Appendix

A Proof of Theorem 1

We first introduce some additional notation used throughout the proof. We denote the individual's asset path by $a_t = w + \int_0^t (s_\tau - c_\tau) d\tau$, which reduces the no-borrowing constraint to $a_t \geq 0$ for all t. We use $\mu(\cdot)$ to denote the Lebesgue measure on the real line, and $P_t = \int_t^\infty p_\tau d\tau$ to represent the integral of the survival probability path starting at t.

We begin with two lemmas that allow us to restrict our search to the space of deferred annuities, and then show that there exists a unique optimal deferred annuity with the stated properties.

Our first lemma establishes that there are two (almost everywhere) mutually exclusive and exhaustive phases: one where the individual consumes from strictly positive savings, and one where the government pays out a positive annuity.

Lemma 1. *It is without loss of optimality to focus on transfer schedules* $\{s_t\}$ *such that, except on a set of measure zero,* $a_t > 0$ *if and only if* $s_t = 0$.

Proof. Fix some $\{s_t\}$. The "only if" direction amounts to showing that if $s_t > 0$ on a set of positive measure, then the individual does not save almost everywhere along that set, i.e. $M = \{t : s_t > 0, a_t > 0\}$ has measure zero. Suppose towards a contradiction that $\mu(M) > 0$. We aim to construct a profitable deviation from $\{s_t\}$ that shows that this transfer schedule cannot be optimal.

Note that if M has strictly positive measure, there are values $\underline{s}$, $\underline{a}$, and a set $\tilde{M} = \{t : s_t > \underline{s}, a_t > \underline{a}\}$ such that $\tilde{M}$ also has strictly positive measure.[8] By definition of the Lebesgue measure, there exists at least one point $\tilde{t} \in \tilde{M}$ and $r > 0$ such that $\mu(B_r(\tilde{t}) \cap \tilde{M}) > 0$, where $B_r(\cdot)$ denotes the open ball of radius r. Note too that we can choose r sufficiently small so that for all t in $B_r(\tilde{t}), a_t > \underline{a}$.[9]

Let $A = (\tilde{t} - r, \tilde{t})$ and $B = [\tilde{t}, \tilde{t} + r)$. Note that it is without loss that both $A \cap \tilde{M}$ and $B \cap \tilde{M}$ have strictly positive measure.[10] Denote by $\mu_P(\cdot)$ the measure $\int_{s \in S} p(s) d\mu(s)$ induced by the survival function $p(\cdot)$. Note that $\mu_P(B \cap \tilde{M}) < p(\tilde{t})\mu(B \cap \tilde{M})$ and $\mu_P(A \cap \tilde{M}) > p(\tilde{t})\mu(A \cap \tilde{M})$ by strict decreasingness of $p(\cdot)$, so that $\frac{\mu(B \cap \tilde{M})}{\mu_P(B \cap \tilde{M})} > \frac{\mu(A \cap \tilde{M})}{\mu_P(A \cap \tilde{M})}$.

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[8] This follows because $\{s_t > 0\} = \bigcup_{q \in \mathbb{Q} \setminus 0} \{s_t > q\}$ (where $\mathbb{Q}$ denotes the rationals), and the same decomposition applies to the set $\{a_t > 0\}$), so that countable subadditivity implies that the measure of at least one of the sets in the union is strictly positive.

[9] To see this, note first that since s_t, c_t are measurable and locally integrable, the asset path a_t is almost everywhere differentiable by the fundamental theorem of calculus for Lebesgue integrals, and a fortiori continuous, so it maps open sets to open sets. Therefore, the set $\{t : a_t > 0\}$ is open in the real line, so it can be uniquely expressed as the countable union of disjoint open intervals. Since $\tilde{t} \in \tilde{M}$, $a(\tilde{t}) > \underline{a}$, so there exists a sufficiently small open interval around this point satisfying the desired property.

[10] This follows from choosing $\tilde{t}$ to be a density point of $\tilde{M}$, i.e., a point such that $\lim_{r \to 0} \frac{\mu(\tilde{M} \cap \overline{B}_r(\tilde{t}))}{\mu(\overline{B}_r(\tilde{t}))} = 1$ where $\overline{B}_r(\tilde{t})$ is the *closed* ball of radius r around $\tilde{t}$. Lebesgue's density theorem states that almost every point in $\tilde{M}$ is a density point, so that if $\tilde{M}$ has strictly positive measure at least one such point exists, and such a point must have an intersection of positive measure with both $[\tilde{t} - r, t]$ and $[\tilde{t}, \tilde{t} + r]$ for r sufficiently small, since otherwise $\frac{\mu(\tilde{M} \cap \overline{B}_r(\tilde{t}))}{\mu(\overline{B}_r(\tilde{t}))}$ would be bounded above by $1/2$. Since the measures of closed and open intervals with the same endpoints are the identical, this delivers the desired statement.

Now consider the following alternative paths of transfer and consumption (and therefore assets):

- For $t \leq \tilde{t} - r$ and $t > \tilde{t} + r$, as well as $(A \cup B) \setminus \tilde{M}$, leave consumption and transfers unchanged.
- Over $A \cap \tilde{M}$, reduce the transfer from y to $y - \frac{\epsilon}{\mu_P(A \cap \tilde{M})}$ and leave consumption unchanged.
- Over $B \cap \tilde{M}$, change the transfer from y to $y + \frac{\epsilon}{\mu_P(B \cap \tilde{M})}$ and change consumption from c to $c + \frac{\theta\epsilon}{\mu_P(B \cap \tilde{M})}$ where $\theta = 1 - \frac{\mu(A \cap \tilde{M})\mu_P(B \cap \tilde{M})}{\mu(B \cap \tilde{M})\mu_P(A \cap \tilde{M})}$.

If such a deviation is feasible, it strictly increases utility, since consumption is strictly higher on a set of positive measure and unchanged elsewhere.[11] We now show it is feasible:

- Government budget constraint: by construction, the augmented transfer schedule costs the same discounted amount as the original one, so if the original path is feasible, so is the augmented path.
- Transfer schedule non-negativity: the only potentially problematic region is over $A \cap \tilde{M}$, where transfers are reduced. Since $s(\cdot) > \underline{s} > 0$ in $\tilde{M}$, non-negativity can be maintained by choosing ϵ sufficiently small.
- Asset path non-negativity and consistency with consumption path:

  For $t < \tilde{t} - r$, the asset and consumption paths are unchanged.

  For $t \in B_r(\tilde{t})$, by choosing ϵ sufficiently small, the no-borrowing constraint continues to hold.[12]

  Lastly, at $t = \tilde{t} + r$ the augmented asset paths $a^{aug}(\cdot)$ and the original asset path $a(\cdot)$ have the same value, so that it is feasible to keep the asset path unchanged thereafter.[13]

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[11] This follows since $\theta \in (0, 1)$, given that $\frac{\mu(B \cap \tilde{M})}{\mu_P(B \cap \tilde{M})} > \frac{\mu(A \cap \tilde{M})}{\mu_P(A \cap \tilde{M})}$

[12] In particular, letting c_s be the original consumption path, the consumer's no-borrowing constraint (which corresponds to the asset path being nonnegative) is

$$\int_0^t \left[c_\tau + \frac{\theta\epsilon\mathbf{1}(\tau \in B \cap \tilde{M})}{\mu_P(B \cap \tilde{M})} \right] d\tau \leq w + \int_0^t \left[s_\tau - \frac{\epsilon\mathbf{1}(\tau \in A \cap \tilde{M})}{\mu_P(A \cap \tilde{M})} + \frac{\epsilon\mathbf{1}(\tau \in B \cap \tilde{M})}{\mu_P(B \cap \tilde{M})} \right] d\tau$$

since, by feasibility of the original asset path, $a_t = w + \int_0^t [s_\tau - c_\tau] d\tau \geq \underline{a}$ for all $t \in B_r(\tilde{t})$, this is equivalent to

$$\epsilon \left(\int_0^t \left[\frac{\mathbf{1}(\tau \in A \cap \tilde{M})}{\mu_P(A \cap \tilde{M})} - \frac{\mathbf{1}(\tau \in B \cap \tilde{M})(1 - \theta)}{\mu_P(B \cap \tilde{M})} \right] d\tau \right) < \underline{a}$$

the expression inside parentheses is a continuous function of t by Lebesgue's differentiation theorem, so it obtains a maximum over the compact set $[\tilde{t} - r, \tilde{t} + r]$, implying that the inequality above can be made to hold uniformly over $B_r(\tilde{t})$ by choosing ϵ sufficiently small.

