

APPROXIMATE NEWTON POLICY GRADIENT ALGORITHMS*

HAOYA LI[†], SAMARTH GUPTA[‡], HSIANGFU YU[§], LEXING YING[¶], AND
INDERJIT DHILLON^{||}

Abstract. Policy gradient algorithms have been widely applied to Markov decision processes and reinforcement learning problems in recent years. Regularization with various entropy functions is often used to encourage exploration and improve stability. This paper proposes an approximate Newton method for the policy gradient algorithm with entropy regularization. In the case of Shannon entropy, the resulting algorithm reproduces the natural policy gradient algorithm. For other entropy functions, this method results in brand-new policy gradient algorithms. We prove that all these algorithms enjoy Newton-type quadratic convergence and that the corresponding gradient flow converges globally to the optimal solution. We use synthetic and industrial-scale examples to demonstrate that the proposed approximate Newton method typically converges in single-digit iterations, often orders of magnitude faster than other state-of-the-art algorithms.

Key words. policy gradient algorithm, approximate Newton method, quadratic convergence, Markov decision process, entropy regularization, reinforcement learning

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1. Introduction. Consider an infinite-horizon Markov decision process (MDP) [4, 33] $\mathcal{M} = (S, A, P, r, \gamma)$, where S is a set of states of the system studied, A is a set of actions made by the agent, P is a transition probability tensor with P_{st}^a being the probability of transitioning from state s to state t when taking action a , r is a reward tensor with r_s^a being the reward obtained when taking action a at state s , and $0 < \gamma < 1$ is a discount factor. Throughout the paper, the state space S and the action space A are assumed to be finite. A policy π is a randomized rule of action-selection where π_s^a denotes the probability of choosing action a at state s . For a given policy π , the value function v_π is defined as

$$(1.1) \quad (v_\pi)_s = \mathbb{E} \sum_{k=0}^{\infty} (\gamma^k r_{s_k}^a \mid s_0 = s),$$

which satisfies the Bellman equation,

$$(1.2) \quad (I - \gamma P_\pi)v_\pi = r_\pi,$$

where $(P_\pi)_{st} = \sum_{a \in A} \pi_s^a P_{st}^a$, $(r_\pi)_s = \sum_{a \in A} \pi_s^a r_s^a$, and I is the identity operator.

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[†]Department of Mathematics, Stanford University, Stanford, CA 94305 USA (lihaoya@stanford.edu).

[‡]Department of Electrical and Computer Engineering, Carnegie Mellon University, Pittsburgh, PA 15213 USA (samarthg@andrew.cmu.edu).

[§]Amazon Search, Palo Alto, CA 94303 USA (rofu.yu@gmail.com).

[¶]Department of Mathematics, Stanford University, Stanford, CA 94305-2125 USA, and Amazon Search, Palo Alto, CA 94303 USA (lexing@gmail.com).

^{||}Department of Computer Sciences, University of Texas at Austin, Austin, TX 78712-1188 USA, and Google (inderjit@cs.utexas.edu).

In order to promote exploration and enhance stability, one often regularizes the problem with a function h_π such as the negative Shannon entropy $(h_\pi)_s = \sum_{a \in A} \pi_s^a \log \pi_s^a$. With the regularization h_π , the original reward r_π is replaced with the regularized reward $r_\pi - \tau h_\pi$ where $\tau > 0$ is a regularization coefficient and (1.2) becomes

$$(1.3) \quad (I - \gamma P_\pi)v_\pi = r_\pi - \tau h_\pi,$$

where we overload the notation v_π for the regularized value function. Other continuously differentiable entropy functions can also be used, as we will show later. Since $\gamma < 1$ and P_π is a transition probability matrix, $(I - \gamma P_\pi)$ is invertible, and

$$(1.4) \quad v_\pi = (I - \gamma P_\pi)^{-1}(r_\pi - \tau h_\pi).$$

In a policy optimization problem, we seek a policy π that maximizes the value function v_π . According to the theory of regularized MDPs [9], when the regularization is strongly convex, there is a unique optimal policy π^* such that $(v_{\pi^*})_s \geq (v_\pi)_s$ for any policy π and state s . It thus suffices to maximize $\rho^\top v_\pi$ for any positive weight vector $\rho \in \mathbb{R}_+^{|S|}$. Using (1.4), the problem can be stated as

$$(1.5) \quad \max_{\pi} \rho^\top (I - \gamma P_\pi)^{-1}(r_\pi - \tau h_\pi).$$

This problem can be solved by, for example, the policy gradient (PG) method. However, the vanilla PG method converges quite slowly. In [1], for instance, the vanilla PG method is shown to have the $O(T^{-1})$ convergence rate, where T denotes the number of iterations. A widely used variant of PG is the softmax policy gradient (SPG) method, where a softmax parameterization is applied before taking gradient updates, which has been shown in [15] to require $O(|S|^{2^{\frac{1}{1-\gamma}}})$ iterations to converge for certain MDPs without regularization. For the PG method with entropy regularization and some of its variants, the convergence rate can be improved to $O(e^{-cT})$, i.e., linear convergence [17], which can still be slow since the constant c in the linear convergence rate $O(e^{-cT})$ is in general close to 0. It is also demonstrated in numerical examples that these algorithms with linear rates can experience slow convergence. For example, in the example in [40], thousands of iterations are needed for the algorithm to converge, even though the model is relatively small and sparse. Therefore, there is a clear need for designing new methods with faster convergence; one idea is to consider the problem's geometry. The Newton method, for example, preconditions the gradient with the Hessian matrix and obtains second-order local convergence. Since the exact Hessian matrix is usually too computationally expensive to obtain, the approximate Newton methods (including quasi-Newton methods), which use structurally simpler approximations of the Hessian instead, are more widely adopted in generic optimization problems and are known to enjoy superlinear convergence [25, 26].

1.1. Contributions. This paper investigates the approximate Newton approach for solving (1.5). The main contributions of this paper are the following.

- First, we present a unified approximate Newton method for the policy optimization problem. The main observation is to decompose the Hessian as a sum of a diagonal matrix and a remainder that vanishes at the optimal solution. This inspires us to use only the diagonal matrix in the approximate Newton method. As a result, the proposed method not only leverages the second-order information but also enjoys low computational cost due to the diagonal structure of the preconditioner used. When the negative Shannon entropy is used, this method reproduces the natural policy gradient (NPG) algorithm. This method results in brand-new policy gradient algorithms for other forms of entropy regularization.

- Second, we analyze the convergence property of the proposed approximate Newton algorithms and demonstrate local quadratic convergence both theoretically and numerically. By leveraging the framework of Newton-type methods (see [8], for example), we provide a simple and straightforward proof for quadratic convergence near the optimal policy. In the numerical tests, we verify that the proposed method leads to fast quadratic convergence even under small regularization and large discount rates (close to 1). Even for industrial-size problems with hundreds of thousands of states, the approximate Newton method converges in single-digit iterations and within a few minutes on a regular laptop. We also prove the global convergence of the approximate Newton gradient flow to the optimal solutions.

1.2. Background and related work. A major workhorse behind the recent success of reinforcement learning (RL) is the large family of policy gradient (PG) methods [38, 34]—for example, the natural policy gradient (NPG) method [12], the actor-critic method [13], the asynchronous advantage actor-critic (A3C) method [19], the deterministic policy gradient (DPG) method [32], the trust region policy optimization (TRPO) [28], the generalized advantage estimation (GAE) [29], and proximal policy optimization (PPO) [30], to mention but a few. The NPG method is known to be drastically faster than the original PG method because the policy gradient in NPG is preconditioned by the Fisher information (an approximation of the Hessian of the KL divergence) matrix to fit the problem geometry better. This idea is extended in TRPO and PPO, where the problem geometry is taken into consideration via trust region constraints (in terms of KL divergence) and a clipping function of the relative ratio of policies in the objective function, respectively. These implicit ways (in the sense that they do not adjust the gradient by an explicit preconditioner) of adjusting the policy gradient are similar to the mirror descent (MD) method [20] in generic optimization problems.

