Homological Mirror Symmetry for the 2-Torus
 Mostly non-serious notes by Sarah McConnell

References:

We prove that the 2-torus $E_\tau$ is mirror dual to itself. We will build an explicit correspondence between holomorphic vector bundles over $E_\tau$ (the B-side) and Lagrangians equipped with flat vector bundles (the A-side). We then extend this correspondence to an equivalence between the bounded derived category of coherent sheaves on $E_\tau$ and (an enlargement of) the Fukaya category of $E_\tau$.

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Recall: $E_\tau = \mathbb{C}/(1, \tau)$, where $\text{Im}(\tau) > 0$. In general, $E_\tau$ will denote the algebro-geometric object, and $E_\tau^r$ (the same space) will denote the symplectic object. If $q = e^{2\pi i \tau}$, then

$$E_\tau = E_q = \mathbb{C}^*/\mathbb{Z},$$

where $1 \in \mathbb{Z}$ acts on $\mathbb{C}^*$ as multiplication by $q$.

We define the $\theta$-functions $\mathbb{C} \to \mathbb{C}$ to be

$$\theta[a, z_0](\tau, z) = \sum_{m \in \mathbb{Z}} e^{\pi i [(m+a)^2 \tau + 2(m+a)(z+z_0)]},$$

and $\theta_\tau(z) = \theta[0, 0](\tau, z)$. We can use these to classify all meromorphic functions on $E_\tau$.

Let $t_x \in \text{Aut}(E_\tau)$ be translation by $x$. If we fix a degree 1 line bundle $\mathcal{L}$ on $E_\tau$, then degree $d$ bundles take the form $t_x^* \mathcal{L} \otimes \mathcal{L}^{d-1}$. For $\varphi : \mathbb{C}^* \to \mathbb{C}^*$ holomorphic, define

$$\mathcal{L}_q(\varphi) = \mathbb{C}^* \times \mathbb{C}/(u, v) \sim (uq, \varphi(u)v).$$

**Theorem 1.** There is a functor $\Phi_\tau$ from $\mathcal{D}^b(E_\tau) = \mathcal{D}^b\text{Coh}(E_\tau)$ to $\mathcal{F}\mathcal{K}^0(E_\tau) = H^0(\text{Fuk}(E_\tau))$ which is an equivalence of additive categories; $\Phi_\tau$ is compatible with the shift functor.

Here are the steps of the construction:
1 Derived Category of Coherent Sheaves

A coherent sheaf is a generalization of a vector bundle. The general definition is annoying, but we don’t need it in this case because every indecomposable coherent sheaf over a Riemann surface is either an indecomposable vector bundle or an indecomposable torsion sheaf supported at a point.

The derived category of coherent sheaves $D^bCoh$ has as objects bounded chain complexes of coherent sheaves, and morphisms $X^\bullet \to Y^\bullet$ are triples

\[
\begin{array}{ccc}
Z^\bullet & \xrightarrow{f} & Y^\bullet \\
\downarrow{s} & & \\
X^\bullet & & \\
\end{array}
\]

where $s$ is a quasi-isomorphism.

**Theorem 2.** If $C$ is a compact Riemann surface, there is a full subcategory $\mathcal{S}$ of $D^bCoh(C)$ whose objects are finite sums

\[
\bigoplus_{j=1}^{N} \mathcal{F}_j[n_j],
\]

where $\mathcal{F}_j$ is an indecomposable coherent sheaf and $\mathcal{F}[n]$ is the chain complex which has $\mathcal{F}$ in degree $n$ and zero everywhere else. Moreover, this subcategory $\mathcal{S}$ is equivalent to $D^bCoh(C)$.

