in particular we will need a construction on abelian categories called localization. Suppose that we have an abelian category \( \mathcal{A} \) with Serre subcategory \( \mathcal{B} \), i.e. an abelian subcategory closed under subobjects, quotients and extensions. In that case we can form the quotient abelian category \( \mathcal{A}/\mathcal{B} \). This is an abelian category with the same objects as \( \mathcal{A} \) but morphisms \( A_1 \to A_2 \) are now equivalence classes of zigzags \( A_1 \leftarrow A \to A_2 \) with the arrow \( \leftarrow \) having kernel and cokernel in \( \mathcal{B} \). Two zigzags \( A_1 \leftarrow A \to A_2 \) and \( A_1 \leftarrow A' \to A_2 \) are equivalent if there exists a commutative diagram

\[
\begin{array}{ccc}
A & \rightarrow & A_2 \\
\downarrow & & \downarrow \\
A_1 & \leftrightarrow & A'' \\
\downarrow & & \downarrow \\
A' & \rightarrow & A_2
\end{array}
\]

with both vertical arrows having kernel and cokernel in \( \mathcal{B} \). We can now state the localization theorems for \( K \)-theory of abelian categories [Wei13, theorem V.5.1].

**Theorem 2.1** (Localization). Let \( \mathcal{A} \) be an abelian category and \( \mathcal{B} \) be a Serre subcategory. Then we have a fiber sequence

\[
K(\mathcal{B}) \to K(\mathcal{A}) \to K(\mathcal{A}/\mathcal{B})
\]

and thus a long exact sequence of \( K \)-theory groups.

We want to apply this construction to get \( K \)-theory of our ring \( R \). But disaster strikes! The exact category \( \mathcal{P}(R) \) is not abelian, since kernels of maps between projective modules need not be projective. For example, if \( R = \mathbb{Z} \), take \( 2 : \mathbb{Z} \to \mathbb{Z} \) with kernel \( \mathbb{Z}/2\mathbb{Z} \), which clearly can’t have a diagonal dotted lift in

\[
\begin{array}{ccc}
& & \mathbb{Z} \\
& \downarrow & \\
\mathbb{Z}/2\mathbb{Z} & \rightarrow & \mathbb{Z}
\end{array}
\]

Instead we’ll work with the abelian category \( \mathcal{M}(R) \) of finitely generated \( R \)-modules. This is an abelian category if \( R \) is Noetherian (one definition of which exactly is that all submodules of a finitely generated module are finitely generated, exactly what we need for kernels to exist). We define \( G(R) = BQ\mathcal{M}(R) \) and \( G_n(R) = \pi_{n+1} BQ\mathcal{M}(R) \). This is closely related to \( K \)-theory, via the obvious map \( K(R) = BQ\mathcal{P}(R) \to BQ\mathcal{M}(R) = G(R) \) coming from the inclusion \( \mathcal{P}(R) \hookrightarrow \mathcal{M}(R) \) of exact categories. In many cases this map induces an isomorphism on homotopy groups.

**Proposition 2.2.** If \( R \) is Noetherian and regular (every finitely generated module has a finite projective resolution), then the natural map \( K_n(R) \to G_n(R) \) is an isomorphism.
Proof. Suppose that $\mathcal{M}$ is an exact category with a full exact category $\mathcal{P}$ that is closed under extensions and kernels of admissible epimorphisms and that each object $M$ of $\mathcal{M}$ admits a finite resolution by objects in $\mathcal{P}$:

$$0 \to P_k \to \ldots \to P_1 \to P_0 \to M$$

Then the resolution theorem for algebraic K-theory [Wei13, theorem V.3.1] says that the inclusion $\text{BQ}\mathcal{P} \to \text{BQ}\mathcal{M}$ is a weak homotopy equivalence.

Consider $\mathcal{P}(R)$ inside $\mathcal{M}(R)$. This satisfies the conditions of the resolution theorem if $R$ is Noetherian and regular. \hfill \Box

Examples of regular Noetherian rings are fields, discrete valuation rings and Dedekind domains. We will recall the definitions of these when relevant.