This similarity in addressing the inherent geometry of the problem is noticed by a line of recent work including [22, 9, 31, 35, 14], and the analysis techniques in MD methods have been adapted to the PG setting. The connection was first built explicitly in [22]. The authors consider a linear program formulation where the objective function is the average reward and the domain is the set of stationary state-action distributions, in which case the TRPO method can be viewed as an approximate mirror descent method and the A3C method as an MD method for the dual-averaging [21] objective. As a complement, [9] considers an actor-critic type method where the policy is updated via either a regularized greedy step or an MD step, and the value function is updated by a regularized Bellman operator, which also includes TRPO as a special case, and error propagation analysis is provided. In [31], an adaptive scaling that naturally arises in the policy gradient is applied to the proximity term of the MD formulation, and the sublinear convergence result is proved with a properly decreasing learning rate. In [35], the application to the nonstochastic setting is enabled by parameterizing the policy and applying MD to the policy parameters, and the corresponding sublinear convergence result is presented.

Regularization, a strategy that considers the modified objective function with an additional penalty term on the policy, is another crucial component in developing PG-type methods. Intuitively, regularization can encourage exploration in the policy iteration process and thus avoid local minima. It is also suggested [2] that regularization makes the optimization landscape smoother and thus enables possibly faster convergence. Linear convergence results are then established for regularized PG and

NPG methods [1, 17, 6]. In these relatively earlier works [1, 17, 6], the regularization usually takes the form of (negative) entropy or relative entropy. In the more recent works [14] and [40], which follow the MD type methods, the regularization is extended to general convex functions with the resulting Bregman divergences different from the KL divergence, and linear convergence is guaranteed as well.

However, most of these algorithms are of either sublinear or linear convergence except the entropy regularized NPG with full step length (which is a special case of the approximate Newton method we propose), and even the linear convergence rate $O(e^{-cT})$ can be slow since c can be close to zero. This motivates us to introduce the approximate Newton policy gradient method in section 2.

2. Approximate Newton method.

2.1. Approximate Newton method and entropy regularized natural policy gradient. This section derives the approximate Newton method for the entropy regularized policy optimization problems. The idea is to approximate the Hessian with a simpler matrix whose inverse is easy to compute. We start with the negative Shannon entropy $(h_\pi)_s = \sum_{a \in A} \pi_s^a \log \pi_s^a$.

In what follows, it is assumed that π^* is the optimizer of the problem stated in (1.5). By introducing $Z_\pi := I - \gamma P_\pi$, the objective function can be written as

$$(2.1) \quad E(\pi) \equiv \rho^\top (I - \gamma P_\pi)^{-1} (r_\pi - \tau h_\pi) = \rho^\top Z_\pi^{-1} (r_\pi - \tau h_\pi) = w_\pi^\top (r_\pi - \tau h_\pi),$$

where $w_\pi := Z_\pi^{-\top} \rho$.

Let us first outline the main idea of the approximate Newton method. The gradient $\nabla_\pi E$ in $\mathbb{R}^{|S||A|}$ of $E(\pi)$ has entries given by

$$(2.2) \quad \frac{\partial E}{\partial \pi_s^a} = (r_s^a - \tau(\log \pi_s^a + 1) - [(I - \gamma P^a)v_\pi]_s + c_s(\pi))(w_\pi)_s,$$

where $c_s(\pi)$ is a multiplier associated with the constraint $\sum_{a \in A} \pi_s^a = 1$ that depends on s . Our key observation is to decompose the Hessian matrix $D^2 E(\pi)$ in $\mathbb{R}^{|S||A| \times |S||A|}$ into two parts

$$(2.3) \quad D^2 E(\pi) = \Lambda(\pi) + \Delta(\pi),$$

where $\Lambda(\pi)$ is a *diagonal* matrix given by $\Lambda_{(sa), (tb)} = -\tau \delta_{\{(sa), (tb)\}} \frac{(w_\pi)_s}{\pi_s^a}$ and $\Delta(\pi)$ is a remainder that *vanishes* at $\pi = \pi^*$, i.e., $\Delta(\pi) = O(\|\pi - \pi^*\|)$ (shown in Theorem 2.1). We emphasize that $\Lambda(\pi)$ is generally not the diagonal part of the Hessian matrix $D^2 E(\pi)$, but a diagonal approximation to it. With this decomposition, we can approximate the Hessian matrix $D^2 E(\pi)$ by $\Lambda(\pi)$ and obtain the following *approximate Newton flow*:

$$\begin{aligned} \frac{d\pi_s^a}{dt} &= -(\Lambda^{-1} \nabla_\pi E)_{sa} = -(\Lambda_{(sa), (sa)})^{-1} \frac{\partial E}{\partial \pi_s^a} \\ &= \pi_s^a (r_s^a - \tau(\log \pi_s^a + 1) - [(I - \gamma P^a)v_\pi]_s + c_s(\pi)) / \tau. \end{aligned}$$

By introducing the parameterization $\theta_s^a = \log \pi_s^a$ and discretizing in time with learning rate η , we arrive at

$$\theta_s^a \leftarrow \eta(r_s^a - \tau - [(I - \gamma P^a)v_\pi]_s + c_s(\pi)) / \tau + (1 - \eta)\theta_s^a.$$

Writing this update back in terms of π_s^a leads to the following update rule:

$$\pi_s^a \propto (\pi_s^a)^{1-\eta} \exp(\eta(r_s^a + (\gamma P^a v_\pi)_s) / \tau),$$

which coincides with the NPG scheme with entropy regularization. This result is summarized in the following theorem with the proof given in subsection 5.1.

THEOREM 2.1. *Let $h_\pi \in \mathbb{R}^{|S|}$ be the negative Shannon entropy $(h_\pi)_s = \sum_{a \in A} \pi_s^a \log \pi_s^a$.*

(a) *There exists a diagonal approximation $\Lambda(\pi)$ of the Hessian matrix $D^2 E(\pi)$ given by $\Lambda_{(sa),(tb)} = -\tau \delta_{\{(sa),(tb)\}} \frac{(w_\pi)_s}{\pi_s^a}$ such that*

$$(2.4) \quad \Lambda(\pi) - D^2 E(\pi) = O(\|\pi - \pi^*\|).$$

(b) *The approximate Newton flow from $\Lambda(\pi)$ is*

$$(2.5) \quad \frac{d\pi_s^a}{dt} = \pi_s^a (r_s^a - \tau(\log \pi_s^a + 1) - [(I - \gamma P^a)v_\pi]_s + c_s(\pi))/\tau.$$

With a learning rate η , the gradient update is

$$(2.6) \quad \pi_s^a \leftarrow \frac{(\pi_s^a)^{1-\eta} \exp(\eta(r_s^a + (\gamma P^a v_\pi)_s)/\tau)}{\sum_{a \in A} (\pi_s^a)^{1-\eta} \exp(\eta(r_s^a + (\gamma P^a v_\pi)_s)/\tau)}.$$

Remark 2.2. The policy update scheme (2.6) is the same as the entropy regularized natural policy gradient scheme in, for example, [6].

Historical note. The natural gradient methods (including the NPG method) were traditionally developed as a way of implementing the vanilla gradient descent method with an intrinsic metric that is invariant to the choice of parameters (cf. [16]), and entropy regularization was originally motivated as a way of encouraging exploration and avoid the suboptimality caused by greedy solvers (cf. [22]). In this regard, it was more or less a coincidence that the algorithm combines the two methods—the regularized NPG obtains a fast quadratic convergence (cf. [6]). The reason behind this coincidence is that the preconditioner used in the natural gradient method approximates the second-order derivatives introduced by the entropy regularization, in this case, though the Fisher information matrix was not designed to approximate any second-order information in the classical natural gradient literature.