[13] This follows from noting that at $\tilde{t} + r$, we have

$$a^{aug}_{t^B} = a_{\tilde{t}+r} - \epsilon \left(\frac{\mu(A \cap \tilde{M})}{\mu_P(A \cap \tilde{M})} - \frac{\mu(B \cap \tilde{M})(1 - \theta)}{\mu_P(B \cap \tilde{M})} ds \right)$$

The second part of the lemma states that if $s_t = 0$ on a set of positive measure, then $a_t > 0$ almost everywhere along that set. Supposing towards a contradiction that $a_t = 0$ on a subset of $\{t : s_t = 0\}$ that has positive measure, note that this implies the individual consumes $c_t = 0$ on a subset of positive measure. This contradicts the Inada condition: since the consumer starts with positive assets, a reallocation of consumption from an earlier time where $c_t > 0, a_t > 0$ would yield strictly higher utility. □

Our first lemma showed that it is without loss of optimality to consider policies with two distinct phases: positive savings and no government transfer, and positive government transfers and no savings. Our second lemma shows that optimally, the switch between these two phases occurs exactly once, starting with the positive savings region and ending with the positive transfer regime, that the transfers fully exhaust the government's budget constraint; and that within the positive transfer phase, the transfer is constant.

Lemma 2. *It is without loss of optimality to consider deferred annuities $\{s_t\}$ of the form*

$$s_t = \begin{cases} 0 & t < t^* \\ P_t^{-1} S & t \geq t^*. \end{cases}$$

Proof. Fix some $\{s_t\}$. First, note that at the optimum, $s_t > 0$ on a set of positive measure. Otherwise, offering any government transfer stream with strictly positive resources weakly loosens the budget constraint at some times t and strictly loosens it at others, implying a deviation towards a superior consumption path. The same logic rules out optimal transfer schedules that do not exhaust the government budget constraint (i.e. those with $\int_0^\infty p_t s_t < S$).

Having established that $s_t > 0$ on a set of positive measure, we claim that $\sup\{t : s_t > 0\} = \infty$. Suppose towards a contradiction that $t^{s\,max} := \sup\{t : s_t > 0\} < \infty$. Note that per Lemma 1, $a_t = 0$ almost everywhere when $s_t > 0$, so $a(t^{s\,max}) - \epsilon = 0$ for all sufficiently small $\epsilon > 0$, so that at all times $s \geq t^{s\,max}$ the consumer is forced to consume nothing. The Inada condition and the fact that under any government transfer a consumption path that never hits zero is feasible implies that zero consumption on a set of positive measure is never optimal. Furthermore, Lemma 1 also implies that the set $\{t : s_t > 0\}$ must contain a single interval. To see this, suppose toward contradiction that this is not the case. Consider t' such that $s_{t'} = 0$ but there exists $t_1 < t'$ with $s_{t_1} > 0$. Let $t^* = \sup\{t : s_t > 0, t < t'\}$. Lemma 1 implies that $a_{t^*} = 0$ but then becomes positive immediately after t^*, which is a contradiction because $a_{t^*} = 0$ means that $a_\tau = 0$ for any τ in the neighborhood of t^*. Therefore, if the set $\{t : s_t > 0\}$ has positive measure, it is almost everywhere an interval of the form $[a, \infty)$.

To summarize, an optimal transfer schedule satisfies $\int_0^\infty p_t s_t = S$ and pays out over some interval $[t^*, \infty)$ in which $a_t = 0$ so that $c_t = s_t$. The last remaining claim is that it is without loss of optimality to consider deferred annuities, i.e. constant positive payment streams. By Jensen's inequality with respect to the probability measure $\mu_{p_t} = \frac{p_t}{\int_{t^*}^\infty p_\tau d\tau}$, for any possibly

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so that setting $\theta = 1 - \frac{\mu(A \cap \tilde{M})\mu_P(B \cap \tilde{M})}{\mu(B \cap \tilde{M})\mu_P(A \cap \tilde{M})}$ yields $a^\epsilon_{t^B} = a_{t^B}$ as desired.

optimal transfer schedule, individuals' utility over this region satisfies

$$\int_{t^*}^{\infty} p_t u\left(s_t\right) dt \leq \left(\int_{t^*}^{\infty} p_\tau \, d\tau\right) u\left(\frac{\int_{t^*}^{\infty} p_t s_t dt}{\int_{t^*}^{\infty} p_\tau \, d\tau}\right) = P_{t^*} u\left(P_{t^*}^{-1} S\right),$$

with strict inequality unless s_t is constant over $[t^*, \infty)$. The deferred annuity $s_t = I\{t \geq t^*\} P_{t^*}^{-1} S$ that pays out over $[t^*, \infty)$ delivers the utility on the RHS above and is feasible, so it is optimal in the sense that other transfer schemes can do no better. □

From Lemma 1 and Lemma 2, we now know that any optimal transfer schedule essentially takes the form of a deferred annuity, and that the individual always has positive assets before the deferred annuity begins and always has zero assets thereafter. The set of possibly optimal transfer schedules are then the deferred annuities indexed by starting date t^*, and to find which is optimal we analyze the individual's indirect utilities when they consume $\overline{s}(t^*)$ (the fixed annuity) over $[t^*, \infty)$, and fully exhaust their budget constraint at time t^* while optimally smoothing consumption.

We can therefore write the indirect utility among all such transfer schedules as $V(t^*) = U(t^*) + W(t^*)$, where

$$U(t^*) = \sup_{c(\cdot) \geq 0, \int_0^{t^*} c_t dt = w} \int_0^{t^*} p_t u(c_t) dt$$

and $W(t^*) = P_{t^*} u(P_{t^*}^{-1} S)$. Proving that there exists a unique optimal transfer schedule then reduces to showing that $V(t^*)$ obtains a unique maximum on $[0, \infty)$. To do this, we show that $V'(t^*)$ crosses zero exactly once and from above at some point t^{**}, implying that $V(t^*)$ obtains its unique global maximum at that point. We now differentiate $V(t^*)$.

Lemma 3. *The derivative of the value function is*

$$V'(t^*) = U'(t^*) + W'(t^*) = p_{t^*}[g(c_{t^*}(t^*)) - g(\overline{s}(t^*))]$$

where $c_{t^*}(t^*)$ denotes the period-t^* consumption rate when the individual optimally smooths up to period t^* and exhausts their budget constraint at t^*, $g(x) := u(x) - xu'(x)$, and $\overline{s}(t^*) := P_{t^*}^{-1} S$.