We now want to prove a localization theorem for $G$-theory. We will do a special case of it, where we localize $R$ at an non-zero divisor $s$. Let $\mathcal{M}_s(R)$ be the abelian category of finitely generated $R$-modules $M$ such that $s^nM = 0$ for some $M$. This is easily seen to be a Serre subcategory of $\mathcal{M}(R)$. We claim that $\mathcal{M}(R)/\mathcal{M}_s(R)$ is equivalent to $\mathcal{M}(s^{-1}R)$. The functor $\mathcal{M}(s^{-1}R) \to \mathcal{M}(R)/\mathcal{M}_s(R)$ is induced by the inclusion, its inverse $\mathcal{M}(R)/\mathcal{M}_s(R) \to \mathcal{M}(s^{-1}R)$ by localizing at $s$. The localization theorem almost proves the following corollary:

**Theorem 2.3.** There is a fiber sequence

$$G(R/sR) \to G(R) \to G(s^{-1}R)$$

inducing a long exact sequence on homotopy groups.

**Proof.** We just need to prove that $\text{BQ}\mathcal{M}_s(R)$ is weakly equivalent to $\text{BQ}\mathcal{M}(R/sR)$. This is a consequence of the devissage theorem [Wei13, theorem V.4.1], which says that if $\mathcal{P}$ is an exact abelian subcategory of $\mathcal{M}$ such that each object $M$ of $\mathcal{M}$ has a finite filtration

$$0 \to M_k \to \ldots \to M_1 \to M_0 = M$$

with successive quotients $M_i/M_{i+1}$ lying in $\mathcal{P}$, then $\text{BQ}\mathcal{P} \to \text{BQ}\mathcal{M}$ is a weak equivalence.

Indeed, all we need to do is note that a finitely generated $R$-module annihilated by $s^n$ has a finite filtration with successive quotients annihilated by $s$. \hfill \Box

There is a more advanced version localizing at any set $S$, not just $S = \{s, s^2, \ldots\}$. In that case $G(R/sR)$ is replaced by the filtered colimit $\text{colim}_{i \in S} G(R/sR)$, coming from the definition $\mathcal{M}_S(R) = \text{colim}_{i \in S} \mathcal{M}_i(R)$ (forced upon us because we know what we have $\mathcal{M}(R)/\mathcal{M}_S(R)$ to be).

We are interested in a special case of this: $R$ a Dedekind domain and $S = R\setminus\{0\}$. Recall that a Dedekind domain is an integral domain where every ideal factors as a product of prime ideals. In this case we can use a different description of $\mathcal{M}_S(R)$. We claim it is equal to $\bigoplus_{p \subset R} \mathcal{M}_p(R)$, where the $p$ are a prime ideals and $\mathcal{M}_p(R)$ consists of those modules annihilated by a power of $p$. This uses the classification of finitely generated torsion modules over a Dedekind domain: they are all of the form $\bigoplus_{i=1}^r R/p_i^{n_i}$. More generally every finitely generated module is a direct sum of rank one projective modules and a torsion module, and thus Dedekind domains are regular. Indeed it suffices to tell the reader how to resolve a module of the form $R/p_i^{n_i}$. Consider the short exact sequence $p_i^{n_i} \to R \to R/p_i^{n_i}$. A Dedekind domain has the property that every submodule of a projective module is projective, so this is a projective resolution. In particular we see that every module has a projective resolution of length 1. We get the following corollary:

**Corollary 2.4.** Let $R$ be a Dedekind domain and $S = R\setminus\{0\}$, then we have a fiber sequence

$$\prod_p K(R/pR) \to K(R) \to K(F)$$

with $F$ the quotient field of $R$. This induces a long exact sequence on homotopy groups.
3. Specialization maps, transfer maps and their properties

In this section we describe specialization and transfer maps between the $K$-theory of different rings. We start by remarking that if $f : R \to S$ is a flat map there is a standard induced map $f_* \in G$-theory, coming from the exact functor $S \otimes_R - : \mathcal{M}(R) \to \mathcal{M}(S)$, and hence in $K$-theory for fields or Dedekind domains.