2.2. Extension to other entropy functions. Theorem 2.1 can be extended to more general entropy functions. It yields brand-new algorithms with quadratic convergence. Here we consider the entropy functions of the form

$$(2.7) \quad (h_\pi)_s = \sum_{a \in A} \phi\left(\frac{\pi_s^a}{\mu_s^a}\right) \mu_s^a,$$

where ϕ is convex on $(0, +\infty)$ and $\phi(1) = 0$, and μ_s is a prior distribution over A such that $\mu_s^a > 0$. The term $(h_\pi)_s$ is also called the “ f -divergence” between π_s and μ_s [24, 3]. If there is no prior knowledge of the policy, one can use the uniform prior, i.e., $\mu_s^a = 1/|A|$ for all a . We further assume that ϕ is twice continuously differentiable and strongly convex and that $\phi'(x) \rightarrow -\infty$ as $x \rightarrow 0$. Here are some examples:

- When $\phi(x) = x \log x$, $(h_\pi)_s = \sum_{a \in A} \left(\frac{\pi_s^a}{\mu_s^a} \log \frac{\pi_s^a}{\mu_s^a}\right) \mu_s^a = \sum_{a \in A} \pi_s^a \log \frac{\pi_s^a}{\mu_s^a}$. When the uniform prior is used, we recover the negative Shannon entropy regularization $\sum_{a \in A} \pi_s^a \log \pi_s^a$ used in Theorem 2.1 after omitting the constant $\log \frac{1}{|A|}$.
- When $\phi(x) = \frac{4}{1-\alpha^2}(1 - x^{\frac{1+\alpha}{2}})$ ($\alpha < 1$), we obtain the α -divergence:

$$(2.8) \quad (h_\pi)_s = \frac{4}{1-\alpha^2} - \frac{4}{1-\alpha^2} \sum_{a \in A} \mu_s^a (\pi_s^a / \mu_s^a)^{\frac{1+\alpha}{2}}.$$

In particular, when $\alpha = 0$ we obtain the Hellinger divergence $(h_\pi)_s = 2 - 2\sum_{a \in A} \sqrt{\mu_s^a \pi_s^a}$ after dividing by 2. When $\alpha \rightarrow -1$ we obtain the reverse KL divergence $(h_\pi)_s = \sum_{a \in A} \mu_s^a \log \frac{\mu_s^a}{\pi_s^a}$. Also, when $\alpha \rightarrow 1$, we obtain the KL divergence $(h_\pi)_s = \sum_{a \in A} \pi_s^a \log \frac{\pi_s^a}{\mu_s^a}$, though the limit of $\phi(x)$ does not exist when $\alpha \rightarrow 1$.

In the following theorem, we extend the approximate Newton method in Theorem 2.1 to the entropy functions described above. The proof of this theorem can be found in subsection 5.2.

THEOREM 2.3. *Assume that π^* is the optimizer of (1.5), where h_π is the entropy function defined in (2.7).*

(a) *The Hessian matrix $D^2E(\pi)$ can be approximated by a diagonal matrix $\Lambda(\pi)$ given by*

$$(2.9) \quad \Lambda_{(sa),(tb)} = -\tau \delta_{\{(sa),(tb)\}} \frac{(w_\pi)_s \phi''(\pi_s^a / \mu_s^a)}{\mu_s^a}$$

near π^ such that $\Lambda(\pi) - D^2E(\pi) = O(\|\pi - \pi^*\|)$.*

(b) *The approximate Newton flow from Λ is*

$$(2.10) \quad \frac{d\pi_s^a}{dt} = \mu_s^a (\phi''(\pi_s^a / \mu_s^a))^{-1} (r_s^a - \tau \phi'(\pi_s^a / \mu_s^a) - [(I - \gamma P^a)v_\pi]_s + c_s(\pi)) / \tau.$$

With parameterization $\theta_s^a = \phi'(\pi_s^a / \mu_s^a)$, the approximate Newton method from $\Lambda(\pi)$ can be expressed as

$$(2.11) \quad \theta_s^a \leftarrow \eta(r_s^a - [(I - \gamma P^a)v_\pi]_s + c_s(\pi)) / \tau + (1 - \eta)\theta_s^a,$$

where $0 < \eta \leq 1$ is the learning rate and $c_s(\pi)$ is a multiplier introduced by the constraint $\sum_{a \in A} \pi_s^a = 1$.

For particular choices of ϕ , the corresponding approximate Newton update scheme can be obtained directly by plugging ϕ into (2.11).

- For the case $\phi(x) = x \log x$ and $(h_\pi)_s = \sum_{a \in A} \pi_s^a \log \frac{\pi_s^a}{\mu_s^a}$, one can solve the multipliers $c_s(\pi)$ explicitly as in Theorem 2.1 and obtain the NPG method with prior distribution μ :

$$(2.12) \quad \pi_s^a \leftarrow \frac{(\mu_s^a)^\eta (\pi_s^a)^{1-\eta} \exp(\eta(r_s^a + (\gamma P^a v_\pi)_s) / \tau)}{\sum_{a \in A} (\mu_s^a)^\eta (\pi_s^a)^{1-\eta} \exp(\eta(r_s^a + (\gamma P^a v_\pi)_s) / \tau)}.$$

- For the case $\phi(x) = \frac{4}{1-\alpha^2} (1 - x^{\frac{1+\alpha}{2}})$ ($\alpha < 1$) and h_π given by the α -divergence (2.8), we have $\theta_s^a = -\frac{2}{1-\alpha} (\pi_s^a / \mu_s^a)^{\frac{\alpha-1}{2}}$; thus by (2.11) the update scheme is

$$(2.13) \quad \pi_s^a \leftarrow \mu_s^a \left((1 - \eta) (\pi_s^a / \mu_s^a)^{\frac{\alpha-1}{2}} + \frac{\alpha-1}{2} \eta (r_s^a - [(I - \gamma P^a)v_\pi]_s + c_s(\pi)) / \tau \right)^{\frac{2}{\alpha-1}}.$$

The remaining problem in the update schemes is the determination of the multipliers $c_s(\pi)$, since they cannot be solved explicitly, as in the case of the negative Shannon entropy (cf. Theorem 2.1). Since ϕ is strongly convex, we know that ϕ' is strictly increasing, and thus $-\phi'$ is a strictly decreasing function mapping from $(0, +\infty)$ to $(-\sup \phi', +\infty)$ since $\lim_{x \rightarrow 0+0} \phi'(x) = -\infty$. Let $\psi := (-\phi')^{-1}$; then $\psi : (-\sup \phi', +\infty) \rightarrow (0, +\infty)$ is a strictly decreasing function that satisfies

$\lim_{x \rightarrow -\sup \phi' + 0} \psi(x) = +\infty$ and $\lim_{x \rightarrow +\infty} \psi(x) = 0$. From (2.11), the equation of the multiplier $c_s(\pi)$ corresponding to $\sum_{a \in A} \pi_s^a = 1$ is

$$(2.14) \quad \sum_{a \in A} \mu_s^a \psi \left(-\frac{\eta}{\tau} c_s(\pi) - (1 - \eta) \phi'(\pi_s^a / \mu_s^a) - \frac{\eta}{\tau} (r_s^a - [(I - \gamma P^a) v_\pi]_s) \right) = 1,$$

or equivalently,

$$(2.15) \quad \sum_{a \in A} \mu_s^a \psi(\tilde{c}_s + x_a) = 1,$$

where

$$(2.16) \quad \tilde{c}_s = -\frac{\eta}{\tau} c_s(\pi), \quad x_a = -(1 - \eta) \phi'(\pi_s^a / \mu_s^a) - \frac{\eta}{\tau} (r_s^a - [(I - \gamma P^a) v_\pi]_s).$$

We claim that the determination of \tilde{c}_s in (2.15) (and thus the determination of $c_s(\pi)$) can be done similarly as in [39] based on the following lemma. The proof of this lemma can be found in subsection 5.3.