Proof. We consider $U'(t^*)$ first. This is the derivative of indirect utility in the phase where the consumer uses their own savings, so the consumer faces a standard consumption-savings problem of the form

$$\max_{\{a_t, c_t\}} \int_0^{t^*} p_t u(c_t) dt,$$

subject to $a_0 = w$, $\dot{a}_t = -c_t$, and $\int_0^{t^*} c_t dt = w$.[14]

Because this is a standard consumption-savings problem, the solution exists, is unique,

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[14] The constraint $a_t > 0$ always holds at the optimum since the individual never optimally consumes 0 by the Inada condition.

and coincides with the maximum of the Lagrangian,[15] which is given by

$$\mathcal{L} = \int_0^{t^*} [p_t u(c_t) - \lambda_t c_t] dt + \lambda_0 w,$$

where $\lambda_t \geq 0$ is the costate on the law of motion for assets. Differentiating the maximized Lagrangian with respect to t^* yields, via Leibniz's rule, $U'(t^*) = p_{t^*} u(c_{t^*}(t^*)) - \lambda_{t^*} c_{t^*}(t^*)$, where $c_{t^*}(t^*)$ denotes the period-t^* consumption rate when the individual optimally smooths up to period t^* (and exhausts their budget constraint at t^*).

To characterize λ_t, note that by Pontryagin's Maximum Principle, the solution to this consumption-savings problem is also characterized by the Hamiltonian, $\mathcal{H} = pu(c) - \lambda c$. Optimality with respect to the control variable c then yields $pu'(c) = \lambda$, while optimality with respect to the state variable a yields $\dot{\lambda} = 0$. Therefore, at the optimum we have that $\lambda_t = \lambda$ is constant and $p_t u'(c_t) = \lambda$ for all t, so that $\lambda_t = p_t u'(c_t)$. Overall then, we have $U'(t^*) = p_{t^*}[u(c_{t^*}(t^*)) - c_{t^*}(t^*) u'(c_{t^*}(t^*))]$.

To differentiate the second term, $W(t^*)$, recall that $P_{t^*} = \int_{t^*}^{\infty} p_t dt$ so that $P'(t^*) = -p_{t^*}$. Moreover, write $\overline{s}(t^*) = P_{t^*}^{-1} S$ for the deferred annuity consistent with a start date of t^* and note that $\overline{s}'(t^*) = P_{t^*}^{-2} p_{t^*} S = P_{t^*}^{-1} p_{t^*} \overline{s}(t^*)$. Therefore, $W'(t^*) = p_{t^*}[u'(\overline{s}(t^*))\overline{s}(t^*) - u(\overline{s}(t^*))]$.

Combining both terms of $V'(t^*)$ yields the desired result. □

Note that $g'(x) = -xu''(x) > 0$ by strict concavity, so that $g(\cdot)$ is strictly increasing. Thus, the sign of the derivative depends only on the values of $c_{t^*}(t^*)$ and $\overline{s}(t^*)$. If $c_{t^*}(t^*) > \overline{s}(t^*)$, then $V'(t^*) > 0$; if $c_{t^*}(t^*) < \overline{s}(t^*)$, then $V'(t^*) < 0$, with a critical point $V'(t^*) = 0$ only if the two terms are the same.

The rest of the proof amounts to showing that there exists a unique t^{**} such that the implied optimal consumption there equals the payout from the deferred annuity consistent with a start date of t^{**}, and consumption only crosses this annuity once, so that t^{**} is the unique optimum:

Lemma 4. *There is a unique time t^{**} such that the implied optimal consumption at t^{**} is $c_{t^{**}}(t^{**}) = \overline{s}(t^{**})$. After this time, $c_{t^*}(t^{**}) < \overline{s}(t^{**})$, while before t^{**} the inequality is reversed.*

Proof. Note first that from the Hamiltonian of the consumer's optimal savings problem, $p_t u'(c_t)$, is constant for all $t \in [0, t^*]$. For a given level of terminal consumption c_{t^*}, then, there is a unique level of consumption $c_t(c_{t^*}, t^*) = (u')^{-1}\left(\frac{u'(c_{t^*})p_{t^*}}{p_t}\right)$ consistent with the individual's optimization. Therefore, $c_{t^*}(t^*)$ is the unique level of terminal consumption so that $\int_0^{t^*} c_t(c_{t^*}(t^*), t^*) dt = w$.[16] Note that since $u(\cdot)$ is strictly concave and p_t is strictly decreasing, $c_t(c_{t^*}, t^*) > c_s(c_{t^*}, t^*)$ whenever $t < s$, and $c_t(c_{t^*}, t^*)$ is increasing in c_{t^*} for all t.

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[15] See for instance Theorems 7.8.1 and 7.9.1 in Léonard and Long (1992).

[16] Such a level of c_{t^*} exists and is unique for any $w > 0$. To see this, note that by the Inada condition, as $c_{t^*} \to 0$, $u'^{-1}(\frac{u'(c_{t^*})p_{t^*}}{p_t}) \to 0$ pointwise and the induced consumptions paths are monotonically decreasing. The dominated convergence theorem thus implies that as $c_{t^*} \to 0$, $\int_0^{t^*} u'^{-1}(\frac{u'(c_{t^*})p_{t^*}}{p_t}) dt \to 0$. Meanwhile, as $c_{t^*} \to \infty$, since $c_t(c_{t^*}) > c_{t^*} \forall t \leq t^*, \lim_{c_{t^*} \to \infty} \int_0^{t^*} c_t dt \geq \lim_{c_{t^*} \to \infty} \int_0^{t^*} c_{t^*} dt = \infty$. Since $c_t(c_{t^*}, t^*)$ is continuous for any $c_{t^*} > 0$, $\int_0^{t^*} c_t(c_{t^*}, t^*) dt$ is continuous, so by the intermediate value theorem a $c_{t^{**}}$ exists so that $\int_0^{t^{**}} c_t(c_{t^{**}}, t^{**}) dt = w$. Since $c_t(c_{t^*}, t^*)$ is strictly increasing in its first argument, $c_{t^{**}}$ is unique.

To show the existence of a unique t^{**} satisfying the desired properties, we claim that

1. as $t^* \to 0$, $\overline{c}_{t^*}(t^*) > \overline{s}(t^*)$,
2. as $t^* \to \infty$, $\overline{c}_{t^*}(t^*) < \overline{s}(t^*)$,
3. $\overline{s}'(t^*) > 0$ for all t^*, and
4. $\frac{d}{dt^*}c_{t^*}(t^*) < 0$ for all t^*.

To prove claim (1), first note that as $t^* \to 0$, $\overline{s}(t^*) \to P_0^{-1}S < \infty$ by assumption. Meanwhile, since $w = \int_0^{t^*} c_t(c_{t^*}(t^*), t^*)dt < \int_0^{t^*} c_0(c_{t^*}(t^*), t^*)dt = t^* c_0(c_{t^*}(t^*), t^*)$, we have $c_0(c_{t^*}(t^*), t^*) \to \infty$ as $t^* \to 0$. Since $u'(c_0) = p_t u'(c_{t^*}(t^*))$ by the individual's Euler equation, this implies $c_{t^*}(t^*) \to \infty$.

Claim (2) follows because as $t^* \to \infty$, $\overline{s}(t^*) \to \infty$ since $\int_{t^*}^{\infty} p_t dt \to 0$ by the fact that p_t is integrable. Meanwhile, since $w = \int_0^{t^*} c_t(c_{t^*}(t^*), t^*)dt > \int_0^{t^*} c_{t^*}(t^*)dt = t^* c_{t^*}(t^*) \implies c_{t^*}(t^*) < \frac{w}{t^*}$, as $t^* \to \infty$, $c_{t^*}(t^*) \to 0$.