3.1. Specialization maps. We start by defining the specialization map for a discrete valuation ring $R$. This is a ring $R$ with discrete valuation $\nu : F \to \mathbb{Z}$ from its field of fractions and $R = \nu^{-1}(\mathbb{N} \cup \{0\})$. This is a ring with a unique maximal ideal $\nu^{-1}(\mathbb{N})$, necessarily of the form $(s)$ for any uniformizer $s \in R$. There are two natural fields associated to a discrete valuation ring: the quotient field $F$ and the residue field $k = R/(s)$. Any discrete valuation ring is in particular a Dedekind domain and conversely the localization of a Dedekind at any non-zero prime ideal is a discrete valuation ring.

Note that $K_n(F)$ is a $K_*(R)$-module and corollary 2.4 gives a map $\partial : K_n(F) \to K_{n-1}(R/(s)) = K_{n-1}(k)$. For a discrete valuation ring $R$ we get a specialization map
\[
\lambda_s : K_n(F) \to K_n(k)
\]
\[a \mapsto \partial(s \cdot a)
\]

This depends on $s$ but not in any important sense. If $s' = us$ for $u \in R^\times$ then $\lambda_{s'}(a) = \lambda_s(a) + \bar{u} \cdot a$, where $\bar{u}$ is image of $u$ in $K_1(F)$.

Let’s say a few things about the specialization map. It turns out that on $K_i$ for $i = 0, 1, 2$, $\partial$ coincides with the tame symbol of Milnor $K$-theory, which a component of the map $K^M_i(F) \to K^M_i(k)[\Pi]/(\{\Pi, -\Pi\})$ induced by the map $F^\times \to k^\times$ sending $u^s$ to $\bar{u} \oplus s\Pi$. Suppose that $k \subset R$, then we can compute $\partial(s \cdot a)$ for $a \in K_n(k)$. Since everything in sight is a $K_*(R)$-module, $\partial$ is $K_*(k)$-linear so that $\partial(s \cdot a) = \partial(s) \cdot a$. But $\partial(s)$ is simply 1, the class corresponding to $k$ as a $k$-module.

This implies the following result:

**Theorem 3.1.** Suppose that $R$ is a discrete valuation ring containing a field $k_0$ such that $[k : k_0]$ is finite. Then the long exact sequence of corollary 2.4 becomes a split short exact sequence
\[0 \to K_n(R) \to K_n(F) \to K_{n-1}(k) \to 0
\]

Moreover for $\pi : R \to k$ we have that $\pi_* : K_n(R) \to K_n(k)$ (the natural map on $G$-theory induced by $k \otimes_R -$) factors over the specialization map.

**Proof.** If $k \subset R$ then the splitting is given by $a \in K_{n-1}(k) \mapsto s \cdot a$, which our previous calculation shows to be a splitting. In the general case one passes to the integral closure of $R$ in a finite field extension of $F$ designed to replace $k_0$ by $k$ as the field including in $R$.

For the second case we note that the sequence is one $K_*(R)$-modules, so that for $i : R \to F$ the inclusion we compute that $\lambda_s(i_\ast a) = \partial(s \cdot i_\ast a) = \partial(s) \cdot \pi_*a = \pi_*a$. In other words, the following diagram commutes:

\[
\begin{array}{ccc}
K_n(R) & \stackrel{\pi_*}{\longrightarrow} & K_n(k) \\
\downarrow{\iota_*} & & \downarrow{\lambda} \\
K_n(F) & & \\
\end{array}
\]

\[
3.2. \textbf{Transfer maps.} \text{ For a field extension } K/k \text{ with inclusion } \iota : k \to K \text{ induces a map } \iota_* : K_n(k) \to K_n(K) \text{ induced by sending a } k\text{-module } V \text{ to } K \otimes_k V. \text{ If } K/k \text{ is finite, there is a transfer map } \bar{\iota}_* : K_n(K) \to K_n(k). \text{ This is multiplication by } [K : k] \text{ on } K_0.
\]