LEMMA 2.4. *Let $L \in \mathbb{R} \cup \{-\infty\}$. Assume that $\psi : (L, +\infty) \rightarrow (0, +\infty)$ is a strictly decreasing function that satisfies $\lim_{x \rightarrow L+0} \psi(x) = +\infty$ and $\lim_{x \rightarrow +\infty} \psi(x) = 0$ and assume also that $\mu_i > 0$; then for any x_1, x_2, \dots, x_k , there is a unique solution to the equation:*

$$(2.17) \quad \mu_1 \psi(x + x_1) + \dots + \mu_k \psi(x + x_k) = 1.$$

Moreover, the solution is on the interval

$$(2.18) \quad \left[\max \left\{ L - \min_{1 \leq i \leq k} x_i, \min_{1 \leq i \leq k} \left\{ \psi^{-1} \left(\frac{1}{k\mu_i} \right) - x_i \right\} \right\}, \max_{1 \leq i \leq k} \left\{ \psi^{-1} \left(\frac{1}{k\mu_i} \right) - x_i \right\} \right].$$

Leveraging Lemma 2.4 and the monotonicity of the function $\mu_1 \psi(x + x_1) + \dots + \mu_k \psi(x + x_k) - 1$, many of the established numerical methods (e.g., bisection) for nonlinear equations can be applied to determine the solution for (2.17). This routine can be used to find \tilde{c}_s in (2.15) and thus the multipliers $c_s(\pi)$ in (2.14) as stated in Proposition 2.5, whose proof is given in subsection 5.4.

PROPOSITION 2.5. *The multipliers $c_s(\pi)$ in the update scheme (2.11) can be determined uniquely such that the updated policy π satisfies $\pi_s^a \geq 0$ and $\sum_{a \in A} \pi_s^a = 1$.*

When the α -divergence is used, we have $\phi = \frac{4}{1-\alpha^2} (1 - x^{\frac{1+\alpha}{2}})$ and $\phi'(x) = \frac{2}{\alpha-1} x^{\frac{\alpha-1}{2}}$; then $L = -\sup \phi' = 0$ and $\psi(x) = (-\phi')^{-1}(x) = (\phi')^{-1}(-x) = (\frac{1-\alpha}{2} x)^{\frac{2}{\alpha-1}}$. The algorithm proposed in this section is summarized in Algorithm 2.1 below.

2.3. Convergence of the approximate Newton gradient flow. Recall from (2.10) that the approximate Newton gradient flow with the general entropy functions is

$$\frac{d\pi_s^a}{dt} = \mu_s^a (\phi''(\pi_s^a / \mu_s^a))^{-1} (r_s^a - \tau \phi'(\pi_s^a / \mu_s^a) - [(I - \gamma P^a) v_\pi]_s + c_s(\pi)) / \tau,$$

from which we can obtain the dynamics of the objective function E :

Algorithm 2.1 Approximate Newton method for the regularized MDP.

Require: The MDP model $\mathcal{M} = (S, A, P, r, \gamma)$, initial policy π_{init} , convergence threshold ϵ_{tol} , regularization coefficient τ , learning rate η , the regularization type (KL or α -divergence).

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1: Initialize the policy  $\pi = \pi_{\text{init}}$ .
2: Set  $\xi = 1 + \epsilon_{\text{tol}}$  and  $k = |A|$ .
3: while  $\xi > \epsilon_{\text{tol}}$  do
4:   Calculate the regularization term  $h_\pi$  by  $(h_\pi)_s = \sum_{a \in A} \mu_s^a \phi(\pi_s^a / \mu_s^a)$ .
5:   Calculate  $P_\pi$  and  $r_\pi$  by  $(P_\pi)_{st} = \sum_{a \in A} \pi_s^a P_{st}^a$ ,  $(r_\pi)_s = \sum_{a \in A} \pi_s^a r_s^a$ .
6:   Calculate  $v_\pi$  by (1.4), i.e.,  $v_\pi = (I - \gamma P_\pi)^{-1}(r_\pi - \tau h_\pi)$ .
7:   if the KL divergence is used then
8:      $(\pi_{\text{new}})_s^a \leftarrow \frac{(\mu_s^a)^\eta (\pi_s^a)^{1-\eta} \exp(\eta(r_s^a + (\gamma P^a v_\pi)_s) / \tau)}{\sum_{a \in A} (\mu_s^a)^\eta (\pi_s^a)^{1-\eta} \exp(\eta(r_s^a + (\gamma P^a v_\pi)_s) / \tau)}$  for
        $a = 1, 2, \dots, |A|$ ,  $s = 1, 2, \dots, |S|$ .
9:   end if
10:  if the  $\alpha$ -divergence is used then
11:    for  $s = 1, 2, \dots, |S|$  do
12:      Set  $L = 0$  and  $\psi(x) = (\frac{1-\alpha}{2}x)^{\frac{2}{\alpha-1}}$ 
13:      Calculate  $x_a = -(1-\eta)\phi'(\pi_s^a / \mu_s^a) - \frac{\eta}{\tau}(r_s^a - [(I - \gamma P^a)v_\pi]_s)$ ,  $a = 1, \dots, |A|$ .
14:      Solve for  $\tilde{c}_s = -\frac{\eta}{\tau}c_s(\pi)$  with the bisection method on the interval
        described in (2.18).
15:      Update  $(\pi_{\text{new}})_s^a \leftarrow \mu_s^a \psi(\tilde{c}_s + x_a)$  for  $a = 1, 2, \dots, |A|$ .
16:    end for
17:  end if
18:   $\xi = \|\pi_{\text{new}} - \pi\|_F / \|\pi\|_F$ .
19:   $\pi = \pi_{\text{new}}$ 
20: end while

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$$\begin{aligned}
(2.19) \quad \frac{dE(\pi)}{dt} &= \sum_{s \in S, a \in A} \frac{\partial E}{\partial \pi_s^a} \frac{d\pi_s^a}{dt} \\
&= \sum_{s \in S, a \in A} \left[(r_s^a - \tau \phi'(\pi_s^a / \mu_s^a) - [(I - \gamma P^a)v_\pi]_s + c_s(\pi))(w_\pi)_s \right. \\
&\quad \cdot \mu_s^a (\phi''(\pi_s^a / \mu_s^a))^{-1} (r_s^a - \tau \phi'(\pi_s^a / \mu_s^a) - [(I - \gamma P^a)v_\pi]_s + c_s(\pi)) / \tau \left. \right] \\
&= \sum_{s \in S, a \in A} \mu_s^a (\tau \phi''(\pi_s^a / \mu_s^a))^{-1} (r_s^a - \tau \phi'(\pi_s^a / \mu_s^a) - [(I - \gamma P^a)v_\pi]_s + c_s(\pi))^2 (w_\pi)_s \\
&\geq 0,
\end{aligned}$$

where we have used the gradient

$$(2.20) \quad \frac{\partial E}{\partial \pi_s^a} = (r_s^a - \tau \phi'(\pi_s^a / \mu_s^a) - [(I - \gamma P^a)v_\pi]_s + c_s(\pi))(w_\pi)_s.$$

As a result, we have shown that $\frac{dE(\pi)}{dt} \geq 0$. Since $E(\pi)$ is upper-bounded by $\rho^\top v_{\pi^*}$, $E(\pi)$ converges. With a closer look, the following theorem states that the limiting policy is exactly the optimal policy π^* , and the proof is given in subsection 5.5.

THEOREM 2.6. *Suppose that ϕ is twice continuously differentiable and strongly convex and that $\phi'(x) \rightarrow -\infty$ as $x \rightarrow 0$; then the approximate Newton flow (2.10) converges globally to the optimal policy π^* .*

3. Quadratic convergence. In this section, we study the quadratic convergence of the approximate Newton method at the learning rate $\eta = 1$, which corresponds to the step size used in the Newton method. Our analysis is inspired by the results in [8, 37]. The following theorem states the second-order convergence when $\eta = 1$, with the proof given in subsection 5.6. For simplicity of notation, we let $f(\pi)_{sa} = -(r_s^a - ((I - \gamma P^a)v_\pi)_s)$, which is the additive inverse of the advantage function.