Claim (3) is immediate after recalling from the proof of Lemma 3 that $\overline{s}'(t^*) = P_{t^*}^{-1} p_{t^*} \overline{s}(t^*) > 0$.

Claim (4) follows from implicitly differentiating the identity $\int_0^{t^*} c_t(c_{t^*}(t^*), t^*)dt = w$ with respect to t^* using Leibniz' rule and the chain rule to yield

$$c_{t^*}(t^*) + c'_{t^*}(t^*) \int_0^{t^*} \frac{\partial c_t(c_{t^*}(t^*), t^*)}{\partial c_{t^*}} dt + \int_0^{t^*} \frac{\partial c_t(c_{t^*}(t^*), t^*)}{\partial t^*} dt = 0.$$

Rearranging, we obtain

$$c'_{t^*}(t^*) = \frac{\int_0^{t^*} \frac{\partial c_t(c_{t^*}(t^*), t^*)}{\partial t^*} dt + c_{t^*}}{-\int_0^{t^*} \frac{\partial c_t(c_{t^*}(t^*), t^*)}{\partial c_{t^*}} dt}.$$

Since each $c_t(\cdot, \cdot)$ is increasing in the first argument c_{t^*}, the denominator above is strictly negative, and $\frac{\partial c_t(c_{t^*}(t^*), t^*, t)}{\partial c_{t^*}} > 0$ in the numerator. The numerator is strictly positive since $c_{t^*}(t^*) > 0$.

Altogether, claims (1) through (4) imply that there exists a unique t^{**} such that $c_{t^{**}}(t^{**}) = \overline{s}(t^{**})$, that for $t^* < t^{**}$ we have $c_{t^*}(t^*) > \overline{s}(t^*)$ (so that $V'(t^*) > 0$), and that for $t > t^{**}$, $c_{t^*}(t^*) < \overline{s}(t^*)$ (so that $V'(t^*) < 0$). □

This implies that t^{**} is the unique local optimum and that neither the left endpoint ($t^* = 0$) nor the right endpoint ($t^* \to \infty$) can be optimal, so that t^{**} is the unique global optimum.

B Comparative statics

Theorem 2. *The date at which the optimal Social Security payment schedule first pays out, t^*, is*

1. *Strictly increasing in initial wealth w.*

2. *Strictly increasing in survival probabilities. That is, consider two survival functions* $\{p_t\}$ *and* $\{q_t\}$*. If* $-\frac{p_t'}{p_t} < -\frac{q_t'}{q_t}$ *for all* t*, then* t^* *is strictly higher under* $\{p_t\}$ *than under* $\{q_t\}$*.*
3. *Strictly increasing in absolute risk aversion. That is, consider two utility functions* $u(c)$ *and* $v(c)$*. If* $-\frac{u''(c)}{u'(c)} > -\frac{v''(c)}{v'(c)}$ *for all* c*, then* t^* *is strictly higher under* $u(\cdot)$ *than under* $v(\cdot)$*.*

Proof. We prove each comparative static result in turn.

Comparative static in wealth. Per the proof of Theorem 1, for a given wealth level w, $t^*(w)$ uniquely solves the equation $c_{t^*}(t^*, w) = P_{t^*}^{-1}S$ where $c_{t^*}(t^*, w)$[17] satisfies

$$\int_0^{t^*} c_t(c_{t^*}(t^*, w), t^*) = w. \tag{10}$$

If we can show that $c(t^*, w)$ is strictly increasing in w for fixed t^*, this establishes the comparative static result by the implicit function theorem, since $c_{t^*}(t^*)$ is strictly decreasing in t^* and $\overline{s}(t^*)$ is strictly increasing in t^*.

Differentiating both sides of eq. (10) with respect to w yields

$$\frac{\partial c_{t^*}(t^*, w)}{\partial w} = \frac{1}{\int_0^{t^*} \frac{\partial c_t(c_{t^*}(t^*,w),t^*)}{\partial c_{t^*}(t^*,w)} dt} > 0$$

since $c_t(c_{t^*}, t^*)$ is increasing in its first argument c_t^*.

Comparative static in survival probabilities. We undertake a similar strategy as for the comparative static in w, showing that $P_{t^*}^{-1}S$ strictly increases in $\{p_t\}$ in the hazard rate order while $c_{t^*}(t^*, \{p_t\})$ strictly decreases.

Recall that the hazard rate of a random variable with continuous differentiable pdf $f(x)$ and CDF $F(x)$ is given by $h(x) = \frac{f(x)}{1-F(x)}$, and that one random variable *hazard rate dominates* (denoted by the order $\succ_{\text{hrd}}$) another with the same support if it has everywhere strictly lower hazard rate.

Suppose we have stochastic survival processes $\{p_t\}$,$\{q_t\}$ and that $p_t \succ_{\text{hrd}} q_t$; that is, $-\frac{p_t'}{p_t} < -\frac{q_t'}{q_t}$ for all t.

Since hazard rate dominance implies first order stochastic dominance (see e.g., Section 1.B.2 in Shaked and Shanthikumar (1994)), we have $q_t < p_t$ for all t. For any fixed t^*, this pointwise bound implies that $\int_{t^*}^{\infty} p_t dt \geq \int_{t^*}^{\infty} q_t dt$. Therefore, $\overline{s}(t^*, q(\cdot)) > \overline{s}(t^*, p(\cdot))$

To see that $c_{t^*}(t^*, p(\cdot)) > c_{t^*}(t^*, q(\cdot))$, differentiate the individual's Euler condition $p_t u(c_t) = \lambda$ with respect to t and use the law of motion for the costate $\lambda_t' = 0$ to yield

$$c_{p(\cdot)}' = -\frac{p'(t)u'(c_{p(\cdot)})}{p_t u''(c_{p(\cdot)})}$$

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[17] Note that, unlike the function $c_t(c_t^*, t^*)$ introduced in the previous section, this is a function with t^* and wealth w as arguments, rather than t^* and consumption.

and the analogous expression for $c'_{q(\cdot)}$, where the subscripts denote the dependence of the optimal consumption path on the survival path.

These are two ordinary differential equations with the same terminal constraint, $\int_0^{t^*} c_t dt = w$. Note that, since c' must be decreasing for both survival paths (since $p'(t) < 0, q'(t) < 0$ and $u''(c) < 0$), the solution $c_{p(\cdot)}$ must intersect the solution $c_{q(\cdot)}$ at least once; otherwise, one of the two consumption paths must lie strictly above the other, which contradicts that they obey the same terminal constraint $\int_0^{t^*} c_t = w$. Therefore there is at least one point t^{**}, c^* with $t^{**} < t^*$ such that $c_{p(\cdot)}(t) = c_{q(\cdot)}(t)$. Since $-\frac{p'(t)}{p_t} > -\frac{q'(t)}{q_t}$ for all t (and thus t^{**}) and $u''(c^*) < 0$, at such a point $c'_{p(\cdot)} < c'_{q(\cdot)}$. It follows, since both consumption paths are strictly decreasing, that $c'_{p(\cdot)}$ intersects $c'_{q(\cdot)}$ exactly once and from below, so that after this unique intersection $c_{p(\cdot)} > c_{q(\cdot)}$ for all t, and thus at t^*. Therefore, $c_{t^*}(t^*, p(\cdot)) > c_{t^*}(t^*, q(\cdot))$.