In fact, we have such a transfer map $f'$s in $G$-theory for any map $f : R \to S$ that exhibits $S$ as a finitely generated $R$-module. Simply note that this means that the functor $\mathcal{M}(S) \to \mathcal{M}(R)$ which considers every finitely-generated $S$-module as an $R$-module is well-defined.
We secretly already met the transfer maps. Note that if $R$ is any ring, then $\pi_p : R \to R/pR$ displays the latter as a finitely generated $R$-module and we get a transfer map. If $R$ was a Dedekind domain then indeed in corollary 2.4 the maps are given by

$$\bigoplus_p \pi_p^*: \bigoplus_p K_n(R/pR) \to K_n(R)$$

Suppose that $R \subset R'$ is an inclusion of Dedekind rings exhibiting $R'$ has a finitely-generated $R$-module. Then the extension $F'/F$ of quotient fields is finite and for a prime ideal $\mathfrak{p}'$ of $R'$ such that $\mathfrak{p} \supset R \cap \mathfrak{p}'$ we have that the map $R/pR \to R'/\mathfrak{p}'R'$ exhibits the latter as a finitely generated module. Then the following diagram commutes

$$
\begin{array}{cccc}
\ldots & \longrightarrow & \bigoplus_p K_n(R/pR) & \longrightarrow & K_n(R) & \longrightarrow & K_n(F) & \longrightarrow & \ldots \\
& & \oplus N_{(R'/\mathfrak{p}'R')/(R/pR)} & & & & N_{R'/R} & & \\
\ldots & \longrightarrow & \bigoplus_{\mathfrak{p}'} K_n(R'/\mathfrak{p}'R') & \longrightarrow & K_n(R') & \longrightarrow & K_n(F') & \longrightarrow & \ldots \\
\end{array}
$$

and thus $\partial$ commutes with norms.

4. The rigidity theorem

Our goal is to prove something about $K$-theory of “families” of fields. Let $C$ be a smooth curve over a field $k$. Then for each closed point $c$ of $C$ the local ring $O_{C,c}$ of germs is a discrete valuation ring with residue field some finite field extension $K$ and thus $\eta$ ring with residue field some finite field extension of $k$. Then the following diagram commutes

$$
\begin{array}{cccc}
\ldots & \longrightarrow & \bigoplus_p K_n(R/pR) & \longrightarrow & K_n(R) & \longrightarrow & K_n(F) & \longrightarrow & \ldots \\
& & \oplus N_{(R'/\mathfrak{p}'R')/(R/pR)} & & & & N_{R'/R} & & \\
\ldots & \longrightarrow & \bigoplus_{\mathfrak{p}'} K_n(R'/\mathfrak{p}'R') & \longrightarrow & K_n(R') & \longrightarrow & K_n(F') & \longrightarrow & \ldots \\
\end{array}
$$

and thus $\partial$ commutes with norms.

The specialization map factors as a map

$$\lambda : \text{Div}(C) \to \text{Hom}(K_*(F;\mathbb{Z}/m\mathbb{Z}), K_*(k;\mathbb{Z}/m\mathbb{Z}))$$

The first important step in our proof is to show that this factors of $\text{Pic}(C)$, by definition the cokernel of the divisor map $F^\times \to \text{Div}(C)$. This map sends a rational function in $F$ to the sum of its poles and roots, counted with multiplicity.

**Proposition 4.1.** The specialization map factors as a map

$$\text{Pic}(C) \to \text{Hom}(K_*(F;\mathbb{Z}/m\mathbb{Z}), K_*(k;\mathbb{Z}/m\mathbb{Z}))$$

**Proof.** Let’s first do the case $C = \mathbb{P}^1$ and show that all specialization maps are the same. Then we use that to prove it for any curve.