(I think this theorem basically says that every coherent sheaf is a finite direct sum of indecomposables, uniquely.)
## 2 Fukaya Category

The Fukaya category $\text{Fuk}(E^\tau)$ has as objects $(L, \alpha, M)$, where

- $L \simeq S^1$ is a (special) Lagrangian submanifold which forms a closed, oriented geodesic,
- $\alpha$ is the slope, so $z(t) = z_0^t e^{\pi i \alpha}$ is a lift of $L$ to $\mathbb{C}$, and
- $M$ is a monodromy operator (vector bundles over $S^1$ with flat connections are in one-to-one correspondence with conjugacy classes of representations $\pi_1(S^1) \to GL_k(\mathbb{C})$).

We define a shift functor by $(L, \alpha, M)[1] = (L, \alpha + 1, M)$. For $p \in L$, let $M_p$ be the fiber of the vector bundle over $p$. If $L = L'$, we define $\text{Hom}(\mathcal{L}, \mathcal{L}') = \text{Hom}(M, M')$ to be the space of vector bundle morphisms, mod isomorphisms of $L$. Otherwise, we define

$$
\text{Hom}(\mathcal{L}, \mathcal{L}') = \bigoplus_{p \in L \cap L'} \text{Hom}(M_p, M'_p).
$$

Technically, composition is not associative, so this is only an $A_\infty$ category, not a true category. We won’t worry about this.

We want to take an appropriate enlargement of our Fukaya category in order to give it the structure of an additive category (i.e., we need biproducts, which we’ll think of as finite direct sums). First, let $\text{Fuk}^0(E^\tau) = H^0(\text{Fuk}(E^\tau))$ be the category whose objects are the same as $\text{Fuk}(E^\tau)$ but with different morphisms. If $L \neq L'$,

$$
\text{Hom}_{\text{Fuk}^0(E^\tau)}(\mathcal{L}, \mathcal{L}') = \begin{cases} 
\bigoplus_{p \in L \cap L'} \text{Hom}(M_p, M'_p), & \alpha' - \alpha \in [0, 1) \\
0, & \text{otherwise}.
\end{cases}
$$

If $L = L'$, then

$$
\text{Hom}_{\text{Fuk}^0(E^\tau)}(\mathcal{L}, \mathcal{L}') = \begin{cases} 
\text{Hom}(M, M'), & \alpha' = \alpha \\
H^1(L, \text{Hom}(M, M')), & \alpha' = \alpha + 1 \\
0, & \text{otherwise}.
\end{cases}
$$

Then in $\text{Fuk}^0$ we get $\text{Hom}(\mathcal{L}_1, \mathcal{L}_2[1]) \cong \text{Hom}(\mathcal{L}_2, \mathcal{L}_1)^*$. To turn $\text{Fuk}^0$ into an additive category, we artificially add direct sums. We define $\mathcal{F}K_0(E^\tau) = (\text{Fuk}^0(E^\tau))$ to be the category whose objects are $k$-tuples of objects in $\text{Fuk}^0(E^\tau)$ ($k \geq 0$), and morphisms are matrices of $\text{Fuk}^0$ morphisms.

## 3 Coherent Sheaves on Elliptic Curves

By Atiyah, we can classify coherent sheaves on $E^\tau$.

For $A \in GL(V)$, we define

$$
F_q(V, A) = \mathbb{C}^* \times V/(u, v) \sim (u, A v).
$$
Every indecomposable vector bundle on $E$ is of the form
\[ \pi_r(\mathcal{L}_q(\varphi) \otimes F_q(V, \exp(N))) \]
where $\varphi : \mathbb{C}^* \to \mathbb{C}^*$ is holomorphic, $N$ is nilpotent with one-dimensional kernel, and $\pi_r : E_q \to E_q$ is the $r$-fold covering.

On $E$, a torsion sheaf $F$ supported at a point $\zeta_0$ is just a finite dimensional vector space $V = F_{\zeta_0}$, along with a nilpotent $N \in \text{End}(V)$. More specifically,

\[ S_r(\zeta_0, V, N) = \mathcal{O}_{\tau_{\zeta_0}} \otimes V/\left\langle \zeta - \zeta_0 - \frac{1}{2\pi i}N \right\rangle, \]

where

\[ \mathcal{O} \xrightarrow{(\zeta - \zeta_0)^r} \mathcal{O} \xrightarrow{r_{\zeta_0}} \]

is exact. Indecomposability is equivalent to $\dim \ker(N) = 1$.