Comparative static in absolute risk aversion. The proof is identical to the comparative static for $p(\cdot)$, except that now $\overline{s}(t^*)$ is unaffected. Supposing that $-\frac{u''(x)}{u'(x)} > -\frac{v''(x)}{v'(x)}$ $\forall x$, the fact that the solutions to the consumption paths under both utility functions must cross at least once at some c^*, t^{**} with $t^{**} < t^*$ implies that $c'_{u(\cdot)}(t^{**}) < c'_{v(\cdot)}(t^{**})$, so that there is a unique intersection between the two consumption paths after which $c_u(t)$ lies everywhere above $c_v(t)$, and therefore $c_{t^*}(t^*, u(\cdot)) < c_{t^*}t(t^*, v(\cdot))$. This implies the result by the same logic as the comparative statics with respect to survival probabilities. $\square$

C Characterizing Solutions for Discrete-Time Settings

Setting and Main Result

Time is discrete and indexed by $t \in \mathcal{T}$, where $\mathcal{T}$ may be finite or countably infinite. Our assumptions on the utility function are identical to the continuous time case.

The (possibly infinite-dimensional) vector $\mathbf{p} = (p_1 = 1, p_2, ..., p_{|\mathcal{T}|-1})$ collects the unconditional probabilities of being alive in period t from the vantage point of period 1, or equivalently, the measure of individuals who are alive in each period. Analogously to continuous time, we impose that p_t is strictly decreasing and strictly positive and that $\sum_{t\in\mathcal{T}} p_t < \infty$, as would be the case, for instance, if there were a constant mortality hazard rate in each period (in which case p_t forms a geometric sequence). We write $P_t = \sum_{i=t}^{|\mathcal{T}|} p_i$.

We allow individuals to save each time period, but they cannot borrow. In keeping with our vector notation, we denote the sequence of consumption choices by $\mathbf{c} = (c_1, c_2, ...)$ and similarly write $u(\mathbf{c})$ for the sequence of induced flow utilities $(u(c_1), u(c_2), ...)$. For a given annuity stream $\mathbf{s}$, consumers therefore solve the consumption-savings problem

$$\mathbf{c}^*(\mathbf{s}) \in \arg\max_{\mathbf{c}\in C(\mathbf{s})} \mathbf{p} \cdot u(\mathbf{c}), \tag{11}$$

where the constraint set

$$C(\mathbf{s}) = \left\{ \mathbf{c} : \sum_{i=1}^{t} c_i \leq w + \sum_{i=1}^{t} s_i, c_i \geq 0, \forall t \in \mathcal{T} \right\} \tag{12}$$

captures the no-borrowing and non-negativity constraints at each date.

We write $V(\mathbf{s})$ for the individuals' indirect utility given annuity stream $\mathbf{s}$, that is their utility when optimally solving their consumption-savings problem given that annuity schedule, $\mathbf{p} \cdot u(\mathbf{c}^*(\mathbf{s}))$.

The government budget constraint is $\mathbf{p} \cdot \mathbf{s} \leq S$ and the non-negativity constraint on the annuity payments. Therefore, the government solves

$$\mathbf{s}^* = \arg\max_{\mathbf{s} \in \mathcal{S}} V(\mathbf{s}) \tag{13}$$

where

$$\mathcal{S} = \{\mathbf{s} : \mathbf{p} \cdot \mathbf{s} \leq S, \mathbf{s} \geq \mathbf{0}\}. \tag{14}$$

Our analog of Theorem 1 in discrete time is the following:

Theorem 3. *In the discrete-time version of our problem, an optimal government transfer schedule exists, is generically unique, and takes the form of a (possibly deferred) annuity with a possible one-period top up. That is, an optimal schedule takes the form*

$$\boldsymbol{s}^* = \begin{cases} s_t = 0 & t < t^* \\ s_t = \tau & t = t^* \\ s_t = \overline{c} & t > t^* \end{cases}$$

for some t^* and τ, where $\tau \leq \overline{s}$ and $p_{t^*}\tau + P_{t^*+1}\overline{s} = S$. In all periods $t \geq t^*$, the no-borrowing constraint binds and the individual consumes $\overline{s}$. In all periods $t < t^* - 1$, the no-borrowing constraint does not bind. The no-borrowing constraint binds in period $t^* - 1$ only if $\tau = \overline{s} = P_{t^*}^{-1} S$. Consumption in periods t^* onward is constant.

Existence and Characterization of the Optimum

We characterize the solution by using the Karush-Kuhn-Tucker (KKT) conditions for both individuals' inner problem and the government's outer problem.

Individual's problem. For the inner problem, we note that the individual's optimization problem in eq. (11) has first-order conditions

$$p_t u'(c_t) - \sum_{s=t}^{\infty} \mu_s + \gamma_t = 0,$$

for all t, where γ_t is the Lagrange multiplier on the non-negativity constraint for c_t and μ_t is the Lagrange multiplier on the individuals' no-borrowing constraint at time t (see Table A1 below).

The Inada condition immediately implies that $\gamma_t = 0$ for all t.[18] Therefore, the above

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[18] Otherwise, if $\gamma_t > 0$, then $c_t = 0$ by complementary slackness. Since $u'(c_t) \to \infty$ as $c_t \to 0$, this cannot hold.

can be written more succinctly as

$$p_t u'(c_t) = \sum_{s=t}^{\infty} \mu_s. \tag{15}$$

In the infinite time horizon case ($\mathcal{T} = \{0, 1, 2, ...\}$), the individual's optimization problem additionally requires a transversality condition (TVC):

$$\lim_{t\to\infty} p_t u'(c_t) a_{t+1} = 0$$

where $a_{t+1} = \sum_{s=1}^{t} s_t - \sum_{s=1}^{t} c_t + w$ is the individual's savings going into period $t+1$.[19]

Government's problem. We next turn to the government's optimality conditions. Note first that the government's problem in eq. (13) is well-defined, following our analysis in the existence section. Note too that $V(\mathbf{s})$ is differentiable by the envelope theorem. As detailed in Table A1, let $\eta_t \geq 0$ be the Lagrange multiplier on the government's nonnegative transfer constraint $s_t \geq 0$. The government's first-order conditions are then

$$\frac{\partial V}{\partial s_t} - \lambda p_t + \eta_t = 0. \tag{16}$$

Next, note that by the envelope theorem applied to the individuals' optimization problem, we have

$$\frac{\partial V}{\partial s_t} = \sum_{s=t}^{\infty} \mu_s. \tag{17}$$

Combined optimality conditions. We combine equations (15), (16), and (17) to yield a succinct characterization of all our optimality conditions:

$$p_t u'(c_t) = \sum_{s=t}^{\infty} \mu_s = \lambda p_t - \eta_t. \tag{18}$$

Along with our constraints on the choice variables and Lagrange multipliers, which we summarize in Table A1 below, the above equation fully characterizes the necessary conditions for a maximum. In Table A1, we list the constraints and their associated multipliers.[20]

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[19] In this problem, the TVC does not rule out any otherwise possibly optimal paths, since it will be implied by the government's FOCs. Nevertheless, it is *necessary*: if our proposed optimal path did not satisfy the TVC, it could not be an optimum.

[20] Each constraint imposes three restrictions: primal feasibility (that the constraint holds at a conjectured optimum), dual feasibility, that the Lagrange multiplier is nonnegative (since the shadow value of relaxing a constraint must be weakly positive for a maximizing individual) , and complementary slackness, which says that at least one of the primal and dual feasibility inequalities must bind: intuitively, if a constraint is slack at the optimum, the shadow value of relaxing it is zero; conversely, if the shadow value of relaxing a constraint is strictly positive, it must be that the constraint is binding at the optimum. Note that while we include the nonnegative consumption constraints below for completeness, they trivially hold at the optimum by the Inada condition, which we already embed in equation (18).