(i) the case $C = \mathbb{P}^1$: If $C = \mathbb{P}^1$, then $F = k(t)$. As in the case of discrete valuation rings, it turns out that for the Dedekind domain $k[t]$ the localization sequence is also split:

$$K_n(k(t);\mathbb{Z}/m\mathbb{Z}) \cong K_n(k[t];\mathbb{Z}/m\mathbb{Z}) \oplus \bigoplus_{a \in k} K_{n-1}(k[T]/(T-a);\mathbb{Z}/m\mathbb{Z})$$

The splitting is given by the map

$$i_* \oplus \bigoplus_{a \in F}((T-a)\cdot -) : K_n(k[t];\mathbb{Z}/m\mathbb{Z}) \oplus \bigoplus_{a \in k} K_{n-1}(k[T]/(T-a);\mathbb{Z}/m\mathbb{Z}) \to K_n(k(t);\mathbb{Z}/m\mathbb{Z})$$

We ignored the question of the choice of uniformizer for the following reason. We saw if we picked a different one then $\lambda_s - \lambda_s'$ is in the image of multiplication by elements of $K_1(k;\mathbb{Z}/m\mathbb{Z})$. But if $k$ algebraically closed, then $K_1(k) = k^\times$ is divisible so that $K_1(k;\mathbb{Z}/m\mathbb{Z}) = 0$. We conclude that in the circumstances above the specialization map is independent of choices if we work with finite coefficients.

More generally by passing to finite coefficients and taking combinations of different $\lambda_c$ we get a specialization map $\lambda$ from the free abelian group $\text{Div}(C)$ on closed points of $c$ to the group homomorphisms between the two $K$-theory groups:

$$\lambda : \text{Div}(C) \to \text{Hom}(K_*(F;\mathbb{Z}/m\mathbb{Z}), K_*(k;\mathbb{Z}/m\mathbb{Z}))$$

The first important step in our proof is to show that this factors of $\text{Pic}(C)$, by definition the cokernel of the divisor map $F^\times \to \text{Div}(C)$. This map sends a rational function in $F$ to the sum of its poles and roots, counted with multiplicity.
Now we recall the fundamental theorem of G-theory [Wei13, V.6.2], which says that for G-theory \( j_s : G_n(R) \to G_n(R[s]) \) is an isomorphism, and that \( k[T]/(T-a) \cong k \). This means that we have an isomorphism.

\[
i_\ast j_\ast \oplus \bigoplus_{a \in F} ((T-a) \cdot -): K_n(k; \mathbb{Z}/m\mathbb{Z}) \oplus \bigoplus_{a \in k} K_n-1(k; \mathbb{Z}/m\mathbb{Z}) \cong K_n(k(t); \mathbb{Z}/m\mathbb{Z})
\]

Let’s compose \( \lambda_c \) with \( i_\ast j_\ast \oplus \bigoplus_{a \in F} ((T-a) \cdot -) \). We use that \( k \subset k[T] \), so that everything is a \( K_\ast(k; \mathbb{Z}/m\mathbb{Z}) \)-module.

The composition \( \lambda_c((T-a) \cdot -) \) is zero, because if the relevant uniformizer is \( s \) then we see a multiplication by \( \partial((T-s) \cdot s) \) using due \( K_\ast(k; \mathbb{Z}/m\mathbb{Z}) \)-linearity. But \( \partial((T-s) \cdot s) \) lies in \( K_1(k; \mathbb{Z}/m\mathbb{Z}) \), which we know is zero.