THEOREM 3.1. *Let*

$$\Phi(\pi) = \tau \sum_{s \in S, a \in A} \mu_s^a \phi\left(\frac{\pi_s^a}{\mu_s^a}\right),$$

where ϕ is twice continuously differentiable and strongly convex and $\phi'(x) \rightarrow -\infty$ as $x \rightarrow 0$. Denote the k th policy obtained in Algorithm 2.1 by $\pi^{(k)}$. For $\eta = 1$, the update scheme in Algorithm 2.1 can be summarized as

$$(3.1) \quad \nabla \Phi(\pi^{(k+1)}) - \nabla \Phi(\pi^{(k)}) = -\left(f(\pi^{(k)}) + \nabla \Phi(\pi^{(k)}) - B^\top c(\pi^{(k)})\right),$$

where $f(\pi)_{sa} = -(r_s^a - ((I - \gamma P^a)v_\pi)_s)$ and we denote by B the $|S|$ -by- $(|S| \times |A|)$ matrix such that $B_{ij} = 1$ for $|A|(i-1) + 1 \leq j \leq |A|i$ and $B_{ij} = 0$ otherwise. Then $\pi^{(k)}$ enjoys a quadratic local convergence to π^* , i.e., $\lim_{k \rightarrow \infty} \pi^{(k)} = \pi^*$ and

$$(3.2) \quad \left\| \pi^{(k+1)} - \pi^* \right\| \leq C \left\| \pi^{(k)} - \pi^* \right\|^2,$$

for some constant C , given that the initial policy $\pi^{(0)}$ is sufficiently close to π^* .

Remark 3.2. It is clear that the quadratic convergence also occurs if $\pi^{(k)}$ is in a sufficiently small neighborhood of π^* for some $k \geq 1$ even if $\pi^{(0)}$ is not. The precise description of this small neighborhood is provided in the proof (see subsection 5.6). For a special case of this result, where $\phi(x) = x \log x$ and $\mu_s^a = 1/|A|$, the algorithm is reduced to the entropy regularized NPG. A similar local convergence result for this special case has been obtained in [6], where the proof leverages the particular structure of Shannon entropy.

Connection with mirror descent. The approximate Newton algorithm (3.1) for $\eta = 1$ has a deep connection with mirror descent. The vanilla mirror descent of $-E(\pi)$ with a learning rate β and the Bregman divergence associated with Φ is given by

$$\begin{aligned} \pi^{(k+1)} &= \arg \min_{\pi} \left\{ -E(\pi^{(k)}) - \nabla E(\pi^{(k)})^\top (\pi - \pi^{(k)}) \right. \\ &\quad \left. + \frac{1}{\beta} (\Phi(\pi) - \Phi(\pi^{(k)}) - \nabla \Phi(\pi^{(k)})^\top (\pi - \pi^{(k)})) \right\} \\ &= \arg \min_{\pi} \left\{ \left((\text{diag}(w_{\pi^{(k)}}) \otimes I_{|A|}) (f(\pi^{(k)}) + \nabla \Phi(\pi^{(k)})) \right)^\top (\pi - \pi^{(k)}) \right. \\ &\quad \left. + \frac{1}{\beta} (\Phi(\pi) - \nabla \Phi(\pi^{(k)})^\top \pi) \right\}, \end{aligned}$$

where $\text{diag}(w_{\pi^{(k)}})$ is the diagonal matrix with the diagonal equal to $w_{\pi^{(k)}} := (I - \gamma P_{\pi^{(k)}}^\top)^{-1} \rho$, \otimes denotes the Kronecker product, and $I_{|A|}$ denotes the $|A|$ by $|A|$ identity matrix. In the last equality, the terms independent of π are dropped and the multiplier term in ∇E is canceled out using $B\pi = B\pi^{(k)} = \mathbf{1}_{|S|}$. The first-order stationary condition of this minimization problem reads

$$(3.3) \quad \nabla \Phi(\pi^{(k+1)}) - \nabla \Phi(\pi^{(k)}) = -\beta(\text{diag}(w_{\pi^{(k)}}) \otimes I_{|A|})(f(\pi^{(k)}) + \nabla \Phi(\pi^{(k)}) - B^\top c(\pi^{(k)})).$$

This suggests that (3.1) can be reinterpreted as an *accelerated* mirror descent method with *adaptive* learning rates $\beta_s \equiv 1/(w_{\pi^{(k)}})_s$ that depend on the state s and the current iterate $\pi^{(k)}$. Observation of the connection between mirror descent and the natural gradient method (which is similar to the approximate Newton method in this paper when the Shannon entropy is used) is given in [23, 10].

In [40], a variant of mirror descent is proposed based on an implicit update scheme:

$$(3.4) \quad (\nabla \Phi(\pi^{(k+1)}))_{sa} - (\nabla \Phi(\pi^{(k)}))_{sa} = -\beta' \left(f(\pi^{(k)})_{sa} + \nabla \Phi(\pi^{(k+1)})_{sa} - (c(\pi^{(k)}))_s \right),$$

with a state-independent learning rate β' . In the next section, we will compare this variant with our approximate Newton method (3.1) and show that the approximate Newton method converges orders of magnitudes faster than the ones in [40].

4. Numerical results.

4.1. Experiment I. We first test the approximate Newton methods derived in section 2 on the model in [40]. For the sake of completeness, we include the model's description here. The MDP considered has a state space S of size 200 and an action space A of size 50. For each state t and action a , a subset S_t^a of S is uniformly randomly chosen such that $|S_t^a| = 20$, and $P_{tt'}^a = 1/20$ for any $t' \in S_t^a$. The reward is given by $r_s^a = U_s^a U_s$, where U_s^a and U_s are independently uniformly chosen on $[0, 1]$. The discount rate γ is set as 0.99, and the regularization coefficient $\tau = 0.001$.

In the numerical experiment, we implement Algorithm 2.1 with the KL divergence, the reverse KL divergence, the Hellinger divergence, and the α -divergence with $\alpha = -3$. We adopt the uniform prior $\mu_s^a = 1/|A|$ to make a fair comparison with the policy mirror descent (PMD) and the general policy mirror descent (GPMD) method in [40]. We set the initial policy as the uniform policy, the convergence threshold as $\epsilon_{\text{tol}} = 10^{-12}$, and the learning rate η as 1. Figure 1(a) demonstrates that, for these four tests, the approximate Newton algorithm converges in 7, 7, 7, and 6 iterations, respectively. In comparison, we apply PMD and GPMD to the same MDP with the same stopping criterion. As also shown in Figure 1(a), many more iterations are needed for GPMD and PMD to reach the same precision: GPMD converges in 14822 iterations, and PMD does not reach the desired precision after 3×10^5 iterations. For the implementation of GPMD and PMD, a quadratic regularization is used, and we have already tuned the hyperparameters to optimize their performance. The number of iterations needed for GPMD and PMD to converge accords with the numerical results provided in [40].

In order to verify the quadratic convergence proved in section 3, we draw the plots of $\log \|\pi - \pi^*\|$ in Figure 1(b), Figure 1(c), Figure 1(d), and Figure 1(e), where π^* is the final policy and the norm used is the Frobenius norm. A green reference line with slope $\log 2$ through the origin is plotted for comparison. If the error converges exactly