Table A1: Summary of Constraints

Constraint	Multiplier	Primal	Dual	Complementary Slackness
Gov't budget	λ	$\mathbf{p}\cdot\mathbf{s}\leq S$	$\lambda>0$	$\lambda(S-\mathbf{p}\cdot\mathbf{s}\geq 0)$
Nonneg. Transfers ($\forall t$)	η	$s_t\geq 0$,	$\eta_t\geq 0$,	$s_t\eta_t=0$
Nonneg. Consump. ($\forall t$)	γ	$c_t\geq 0$	$\gamma_t\geq 0$	$c_t\gamma_t=0$
No borrowing ($\forall t$)	μ	$\sum_{i=1}^{t}(c_i-s_i)\leq w$	$\mu_t\geq 0$	$\left(w+\sum_{i=1}^{t}(s_i-c_i)\right)\mu_t=0$

Sufficient conditions for global optimum. We claim that the above conditions (the FOCs and the conditions imposed by the constraints) are in fact *sufficient* for a global maximum.[21] Note that the government's constraints are linear, and therefore convex, in $\mathbf{s}$. They additionally satisfy a standard constraint qualification,[22] ensuring that if $V(\mathbf{s})$ is concave, the necessary FOC are also sufficient for a global maximum (see Luenberger (1969), ch. 8, Theorem 1). Intuitively, there is no state variable, so unlike in the individual' inner problem, we do not need a TVC.

It remains to show that $V(\mathbf{s})$ is concave. Note that the individuals' no-borrowing constraints are linear in $\mathbf{s}$. Therefore, if $\mathbf{c}=\arg\max V(\mathbf{s})$ and $\mathbf{c}'=\arg\max V(\mathbf{s}')$, then $\lambda\mathbf{c}+(1-\lambda)\mathbf{c}'$ is feasible for $\lambda\mathbf{s}+(1-\lambda)\mathbf{s}'$ and yields weakly higher utility than $\lambda V(\mathbf{s})+(1-\lambda)V(\mathbf{s}')$ for all $\lambda\in(0,1)$, by concavity of u.

Partial characterization of transfer and consumption streams. Having described the sufficient conditions for a global optimum above, we now turn to refining the space of solutions using these conditions. The following three lemmas partially characterize the optimal transfer and consumption streams.

Lemma 5. *The budget constraint binds and* $\lambda>0$.

Proof. If the budget constraint were slack, then $\lambda=0$, and equation (18) demands that $p_t u'(c_t)\leq 0$ for all t. This is a contradiction because $u'(c)>0$ for all c by the Inada condition. ☐

Lemma 6. *If* $s_t>0$, *then* $s_{t+1}>0$.

Proof. If $s_t>0$, $\eta_t=0$ by complementary slackness. Therefore, equation (18) applied at date t implies $\sum_{i=t}^{\infty}\mu_i=\lambda p_t$. Since $\eta_{t+1}\geq 0$, applying equation (18) at date $t+1$ yields

$$\sum_{i=t+1}^{\infty}\mu_i=\lambda p_t-\mu_t\leq\lambda p_{t+1}.$$

Since $\lambda>0$ and $p_t>p_{t+1}$, this requires $\mu_t>0$. By complementary slackness, this implies the no-borrowing constraint binds in period t. Therefore, the no-borrowing constraint in

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[21] That is, we do not risk that the FOC picks out a minimum or a saddle point, or a local rather than global maximum)

[22] In particular, in the infinite dimensional case, Slater's condition holds: since $\sum_{t\in\mathcal{T}}p_tM<\infty$ for any $S>0$, for ϵ sufficiently small and $\mathbf{s}=\epsilon$ we have $\mathbf{s}\cdot\mathbf{p}<S$ and $\inf_{t\in\mathcal{T}}\mathbf{s}>0$, i.e., there is a feasible point on the interior (in the l^∞ norm) of the constraint space. In the finite dimensional case, the constraints are linear and finitely many so the constraint qualification holds trivially.

period $t+1$ is $c_{t+1} \leq s_{t+1}$. Supposing towards a contradiction that $s_{t+1} = 0$, this implies $c_{t+1} = 0$, which can never occur at an optimum by the Inada condition. □

Lemma 7. *At the optimal contract, for all dates t where $s_t > 0$, c_t is constant.*

Proof. Complementary slackness implies $\eta_i = 0$ for all such dates, so equation (18) reduces to $u'(c_i) = \lambda$. Since u' is injective, $c_i = \bar{c}$ must be constant. □

Characterizing when the no-borrowing constraint binds. Note that Lemma 6 implies that if there is ever a t such that $s_t > 0$, then at all $i > t$, $s_i > 0$. Clearly there exists some t such that $s_t > 0$: otherwise the budget constraint could not hold. This being the case, we now divide our analysis into two mutually exclusive and exhaustive sets of time periods: $T_0 = \{t : s_t = 0\}$ and $T_\infty = \{t : s_t > 0\}$.

The following two lemmas characterize the periods in which the no-borrowing constraint does and does not bind. We first show that the no-borrowing constraint is slack in all periods of T_0 except possibly the last:

Lemma 8. *Let t_0^{max} be the latest time period in T_0. The no-borrowing constraint does not bind for all $t < t_0^{max}$.*

Proof. Suppose otherwise that for $t < t_0^{\max}$, the no-borrowing constraint at t binds. Then the no-borrowing constraint at time $t+1$ is $c_{t+1} \leq s_{t+1}$. Since $s_{t+1} = 0$ for such a t by the definition of T_0, this implies $c_{t+1} = 0$, which never holds at the optimum by the Inada condition. □

Lastly, we characterize the borrowing constraint in the tail:

Lemma 9. *Consider the "tail" of dates $\mathcal{T}_\infty = \{t : s_t > 0\}$. The individuals' no-borrowing constraint binds for every period in the tail.*

Proof. Suppose towards a contradiction that for $t \in \mathcal{T}_\infty$ the no-borrowing constraint does not bind. Then $\mu_t = 0$ by complementary slackness. By equation (18), this implies that $p_t u'(c_t) = p_{t+1} u'(c_{t+1})$ and therefore that $c_t > c_{t+1}$ by strict concavity of u and $p_t > p_{t+1}$. But since $s_t > 0$ and $s_{t+1} > 0$ (by the definition of T_∞), we have $c_t = c_{t+1} = \bar{c}$ (as established in Lemma 7), a contradiction. □

We can combine our lemmas to arrive at a sharper characterization. By Lemma 9, since for all dates in T_∞ the budget constraint binds, we have $c_t = s_t$ for all dates $t > t_\infty^{\min} := t_0^{\max} + 1$. And since by Lemma 7, c_t is constant in the tail, so too is s_t for all $t > t_\infty^{\min}$.

Two types of optimal transfer schedules. The sole indeterminacy at this point is whether or not the consumer's no-borrowing constraint binds in period $t_0^{\max}$. There are therefore two types of possibly optimal transfer schedules:

1. A "constant contract": that is, the government pays out $P_{t_\infty^{\min}}^{-1} S$ starting in period $t_\infty^{\min}$. The individual's no-borrowing constraint is slack up to period $t_0^{\max} = t_\infty^{\min} - 1$, after which it binds.