That leaves us to figure out what the composition \( \lambda_c(i_\ast j_\ast) \) is. By \( K_\ast(k; \mathbb{Z}/m\mathbb{Z}) \)-linearity we need to figure out what happens when we apply it to \( 1 \in K_0(k; \mathbb{Z}/m\mathbb{Z}) \). It gets send to \( 1 \in K_0(k(t); \mathbb{Z}/m\mathbb{Z}) \) and then to \( \partial(s) = 1 \). This proves that \( \lambda_c(i_\ast j_\ast) = \pm \text{id} \) (there might be a sign issue because the product in K-theory is graded-commutative). We see that \( \lambda_c \) doesn’t depend on \( c \) at all!

(ii) The general case: Let \( R = k[t] \) and let the function field of \( C \) be denoted by \( F \). The elements of \( F \) are rational functions and thus the non-constant \( f \in F \) correspond to finite maps \( f : C \to \mathbb{P}^1 \). Pick an arbitrary \( f \). Let \( R' \) be local ring \( \mathcal{O}_{C,c} \) of germs of rational functions at \( c \) with uniformizer \( s \). Then pullback induces an inclusion \( R \to R' \) exhibiting the latter as a finitely generated \( R \)-module.

A rational function over \( \mathbb{P}^1 \) pulls back to one over \( C \), so we can write \( t = us^e \). Here \( e \) is the ramification of \( f \) at \( c \), i.e. the order of the zero at \( c \). We will compute for \( 0 \in \mathbb{P}^1 \) the specialization \( \lambda_0(N_{F/k(t)}(a)) \). By definition \( \lambda_0(N_{F/k(t)}(a)) = \partial_0(t \cdot N_{F/k(t)}(a)) \) and the projection formula for transfers the latter is \( \partial_0(N_{F/k(t)}(t \cdot a)) \). By compatibility of the \( \partial \)'s with transfer maps we get \( \partial_0(N_{F/k(t)}(t \cdot a)) = \sum_{f(c)=0} N_c \partial_c(t \cdot a) \). Finally, using that \( t = us^e \) we conclude that \( \partial_c(t \cdot a) = e_c \partial(s, a) = e_c \lambda_c(a) \). The end result is

\[
\lambda_0(N_{F/k(t)}(a)) = \partial_0(N_{F/k(t)}(t \cdot a)) = \sum_{f(c)=0} N_c \partial_c(t \cdot a) = \sum_{f(c)=0} e_c \lambda_c(a)
\]

Using our calculation for \( \mathbb{P}^1 \), \( \lambda_0 = \lambda_\infty \) and we conclude that

\[
\sum_{f(c)=0} e_c \lambda_c(a) = \sum_{f(c)=\infty} e_c \lambda_c(a)
\]

Thus we see that \( \sum_{f(c)=0} e_c \lambda_c(a) - \sum_{f(c)=\infty} e_c \lambda_c(a) \) is exactly the specialization of \( \text{div}(f) \), so \( \lambda \circ \text{div} = 0 \).

\[\square\]

**Theorem 4.2** (Rigidity). If \( c_0 \) and \( c_1 \) are two closed points of \( C \), then \( \lambda_{c_0} = \lambda_{c_1} : K_\ast(F; \mathbb{Z}/m\mathbb{Z}) \to K_\ast(k; \mathbb{Z}/m\mathbb{Z}) \).

**Proof.** The degree map \( \text{deg} : \text{Pic}(C) \to \mathbb{Z} \) is a surjective map with kernel \( \text{Pic}^0(C) \). Both the degrees of \( [c_0] \) and \( [c_1] \) are one, so \( \lambda_{c_0} - \lambda_{c_1} \) is obtained by applying \( \lambda \) to an element of \( \text{Pic}^0(C) \). Consider the restriction of \( \lambda \) to \( \text{Pic}^0(C) \). There is a corresponding scheme that is an abelian variety, i.e. a group object in varieties, and over an algebraically closed field its group of closed fields is divisible. Both \( K_\ast(k; \mathbb{Z}/m\mathbb{Z}) \) and \( K_\ast(F; \mathbb{Z}/m\mathbb{Z}) \) are annihilated by \( m^2 \), so that every element \( \text{Pic}^0(C) \) has to be mapped to zero by \( \lambda_{c_0} - \lambda_{c_1} \). This completes the proof. \[\square\]

In other words, we have proved that the family of maps \( \lambda_c \) over \( C \) is constant.