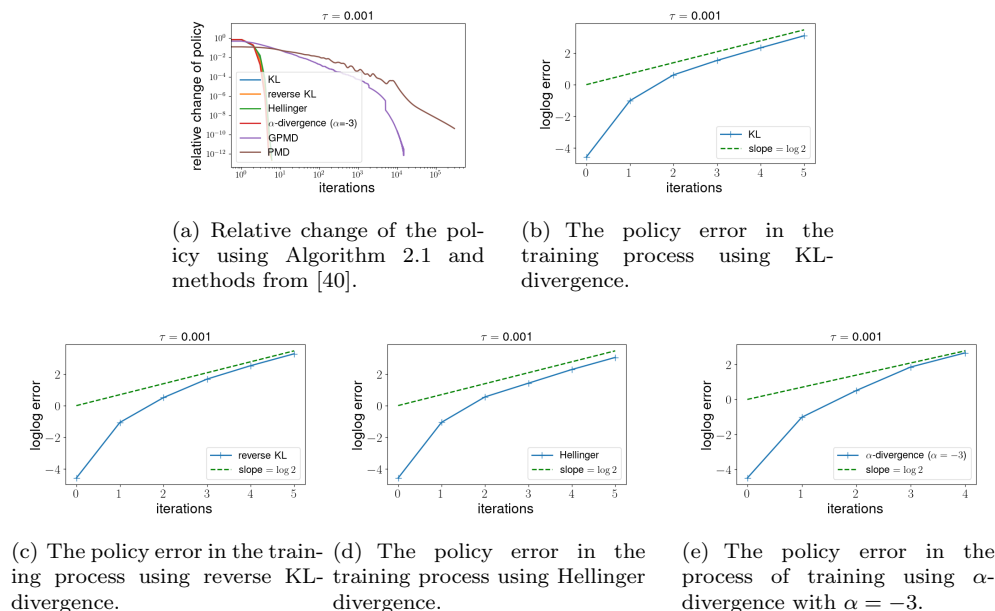


FIG. 1. Figures for the synthetic medium scale MDP. (a) Relative change of the policy $\|\pi_{\text{new}} - \pi\|_F / \|\pi\|_F$ during training of Algorithm 2.1 compared with PMD and GPMD in [40], with the logarithmic scale used for both axes. Notice that Algorithm 2.1 converges in 6–7 iterations to 10^{-12} in all cases, while PMD and GPMD take more than 10^4 iterations. Here the quadratic regularization is used for PMD and GPMD. (b)–(e) Blue: The convergence of $\log \|\pi - \pi^*\|_F$ in the training process with the KL divergence, the reverse KL divergence, the Hellinger divergence, and the α -divergence with $\alpha = -3$, respectively. Green: A line through the origin with slope $\log 2$. Comparison of the convergence plots with the green reference lines shows a clear quadratic convergence for Algorithm 2.1. (Color available online.)

at a quadratic rate, the plot of $\log \|\pi - \pi^*\|_F$ shall be parallel to the reference line. The convergence curves approach the reference lines at the end (and are even steeper than the reference lines in the beginning), demonstrating clearly a quadratic convergence for all forms of regularization used here.

4.2. Experiment II. Next, we apply the approximate Newton methods derived in section 2 to an MDP model constructed from the search logs of an online shopping store, with two different ranking strategies. Each issued query is represented as a state in the MDP. In response to a query, one can choose one of the two ranking strategies (actions) to return a ranked list of products shown to the customer. Based on the shown products, the customer can refine or update the query, thus entering a new state. The reward at each state-action pair is a weighted sum of the clicks and purchases resulting from the action. Based on the data collected from two separate 5-week periods for both ranking strategies, we construct an MDP with 135k states and a very sparse transition tensor P with only 0.01% nonzero entries. The discount rate γ is set as 0.99, and the regularization coefficient is $\tau = 0.001$. We use the uniform prior $\mu_s^a = 1/|A|$ in the implementation.

When calculating v_π by $v_\pi = (I - \gamma P_\pi)^{-1}(r_\pi - \tau h_\pi)$, we apply the iterative solver Bi-CGSTAB [36], a widely used numerical method with high efficiency and robustness for solving large sparse nonsymmetric systems of linear equations [27, 7], to leverage the sparsity of the transition tensor.

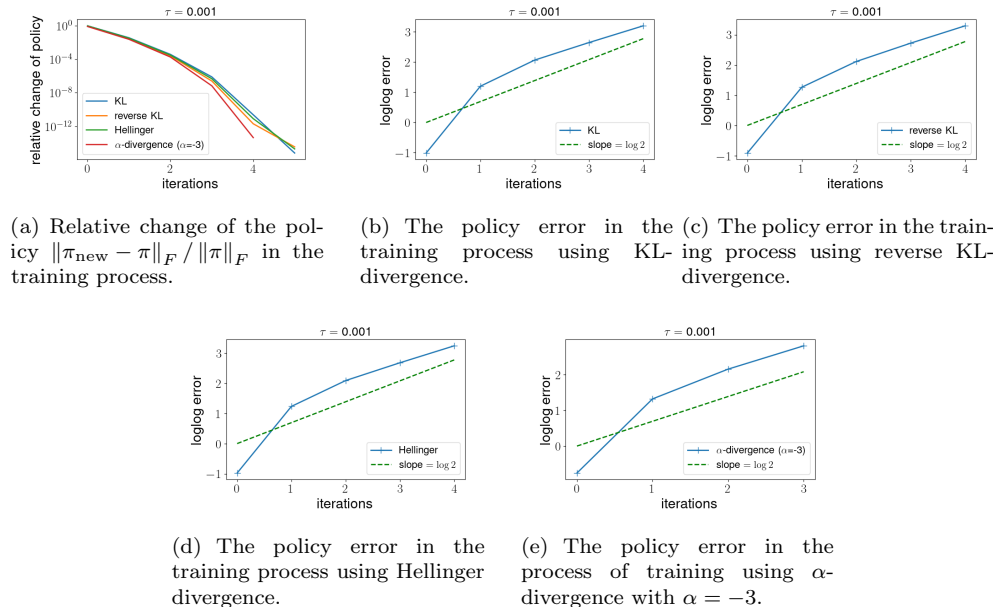


FIG. 2. Figures for the industrial-size MDP. (a) Relative change of the policy $\|\pi_{\text{new}} - \pi\|_F / \|\pi\|_F$ in the training process of Algorithm 2.1. A logarithmic scale is used for the vertical axis. (b)–(e) Blue: The convergence of $\log \|\pi - \pi^*\|_F$ in the training process with the KL divergence, the reverse KL divergence, the Hellinger divergence, and the α -divergence with $\alpha = -3$, respectively. Green: A line through the origin with slope $\log 2$. (Color available online.)

TABLE 1
Number of approximate Newton iterations and BiCGSTAB steps used in the training process.

Regularizer	KL	Reverse-KL	Hellinger	α -divergence ($\alpha = -3$)
Approx-Newton iterations	6	6	6	5
Total Bi-CGSTAB steps	110	109	110	83
Average Bi-CGSTAB steps	18.3	18.2	18.3	16.6

In the numerical experiment, we implement Algorithm 2.1 with the KL divergence, the reverse KL divergence, the Hellinger divergence, and the α -divergence with $\alpha = -3$. We set the initial policy as the uniform policy, the convergence threshold as $\epsilon_{\text{tol}} = 10^{-12}$, and the learning rate η as 1. All the tests have fast convergence, as shown in Figure 2(a), where logarithmic scale is used for the vertical axis. More specifically, the approximate Newton algorithm using the KL divergence, the reverse KL divergence, the Hellinger divergence, and the α -divergence with $\alpha = -3$ converges in 6, 6, 6, 5 iterations, respectively. It is worth noticing that even though the size of the state space S here is some magnitudes larger than the examples in subsection 4.1, the number of approximate Newton iterations used is about the same. The comparison with GPMD and PMD is not given for this example since they are intractable to implement due to the high computational cost caused by the large MDP model.

In Table 1, we report the number of BiCGSTAB steps used in the algorithm. In each approximate Newton iteration, fewer than 20 BiCGSTAB steps are used in order to find v_π . For all four regularizers used here, only about 100 BiCGSTAB steps are needed in the whole training process, thanks to the fast convergence of

TABLE 2
Number of approximate Newton iterations and BiCGSTAB steps used in the training process.

Regularizer	KL	Reverse-KL	Hellinger	α -divergence ($\alpha = -3$)
Approx-Newton iterations	6	6	6	7
Total Bi-CGSTAB steps	370	379	492	452
Average Bi-CGSTAB steps	61.7	63.2	82.0	64.6

the approximate Newton method. As a comparison, the regularized value iteration (a special case for the method in [9]) typically needs thousands of matrix-vector multiplication with the MDP transition matrix, since its convergence rate is $O(\gamma^T)$.