2. A "top up" contract: the government pays out some quantity τ in period $t_\infty^{\min}$ and $\bar{c}$ in periods $t_\infty^{\min}+1$ onward, where $p_{t_\infty^{\min}}\tau + P_{t_\infty^{\min}+1}\bar{c} = S$. The individual's no-borrowing constraint is slack up to period $t_\infty^{\min}$, after which it binds. The individual's savings going into period $t_\infty^{\min}$ are such that consumption in periods $t_\infty^{\min}$ and $t_\infty^{\min}+1$ are equated (as demanded by Lemma 7), that is, $w - \sum_{i=1}^{t_0^{\max}} c_i + \tau = \bar{c}$.

We now show that for fixed primitives $(u(\cdot), \{p_t\}, S, w)$, there is a unique date $t^* = t_\infty^{\min}$ such that there is an optimal contract at the date. We first ask when there is an optimal contract that is a constant contract. We then show that whenever there is *not* an optimal contract that is a constant contract, there is an optimal contract that is a top-up.

Proposition 1. *Let $c_i^{CC}(t), i = 1, 2, ...|\mathcal{T}|$ be the consumer's optimal consumption stream given a constant contract that starts paying out in period t. There exists an optimal constant contract that begins paying out at time $t < |\mathcal{T}|$ iff*

$$p_{t-1}u'(P_t^{-1}S) \geq p_{t-1}u'(c_{t-1}^{CC}(t)) \geq p_t u'(P_t^{-1}S), \tag{19}$$

and there exists an optimal constant contract with first payout time $t = |\mathcal{T}|$ if and only if

$$c_{t-1}^{CC}(t) < P_t^{-1}S. \tag{19$'$}$$

Proof. Since the KKT conditions are necessary and sufficient for an optimum in our setting, we characterize existence of a constant contract by asking when there exist values of the Lagrange multipliers consistent with dual feasibility and complementary slackness when the primal variables take the form of a constant contract that starts paying out at date t.

We first need optimality of the consumption stream across all dates where there is no transfer. For all dates before $t-1$, the consumer's budget constraint is slack, so that $\mu_0 = \cdots = \mu_{t-2} = 0$ by complementary slackness. This implies, via equation (18), that $p_t u'(c_t)$ is constant over $i = 1, \ldots, t-1$.

Next we need optimality of the consumption stream across dates with a transfer. Since $\eta_t = 0$ for such dates, we have $u'(c_i) = \lambda$ for $i \geq t$ by eq. (18) again, so that $\lambda = u'(P_t^{-1}S)$. Dual feasibility of the μ_is (the multipliers on the no-borrowing constraints, which bind from period $t-1$ onward) requires that

$$\mu_i = \sum_{s=i}^{|\mathcal{T}|} \mu_s - \sum_{s=i+1}^{|\mathcal{T}|} \mu_s = p_i u'(c_i) - p_{i+1}u'(P_t^{-1}S) \geq 0, \quad \forall i \geq t-1.$$

Note that if $i = |\mathcal{T}|$, then the second summation is trivially 0 so the condition reduces to the vacuous $p_i u'(c_{|\mathcal{T}|}) \geq 0$. Otherwise, consider $i < |\mathcal{T}|$. The condition above is unproblematic for $i > t-1$, as $c_t = P_t^{-1}S$, so that we can always choose strictly positive μ_t, since $p_{t+1} < p_t$. The potentially problematic constraint is at $t-1$, where we need

$$p_{t-1}u'(c_{t-1}) \geq p_t u'(P_t^{-1}S).$$

This is the first restriction governing optimality of a constant contract. The second restriction comes from the non-negativity constraint on η_t in the periods before the deferred annuity starts. Dual feasibility and eq. (18) imply

$$\eta_i = p_i(\lambda - u'(c_i)) \geq 0, \quad \forall i < t.$$

Concavity of the consumer's problem and strict decreasingness of p_t implies that any optimal consumption path is strictly decreasing, so the constraint is implied for all $i < t$ if it holds at $t-1$, at which point it demands $u'(c_{t-1}) \leq u'(P_t^{-1}S)$. Note this condition must hold even at $t = |\mathcal{T}|$. □

Now we characterize when there is a top-up contract that is optimal.

Proposition 2. *If there is no optimal constant contract, i.e., equation (19) never holds (and in the finite time horizon case (19)′ never holds), then there exists a date t^* such that a unique top-up contract at date t^* is optimal, i.e., there exists*

$$\overline{c} \in \left(P_{t^*}^{-1}S, P_{t^*+1}^{-1}S\right) \tag{20}$$

such that for all $i \geq t^*$,

$$c_i^{\textit{top up}}(t^*, \overline{c}) = \overline{c}.$$

Proof. Recall a top-up contract entails a payment τ in period t and a constant annuity stream $\overline{c}$ in periods $t+1$ onwards such that $p_t\tau + P_{t+1}\overline{c} = S$. Write $\tau(\overline{c})$ for the unique level of τ consistent with a top up of $\overline{c}$; note this function is strictly decreasing. If a top up contract is optimal, then consumption must be the same in periods t and $t+1$ by Lemma 7, and the budget constraint is slack in period $t-1$, which requires that $\tau < \overline{c}$. Note that in principle, there are many possible top-up contracts that could be implemented at date t, since the tail level of consumption is no longer pinned down solely by the government budget constraint. We will show there is only one top up at date t, however, that is possibly consistent with the KKT conditions (in particular, that delivers the same consumption in periods t and period $t+1$).

To characterize the bounds on $\overline{c}$, note that when $\tau = \overline{c}$, the top up reduces to a constant contract that begins in period t, so this is equivalent to the condition $\overline{c} > P_t^{-1}S$. Meanwhile, nonnegativity of government transfers requires that $\tau > 0$, or equivalently that $\overline{c} < P_{t+1}^{-1}S$, the annuity stream that a constant contract starting in period $t+1$ would deliver. We next show that these conditions, along with the consumer being on their Euler path up to date t, imply all remaining KKT conditions.

Optimality of the consumption stream across all dates where there is no transfer is the same as for the constant contract, except that now the consumer's budget constraint is still slack in period $t-1$, so that we have $\mu_0 = \cdots = \mu_{t-1} = 0$ by complementary slackness, so that $p_i u'(c_i)$ is constant over $i = 0, \ldots, t$.

For optimality of the consumption stream across dates with a transfer, we have $u'(c_i) = \lambda$ as before, so that consumption must be constant for all such dates. Since the no-borrowing constraint binds in all periods t onward, this demands that $c_i = \overline{c}$ for all $i \geq t$. We therefore need that $c_t^{\textit{top up}}(t, \overline{c}) = \overline{c}$, where $c_t^{\textit{top up}}(t)$ is the consumption path of an optimizing consumer given a top up contract that starts paying out $\tau(\overline{c})$ in period t and $\overline{c}$ in period $t+1$. Dual feasibility of the μ_is for $i \geq t$ is implied by constant consumption in the tail, which holds if $c_t^{\textit{top up}}(t, \overline{c}) = \overline{c}$. Dual feasibility of the η_is now requires that in period $t-1$, we have

$u'(c_{t-1}) \leq u'(\bar{c}) \iff c_{t-1} \geq \bar{c}$. This is already implied, however, by $\bar{c} = c_t^{top\,up}(t, \bar{c})$ and the fact that c_{t-1}, c_t are on the Euler path, so $c_{t-1} < c_t$ (since utility is strictly concave and $p_t < p_{t-1}$.)

Proving the proposition now amounts to careful choice of t^*.