5. The algebraic K-theory of algebraically closed fields

We can prove the main theorem.
Theorem 5.1. Let $k \subset F$ be an inclusion of algebraically closed fields, then the inclusion $i^* : K_n(k; \mathbb{Z}/m\mathbb{Z}) \to K_n(F; \mathbb{Z}/m\mathbb{Z})$ is an isomorphism.

Proof. We will break it up into two parts:

(i) injectivity: This only needs that $k$ is algebraically closed. Write $F$ has the union of its finitely generated subfields $F_n$, so that (e.g. using the $+$ construction and that homotopy groups commute with colimits) we have that $K_n(F; \mathbb{Z}/m\mathbb{Z}) = \text{colim} K_n(F_n; \mathbb{Z}/m\mathbb{Z})$ and we can assume that $F$ is finitely generated. We do an induction over the transcendence degree of $F$, which is finite. Picking a transcendence basis and writing $F = k(x_1, \ldots, x_n)$ we see that $F$ is the fraction field of a discrete valuation ring $R$ with residue field $E = k(x_1, \ldots, x_{n-1})$ of transcendence degree one lower.

Look at the following diagram

$$
\begin{align*}
K_0(R; \mathbb{Z}/m\mathbb{Z}) &\longrightarrow K_0(F; \mathbb{Z}/m\mathbb{Z}) \\
\downarrow &\downarrow \lambda \\
K_0(k; \mathbb{Z}/m\mathbb{Z}) &\longrightarrow K_0(E; \mathbb{Z}/m\mathbb{Z})
\end{align*}
$$

We saw that $K_0(R; \mathbb{Z}/m\mathbb{Z}) \to K_0(F; \mathbb{Z}/m\mathbb{Z})$ followed by the specialization map for $R$ is the natural map $K_0(R; \mathbb{Z}/m\mathbb{Z}) \to K_0(E; \mathbb{Z}/m\mathbb{Z})$. Precomposing with $K_0(k; \mathbb{Z}/m\mathbb{Z}) \to K_0(R; \mathbb{Z}/m\mathbb{Z})$ we get that $K_0(k; \mathbb{Z}/m\mathbb{Z}) \to K_0(F; \mathbb{Z}/m\mathbb{Z})$ followed by the specialization map is $K_0(k; \mathbb{Z}/m\mathbb{Z}) \to K_0(E; \mathbb{Z}/m\mathbb{Z})$. By the inductive hypothesis this is injective, hence so is $K_0(k; \mathbb{Z}/m\mathbb{Z}) \to K_0(F; \mathbb{Z}/m\mathbb{Z})$.

(ii) surjectivity: Write $F$ as the union of its finitely generated sub-$k$-algebras $A$. Thus every element of $K_0(F; \mathbb{Z}/m\mathbb{Z})$ is the image of $K_0(A; \mathbb{Z}/m\mathbb{Z})$ for some $A$. We may assume that Spec $(A)$ is smooth, as the singular locus is closed and the map $A \to F$ factors over any localization $s^{-1} A$.

Let’s go back to our map $A \to F$, which we denote by $h_0$. There is a second map $h_1 : A \to A/m = k \to F$. These factors as $A \to A \otimes_k F$ composed with $h_1 : A \otimes_k F \to F$. Since smoothness is preserved by basechange along a field extension, Spec $(A \otimes_k F) = \text{Spec}(A) \times_{\text{Spec}(k)} \text{Spec}(F)$ is smooth. It thus suffices to prove that the maps $(h_i)_* : K_0(A; \mathbb{Z}/m\mathbb{Z}) \to K_0(F; \mathbb{Z}/m\mathbb{Z})$ are the same.