4 Functorial Covering Maps

We define $\pi_r : E_{r\tau} \to E_\tau$ to be the $r$-fold cover. Its mirror is $p_r : E^{r\tau} \to E^\tau$. Note that for an object $(L, \frac{n}{m}, M)$ with $\gcd(n, m) = 1$, $p_r|_L$ has degree $d = \gcd(n, r)$.

These maps will commute with $\Phi_\tau$. That is, $\Phi_\tau(\pi_r^* A) = p_r^* \Phi_{r\tau}(A)$ for $A$ an indecomposable vector bundle.

We define $p_{rs}$ on $\mathcal{FK}^0(E^\tau)$ by $p_{rs}(L, \alpha, M) = (p_r(L), \alpha', p_{rs}M)$, where $p_{rs}M(v_1, \ldots, v_d) = (v_2, \ldots, v_d, M\cdot v_1)$ and $\alpha'$ is picked to be the correct slope so that $\alpha$ and $\alpha'$ both lie in some interval $[k - \frac{1}{2}, k + \frac{1}{2})$ for $k \in \mathbb{Z}$.

We also get a map on morphisms $\text{Hom}(\mathcal{L}, \mathcal{L}') \to \text{Hom}(p_{rs}\mathcal{L}, p_{rs}\mathcal{L}')$ which is annoying to write down but not too tricky. This map comes with some gross-looking diagrams.

By our classification of indecomposable vector bundles, we can now reduce to the simple vector bundles described below.

5 Our Functor on Simple Objects

Take some rank $k$ vector bundle $A = \mathcal{L}(\varphi) \otimes F(V, \exp(N))$, where $\varphi = (t_{a\tau + b}\varphi_0) \cdot \varphi_0^{n-1}$. We define $\Phi_\tau(A) = (L, \alpha, M)$, where

- $L$ has a lift parametrized by $t \mapsto (a + t, (n - 1)a + nt)$ (slope $n, y$-intercept $a$),
- $\alpha \in [-\frac{1}{2}, \frac{1}{2})$ has $e^{i\pi\alpha} = (n + im)/\sqrt{1 + n^2}$, and
- $M = e^{-2\pi ib}\exp(N)$.

We can use our functorial covering maps to extend $\Phi_\tau$ to all indecomposable vector bundles. Since we only need to define $\Phi_\tau$ on indecomposable coherent sheaves, all that remains is to define it on torsion sheaves supported at a point. If $S = S(-a\tau - b, V, N)$, let $\Phi_\tau(S) = (L, \frac{1}{2}, e^{-2\pi ib}\exp(N))$, where $L$ is the vertical Lagrangian with $x$-intercept $a$. 

4
6 Morphism Crap

There are two versions of Serre duality. On the A-side, we have the standard version: if $V$ is a vector bundle over $E$, then

$$H^q(E, V) \cong H^{1-q}(E, K \otimes V^*)^*,$$

where $K$ is the canonical bundle. On the B-side, if $F$ and $G$ are coherent sheaves on $E$, then

$$\text{Hom}(F, G) \cong \text{Ext}^1(G, F)^*.$$

For $A$ and $B$ coherent sheaves,

$$\text{Hom}_D(A[m], B[n]) = \text{Hom}_D(A, B[n-m]) = \text{Ext}^{n-m}(A, B).$$

(This is probably easier to see with the alternate definition of the derived category, where we take complexes of injectives.) In particular, this value is zero unless $n - m \in \{0, 1\}$, so we just need to define

$$\varphi : \text{Hom}_D(A_1, A_2) \to \text{Hom}_{FK}(\Phi(\varphi(A_1), \varphi(A_2))$$

$$\varphi : \text{Hom}_D(A_1, A_2[1]) \to \text{Hom}_{FK}(\Phi(\varphi(A_1), \varphi(A_2)[1]).$$