Note first that in the infinite-horizon case, there must be a $\tilde{t}$ that is the earliest time such that the right-side inequality in (19) holds: intuitively, eventually, by paying out later and later, the implied deferred annuity becomes so large that the consumer will optimally consume all their wealth before it begins. To see this formally, note that this corresponds to the inequality

$$u'(c_{t-1}^{CC}(t)) \geq \frac{p_t}{p_{t-1}} u'\left(P_t^{-1} S\right).$$

Since $\sum_{t \in \mathcal{T}} p_t < \infty$ and p_t is strictly decreasing, $\lim_{t \to \infty} p_t = 0$. We also know that $\frac{p_t}{p_{t-1}} \leq 1$. Meanwhile, since $p_t \to 0$, the Inada condition implies that $u'\left(P_t^{-1} S\right) \to 0$. Note that $c_{t-1}^{CC}(t) \leq w$ uniformly (in t) by the consumer's no-borrowing constraint, so $u'(c_{t-1}^{CC}(t)) \geq u'(w) > 0$ in the limit by the concavity of u.

Now, set $t^* = \min\{\tilde{t}, |\mathcal{T}|\} - 1$ (noting that whenever $|\mathcal{T}|$ is infinite, $\tilde{t}$ is finite). Our first claim is that

$$c_{t^*}^{CC}(t^* + 1) < P_{t^*+1}^{-1} S. \tag{21}$$

If $|\mathcal{T}| < \infty$ and $t^* = |\mathcal{T}| - 1$, this is immediate from negative equation (19)$'$. If $t^* = \tilde{t} - 1$, then by the definition of $\tilde{t}$, the right-side inequality in eq. (19) holds at $t^* + 1$, so that the only way a constant contract at date t^* can fail to be optimal is if the left-side inequality fails, which delivers precisely this condition.

Our second claim is that

$$p_{t^*-1} u'(c_{t^*-1}^{CC}(t^*)) < p_{t^*} u'\left(P_{t^*}^{-1} S\right). \tag{22}$$

If $t^* = \tilde{t} - 1$, this is immediate from the definition of $\tilde{t}$. If $t^* = |\mathcal{T}| - 1$, this follows from the fact that $\tilde{t} > |\mathcal{T}|$ only if the right-side inequality of eq. (19) fails for all $t \in \mathcal{T}$.

We can now prove our desired result.

Note that at the left endpoint $\bar{c}^L := P_{t^*}^{-1} S$ of the interval in equation (20), we have a constant contract that first pays out in period t^*. By eq. (22), the budget constraint at period $t^* - 1$ is optimally slack. Therefore, by the consumer's date $t^* - 1$ Euler condition,

$$p_{t^*-1} u'(c_{t^*-1}^{CC}(t^*)) = p_{t^*} u'(c_{t^*}^{CC}(t^*)) < p_{t^*} u'(P_{t^*}^{-1} S),$$

which implies $c_{t^*}^{CC}(t^*) > \bar{c}^L$.

Meanwhile, consider the right endpoint $\bar{c}^H := P_{t^*+1}^{-1} S$ of the interval in equation (20). Then eq. (21) immediately implies

$$c_{t^*}^{CC}(t^* + 1) < \bar{c}^H.$$

Lastly, for $\bar{c}$ in the region in between these two endpoints, we have that the budget constraint optimally binds first in period t^*, so that $c_t^*(\bar{c})$ solves $\sum_{i=1}^{t^*} c_i(c_t^*(\bar{c})) = w + \tau(\bar{c})$. The RHS above is strictly decreasing in $\bar{c}$, while each $c_i(\cdot)$ is strictly increasing in its argument by the consumer's Euler condition, so that $c_t^*(\bar{c})$ is strictly decreasing by the implicit function theorem.[23] The intermediate value theorem and strict monotonicity thus imply a unique $\bar{c}$ in the feasible interval such that

$$c_{t^*}^{top\ up}(t^*, \bar{c}) = \bar{c},$$

as desired. $\square$

Therefore, there is always either a top up contract or a constant contract at some date t^* that satisfies all the necessary and sufficient conditions for an optimum.

Generic Uniqueness

Lastly, we prove generic uniqueness. We claim it suffices to show that for almost all budget levels S, the Lagrange multiplier λ associated with the government's binding budget constraint can take only one value. Suppose this were true. Since at an optimum we have that for all dates t with $s_t > 0$, $u'(\bar{c}) = \lambda$, and u' is injective, this implies that almost everywhere, there is a unique level of optimal "tail consumption" (consumption in periods where the government pays out). Since every p_t is strictly positive (and thus P_t is strictly decreasing), there clearly cannot be two constant contracts with the same level of tail consumption. Since any top up starting at date t has tail consumption in $(P_t^{-1}S, P_{t+1}^{-1}S)$, and these intervals are disjoint, there also cannot be two top-ups with the same tail consumption or a top up and a constant contract with the same tail consumption.

We now prove that λ is S-almost everywhere unique to complete the proof. Consider the function

$$W(S) = \max_{\mathbf{s} \in \mathcal{S}(S)} V(\mathbf{s})$$

where we use the notation $\mathcal{S}(S) = \{\mathbf{s} : \mathbf{p} \cdot \mathbf{s} \leq S, \mathbf{s} \geq \mathbf{0}\}$ to stress the dependence of the feasible set on the budget limit S.[24]

We note first that W is concave; this follows immediately from Berge's Theorem since $V(\mathbf{s})$ itself is concave and $\mathcal{S}(S)$ is convex in S.

We next establish that λ^*, the Lagrange multiplier at any optimizing allocation, is a supergradient of $W(S)$; that is, for all $S > 0$ and $S' > 0$, $W(S') - W(S) \leq \lambda^*(S' - S)$. Since a concave function is almost everywhere differentiable and a differentiable function has derivative equal to its supergradient, this in turn establishes that λ^* is single-valued S-almost everywhere.

To see that λ^* is a supergradient, note first that by the existence proof, an optimizing $\mathbf{s}^*$ exists for a given nonzero budget level S. Since any optimum of the government's problem is fully characterized by the KKT conditions, this implies that $\mathbf{s}^*$ maximizes (among $\mathbf{s} \geq 0$)

$$V(\mathbf{s}) - \lambda^*(\mathbf{p} \cdot \mathbf{s} - S)$$

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[23] This argument ignores potential discontinuities of the optimal consumption at the boundary points $\bar{c}^H$ and $\bar{c}^L$ themselves. This is not an issue because the consumer's optimization problem is differentiable in $\bar{c}$ by the Envelope theorem, and therefore continuous.

[24] Note that we use a maximum in place of a supremum because we have already established the maximum exists.

where λ^* is the Lagrange multiplier associated with $\mathbf{s}^*$. We therefore have, for any alternative $\mathbf{s}$,

$$V(\mathbf{s}) - \lambda^*(\mathbf{p}\cdot\mathbf{s} - S) \leq V(\mathbf{s}^*) - \lambda^*(\mathbf{p}\cdot\mathbf{s}^* - S)$$

for any $\mathbf{s} \geq 0$, including any $\mathbf{s}$ such that $\mathbf{p}\cdot\mathbf{s} \leq S'$ for another budget level $S' > 0$.

Note that by the reasoning above, the budget constraint binds at the optimum, so $\mathbf{p}\cdot\mathbf{s}^* - S = 0$. Therefore, for any $\mathbf{s}$ such that $\mathbf{p}\cdot\mathbf{s} \leq S'$, we have

$$V(\mathbf{s}) \leq V(\mathbf{s}^*) + \lambda^*(S' - S) = W(S) + \lambda^*(S' - S)$$

Since this inequality is weak, it is preserved by taking the supremum over all $\mathbf{s}$ such that $\mathbf{p}\cdot\mathbf{s} \leq S'$, which yields exactly $W(S')$ on the LHS, so that

$$W(S') - W(S) \leq \lambda^*(S' - S)$$

as desired.

This establishes that λ^* is generically single valued.