We now prove that if $h_0, h_1 : A \to F$ are two $k$-algebra homomorphisms, then the maps $K_0(A; \mathbb{Z}/m\mathbb{Z}) \to K_0(F; \mathbb{Z}/m\mathbb{Z})$ are the same. This corresponds to a pair of closed points on $A$. If $A$ is smooth, it is a theorem that there exists an affine smooth curve $C = \text{Spec}(R)$ with a map $C \to A$ and two closed points $c_0, c_1$ on $C$ such that $c_i$ is mapped to $h_i$. Let $E$ be the function field of $C$. By the rigidity theorem the specializations $K_0(E; \mathbb{Z}/m\mathbb{Z}) \to K_0(F; \mathbb{Z}/m\mathbb{Z})$ are the same. But the $(h_i)_* e$ each factor as the composite $K_0(A; \mathbb{Z}/m\mathbb{Z}) \to K_0(R; \mathbb{Z}/m\mathbb{Z}) \to K_0(E; \mathbb{Z}/m\mathbb{Z}) \to K_0(F; \mathbb{Z}/m\mathbb{Z})$ where only the last maps depends $c_i$. This proves the claim.

By our claim the maps $(h_i)_* : K_0(A; \mathbb{Z}/m\mathbb{Z}) \to K_0(F; \mathbb{Z}/m\mathbb{Z})$ are the same. But $h_i^*$ factors as $K_0(A; \mathbb{Z}/m\mathbb{Z}) \to K_0(k; \mathbb{Z}/m\mathbb{Z}) \to K_0(F; \mathbb{Z}/m\mathbb{Z})$. This shows that the image of $K_0(A; \mathbb{Z}/m\mathbb{Z})$ in $K_0(F; \mathbb{Z}/m\mathbb{Z})$ lies in the image of $K_0(k; \mathbb{Z}/m\mathbb{Z})$.

□

Let’s discuss what is known about the $K$-theory of algebraically closed of different characteristics.

Characteristic $p$: Any algebraically closed field of characteristic $p$ non-canonically contains $\bar{\mathbb{F}}_p$ and we know its $K$-theory from Quillen’s computations of $K_n(\bar{\mathbb{F}}_p \bar{\mathbb{F}})$.

Using the compatibility of algebraic $K$-theory with colimits this implies that

$$
K_n(\bar{\mathbb{F}}_p) = \begin{cases} 
\mathbb{Z} & \text{for } n = 0 \\
0 & \text{for } n > 0 \text{ even} \\
\mathbb{Q}/\mathbb{Z}[1/p] & \text{for } n \text{ odd}
\end{cases}
$$
This implies that

\[ K_n(\bar{\mathbb{F}}_p, \mathbb{Z}/p\mathbb{Z}) = \begin{cases} \mathbb{Z}/p\mathbb{Z} & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases} \]

For \( m \) coprime to \( p \), we have

\[ K_n(\bar{\mathbb{F}}_p; \mathbb{Z}/m\mathbb{Z}) = \begin{cases} 0 & \text{for } n \text{ odd} \\ \mathbb{Z}/m\mathbb{Z} & \text{for } n \text{ even} \end{cases} \]

Hence for any algebraically closed field \( F \) of characteristic \( p \) we have \( K_n(F; \mathbb{Z}/p\mathbb{Z}) = 0 \) for \( n > 0 \) and \( K_n(F; \mathbb{Z}/m\mathbb{Z}) = 0 \) if \( n \) is odd and \( \mathbb{Z}/m\mathbb{Z} \) for \( n \) even.

**Characteristic zero:** Let \( F \) be an algebraically closed field of characteristic zero. Similarly we can use \( \bar{\mathbb{Q}} \) to compute that for every \( m \) we have that \( K_n(F; \mathbb{Z}/m\mathbb{Z}) = 0 \) if \( n \) is odd and \( \mathbb{Z}/m\mathbb{Z} \) for \( n \) even.

**References**