As in the previous numerical example, in Figure 2(b), Figure 2(c), Figure 2(d), and Figure 2(e) we verify the quadratic convergence by comparing the plot of $\log \|\pi - \pi^*\|$ with a green reference line through the origin with slope $\log 2$. As the convergence curves are approximately parallel to the reference lines, this verifies that the proposed algorithm converges quadratically with all the regularizations in this example as well.

4.3. Experiment III. In this section, we are concerned with an MDP with a relatively large action space and state space at the same time. We consider the state space and action space with size $|S| = 10000$ and $|A| = 300$ with $(S, A) = (\{0, 1, \dots, |S| - 1\}, \{0, 1, \dots, |A| - 1\})$. Here the transition tensor is defined as $P_{tt'}^a = 1$ when $t' = (t + a) \bmod |S|$, $t \neq |S| - 1$ or $t = t' = |S| - 1$, and $P_{tt'}^a = 0$ otherwise. The reward is set as $r_s^a = 1 - \gamma$ if $s = |S| - 1$ and $r_s^a = 0$ otherwise, where $\gamma = 0.99$.

Similar to the previous tests, we apply the approximate Newton algorithm with the KL divergence, the reverse KL divergence, the Hellinger divergence, and the α -divergence with $\alpha = -3$ and the uniform prior $\mu_s^a = 1/|A|$. For this experiment, we use $\tau = 0.01$ and $\epsilon_{\text{tol}} = 10^{-9}$. Similar to the previous examples, in all 4 tests the algorithm converges with single-digit approximate Newton iterations, as shown in Figure 3(a). The quadratic convergence can be verified in the plots of $\log \|\pi - \pi^*\|$ displayed in Figure 3(b), Figure 3(c), Figure 3(d), and Figure 3(e). Due to the size and sparsity of the transition tensor, we also adopt Bi-CGSTAB for calculating v_π , and the number of Bi-CGSTAB iterations used is reported in Table 2. Around 400 Bi-CGSTAB steps are used, which also involves fewer matrix-vector multiplications with the transition matrix compared to traditional value iteration methods.

5. Proofs.

5.1. Proof of Theorem 2.1.

Proof. Step 1: Expand $E(\pi)$ and prove the first-order condition (5.2). For any $\epsilon \in \mathbb{R}^{|S| \times |A|}$, introduce $r_\epsilon \in \mathbb{R}^{|S|}$ and $Z_\epsilon \in \mathbb{R}^{|S| \times |S|}$ such that

$$(5.1) \quad (r_\epsilon)_s := \sum_{a \in A} \epsilon_s^a r_s^a, \quad (Z_\epsilon)_{st} := \sum_{a \in A} \epsilon_s^a (\delta_{st} - \gamma P_{st}^a),$$

where $\delta_{st} = 1$ if $s = t$ and $\delta_{st} = 0$ otherwise. Then r_ϵ and Z_ϵ are linear with respect to ϵ , which is helpful when expressing the first-order conditions and simplifying the expansion of $E(\pi)$.

Now we proceed to prove that for any ϵ with $\sum_{a \in A} \epsilon_s^a = 0$ and $|\epsilon_s^a| < \pi_s^a$, at $\pi = \pi^*$

$$(5.2) \quad r_\epsilon - \tau Dh_\pi \epsilon - Z_\epsilon Z_\pi^{-1} (r_\pi - \tau h_\pi) = 0,$$

where $Dh_\pi \in \mathbb{R}^{|S| \times |S| \times |A|}$ is the gradient matrix of h_π with respect to π .

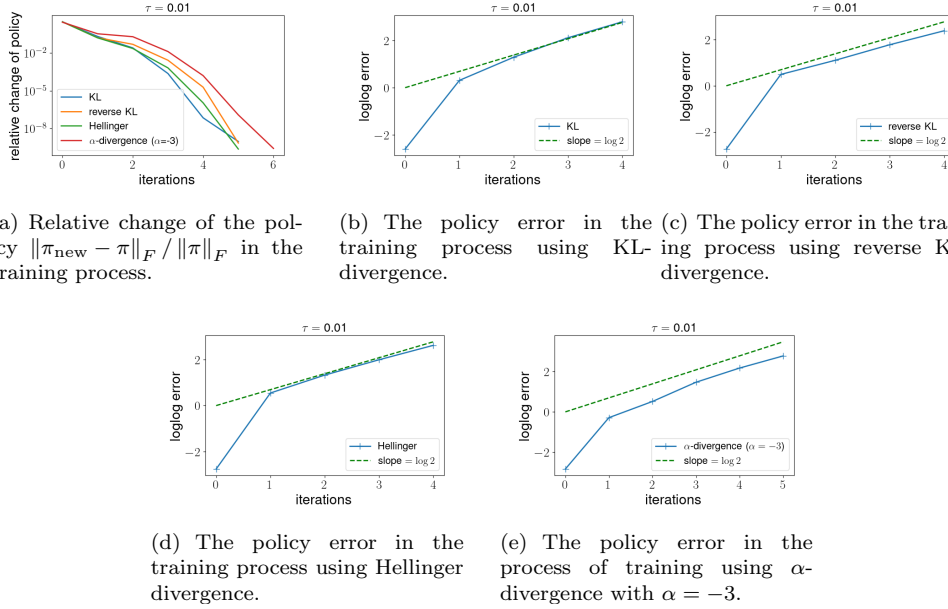


FIG. 3. Figures for an MDP with relatively large state and action spaces. (a) Relative change of the policy $\|\pi_{\text{new}} - \pi\|_F / \|\pi\|_F$ in the training process of Algorithm 2.1. A logarithmic scale is used for the vertical axis. (b)–(e) Blue: The convergence of $\log \|\pi - \pi^*\|_F$ in the training process with the KL divergence, the reverse KL divergence, the Hellinger divergence, and the α -divergence with $\alpha = -3$, respectively. Green: A line through the origin with slope $\log 2$. (Color available online.)

Since π is a policy, $\sum_{a \in A} \pi_s^a = 1$ for any s . Thus

$$(5.3) \quad (Z_\pi)_{st} = \delta_{st} - \gamma \sum_{a \in A} \pi_s^a P_{st}^a = \sum_{a \in A} \pi_s^a (\delta_{st} - \gamma P_{st}^a).$$

Now consider a policy $\pi + \epsilon$, i.e., $\sum_{a \in A} \epsilon_s^a = 0$ and $\pi_s^a + \epsilon_s^a \geq 0$; then, thanks to (5.3), one can obtain

$$(5.4) \quad Z_{\pi+\epsilon} = Z_\pi + Z_\epsilon, \quad r_{\pi+\epsilon} = r_\pi + r_\epsilon,$$

where Z_ϵ and r_ϵ are defined in (5.1), i.e., $(Z_\epsilon)_{st} = \sum_{a \in A} \epsilon_s^a (\delta_{st} - \gamma P_{st}^a)$, $(r_\epsilon)_s = \sum_{a \in A} \epsilon_s^a r_s^a$. Leveraging the linearity (5.4), we obtain the expansion:

$$(5.5) \quad \begin{aligned} E(\pi + \epsilon) &= \rho^\top Z_{\pi+\epsilon}^{-1} (r_{\pi+\epsilon} - \tau h_{\pi+\epsilon}) = \rho^\top (Z_\pi + Z_\epsilon)^{-1} (r_\pi + r_\epsilon - \tau h_{\pi+\epsilon}) \\ &= \rho^\top Z_\pi^{-1} (I - Z_\epsilon Z_\pi^{-1} + Z_\epsilon Z_\pi^{-1} Z_\epsilon Z_\pi^{-1}) \left(r_\pi + r_\epsilon - \tau h_\pi - \tau D h_\pi \epsilon - \frac{1}{2} \epsilon^\top \tau D^2 h_\pi \epsilon \right) \\ &\quad + O(\|\epsilon\|^3) \\ &= E(\pi) + w_\pi^\top [-Z_\epsilon Z_\pi^{-1} (r_\pi - \tau h_\pi) + (r_\epsilon - \tau D h_\pi \epsilon)] + w_\pi^\top \left(-\frac{1}{2} \epsilon^\top \tau D^2 h_\pi \epsilon \right) \\ &\quad + w_\pi^\top [-Z_\epsilon Z_\pi^{-1} (r_\epsilon - \tau D h_\pi \epsilon) + Z_\epsilon Z_\pi^{-1} Z_\epsilon Z_\pi^{-1} (r_\pi - \tau h_\pi)] + O(\|\epsilon\|^3), \end{aligned}$$

where $D h_\pi$ is a second-order tensor that maps from $S \times A$ to S , and $D^2 h_\pi$ is a third-order tensor that maps from $(S \times A)^{\otimes 2}$ to S . With this expansion, one can see that

$$\frac{\partial E}{\partial \pi_s^a} = (r_s^a - \tau(\log \pi_s^a + 1) - [(I - \gamma P^a)v_\pi]_s + c_s(\pi))(w_\pi)_s,$$

where $c_s(\pi)$ is a multiplier that depends only on s . Then at $\pi = \pi^*$,

$$\frac{\partial E}{\partial \pi_s^a} = (r_s^a - \tau(\log \pi_s^a + 1) - [(I - \gamma P^a)v_\pi]_s + c_s(\pi))(w_\pi)_s = 0.$$