But by duality we have

$$\text{Hom}_D(A_1, A_2[1]) = \text{Ext}^1(A, B) = \text{Hom}(B, A)^*$$

$$\text{Hom}_{FK}(\Phi(\varphi(A_1), \varphi(A_2)[1]) = \text{Hom}_{FK}(\Phi(\varphi(A_2), \varphi(A_1))^*,$$

so it’s actually sufficient to define it on $\text{Hom}_D(A_1, A_2)$. Then we throw commutative diagrams at the problem to show that the definition on simple objects extends to all indecomposable coherent sheaves and hence to the subcategory $\mathcal{S}$.

First, we consider the case where $A_1$ and $A_2$ are both vector bundles. We’ll take $A_i = L(\varphi_i) \otimes F(V_i, \exp(N_i))$ (Big commutative diagrams extend it to all indecomposable vector bundles). Then

$$\text{Hom}(A_1, A_2) = H^0(A_2 \otimes A_1^*)$$

$$= H^0(L(\varphi_1^{-1} \varphi_2) \otimes F(V_1^* \otimes V_2, \exp(1 \otimes N_2 - N_1^* \otimes 1)))$$

$$\cong H^0(L(\varphi_1^{-1} \varphi_2) \otimes \text{Hom}(V_1, V_2),$$

where the last isomorphism is canonical (Lemma 7.7 in Port). Now we can take a basis of elements of the form $\theta \otimes f$ and write down $\Phi$ explicitly; it’s pretty nasty (p34 of Port).

Now take two torsion sheaves $S_i = S(-a_i \tau - b_i, V_i, N_i)$. Then $\text{Hom}(S_1, S_2)$ is only non-trivial if $a_1 \tau + b_1 = a_2 \tau + b_2$ (these are sheaves supported at a point). On the other side, $\Phi(S_i) = (L_i, \frac{1}{2}, M_i)$, where $L_i$ is the vertical Lagrangian with $x$-intercept $a_i$. If
$Hom(\Phi_\tau(S_1), \Phi_\tau(S_2)) \neq 0$ then $L_1 \cap L_2 \neq \emptyset$, so $a_1 = a_2$. By linear algebra, it also follows that $b_1 = b_2$. Then we have

$$Hom(S_1, S_2) = \{ f \in Hom(V_1, V_2) \mid f \circ N_1 = N_2 \circ f \}$$

$$= \{ f \in Hom(V_1, V_2) \mid f \circ M_1 = M_2 \circ f \}$$

$$= Hom(\Phi_\tau(S_1), \Phi_\tau(S_2)).$$

Finally, consider the case of a vector bundle $A = L(\varphi) \otimes F(V, \exp(N))$, $\varphi = t^*_\alpha + \beta \cdot \varphi_0 \cdot \varphi^{n-1}$, and a torsion sheaf $S = S(-a\tau - b, V', N')$. We can write $\Phi_\tau(S) = (L_S, \frac{1}{2}, M_S)$ and $\Phi_\tau(A) = (L_A, \alpha_A, M_A)$. Then $Hom(S, A) = 0$ (I think because $S$ is torsion). On the other side, $Hom(\Phi_\tau(S), \Phi_\tau(A)) = 0$ because $\alpha_A < \frac{1}{2}$ (by our construction of $\Phi_\tau$, and because we dropped a bunch of morphisms from the Fukaya category).

Lastly, we consider $Hom(A, S) = Hom(V, V') = V^* \otimes V'$ (since $S$ is supported at a point). Since the Lagrangians in $\Phi_\tau(A)$ and $\Phi_\tau(S)$ intersect in exactly one point, we have $Hom(\Phi_\tau(A), \Phi_\tau(S)) = Hom(V, V') = V^* \otimes V'$. Then we can write down

$$\Phi_\tau = e^{-\pi i (na^2 - 2a\alpha) + 2\pi i (a\beta + b\alpha - nab)} \cdot \exp[(na - \alpha)1_{V'} \otimes N' + \alpha N \ast 1_V].$$