Since $w_\pi = (I - \gamma P_\pi^\top)^{-1} \rho = \rho + \sum_{i=1}^{\infty} \gamma^i (P_\pi^\top)^i e$ and all elements of ρ are positive, we also know that all elements of w_π are positive. Thus at $\pi = \pi^*$,

$$r_s^a - \tau(\log \pi_s^a + 1) - [(I - \gamma P^a)v_\pi]_s + c_s(\pi) = 0.$$

Multiplying the left-hand side with ϵ_s^a and taking the sum over a , we obtain

$$(r_\epsilon - \tau Dh_\pi \epsilon - Z_\epsilon v_\pi)_s + c_s(\pi) \sum_{a \in A} \epsilon_s^a = 0 \quad \forall s, \quad \forall \epsilon.$$

Since $\sum_{a \in A} \epsilon_s^a = 0$ for any s and $v_\pi = Z_\pi^{-1}(r_\pi - \tau h_\pi)$, we have

$$r_\epsilon - \tau Dh_\pi \epsilon - Z_\epsilon Z_\pi^{-1}(r_\pi - \tau h_\pi) = 0 \quad \forall \epsilon,$$

at $\pi = \pi^*$, which proves (5.2).

Step 2: Derive the decomposition (2.3) with the obtained expansion and first-order condition. With (5.2), one can approximate the second-order term in (5.5) for π near π^* :

$$\begin{aligned} w_\pi^\top \left(-\frac{1}{2} \epsilon^\top \tau D^2 h_\pi \epsilon \right) + w_\pi^\top [-Z_\epsilon Z_\pi^{-1}(r_\epsilon - \tau Dh_\pi \epsilon) + Z_\epsilon Z_\pi^{-1} Z_\epsilon Z_\pi^{-1}(r_\pi - \tau h_\pi)] \\ = \frac{1}{2} \epsilon^\top \Lambda(\pi) \epsilon - w_\pi^\top Z_\epsilon Z_\pi^{-1}(r_\epsilon - \tau Dh_\pi \epsilon - Z_\epsilon Z_\pi^{-1}(r_\pi - \tau h_\pi)) \\ \approx \frac{1}{2} \epsilon^\top \Lambda(\pi) \epsilon. \end{aligned}$$

By (5.2) and that h is twice continuously differentiable, the approximate Hessian Λ converges to the true Hessian as π converges to π^* , and their difference $\Lambda(\pi) - D^2 E(\pi) = O(\|\pi - \pi^*\|)$. Hence, the second-order derivatives of $E(\pi)$ can be approximated by

$$(5.6) \quad \frac{\partial^2 E}{\partial \pi_s^a \partial \pi_t^b} \approx \Lambda_{(sa),(tb)} = -\tau \delta_{\{(sa),(tb)\}} \frac{(w_\pi)_s}{\pi_s^a},$$

from which we have shown that Λ is diagonal.

Step 3: Derive the approximate Newton flow and the policy update scheme with the obtained decomposition. Using this approximate second-order derivative as a preconditioner, w_π is canceled out in the policy gradient algorithm, which yields the gradient flow

$$\frac{d\pi_s^a}{dt} = \pi_s^a (r_s^a - \tau(\log \pi_s^a + 1) - [(I - \gamma P^a)v_\pi]_s + c_s(\pi)) / \tau.$$

Adopting the parameterization $\pi_s^a = \exp(\theta_s^a)$, we have

$$(5.7) \quad \frac{d\theta_s^a}{dt} = (r_s^a - \tau(\theta_s^a + 1) - [(I - \gamma P^a)v_\pi]_s + c_s(\pi)) / \tau.$$

With a learning rate η , this becomes

$$(5.8) \quad \theta_s^a \leftarrow \eta(r_s^a - \tau - [(I - \gamma P^a)v_\pi]_s + c_s(\pi))/\tau + (1 - \eta)\theta_s^a,$$

which corresponds to

$$(5.9) \quad \pi_s^a \leftarrow (\pi_s^a)^{1-\eta} \exp(\eta(r_s^a - \tau - [(I - \gamma P^a)v_\pi]_s + c_s(\pi))/\tau),$$

and $c_s(\pi)$ is determined by the condition that $\sum_{a \in A} \pi_s^a = 1$. Equivalently, we have

$$\begin{aligned} \pi_s^a &\leftarrow \frac{(\pi_s^a)^{1-\eta} \exp(\eta(r_s^a - \tau - [(I - \gamma P^a)v_\pi]_s)/\tau)}{\sum_{a \in A} (\pi_s^a)^{1-\eta} \exp(\eta(r_s^a - \tau - [(I - \gamma P^a)v_\pi]_s)/\tau)} \\ &= \frac{(\pi_s^a)^{1-\eta} \exp(\eta(r_s^a + (\gamma P^a v_\pi)_s)/\tau)}{\sum_{a \in A} (\pi_s^a)^{1-\eta} \exp(\eta(r_s^a + (\gamma P^a v_\pi)_s)/\tau)}, \end{aligned}$$

where we cancel out the factors independent of a and obtain (2.6). This finishes the proof. \square

5.2. Proof of Theorem 2.3.

Proof. Similar to (5.2), we first prove that for any ϵ with $\sum_{a \in A} \epsilon_s^a = 0$ and $|\epsilon_s^a| < \pi_s^a$, at $\pi = \pi^*$

$$(5.10) \quad r_\epsilon - \tau D h_\pi \epsilon - Z_\epsilon Z_\pi^{-1} (r_\pi - \tau h_\pi) = 0.$$

Similar to the proof of Theorem 2.1, by direct calculations, one can get

$$(5.11) \quad \frac{\partial E}{\partial \pi_s^a} = (r_s^a - \tau \phi'(\pi_s^a / \mu_s^a) - [(I - \gamma P^a)v_\pi]_s + c_s(\pi))(w_\pi)_s,$$

where $c_s(\pi)$ is a multiplier that only depends on s . Since all elements of w_π are positive, at $\pi = \pi^*$,

$$(5.12) \quad (r_s^a - \tau \phi'(\pi_s^a / \mu_s^a) - [(I - \gamma P^a)v_\pi]_s + c_s(\pi)) = 0.$$

By multiplying (5.12) with ϵ_s^a and summing over a , one can obtain

$$r_\epsilon - \tau D h_\pi \epsilon - Z_\epsilon Z_\pi^{-1} (r_\pi - \tau h_\pi) = 0 \quad \forall \epsilon,$$

at $\pi = \pi^*$, which proves (5.10). Since the only difference between the functional $E(\pi)$ defined here and the $E(\pi)$ in Theorem 2.1 lies in the regularizer h , one can still obtain the expansion:

$$\begin{aligned} E(\pi + \epsilon) - E(\pi) - w_\pi^\top [-Z_\epsilon Z_\pi^{-1} (r_\pi - \tau h_\pi) + (r_\epsilon - \tau D h_\pi \epsilon)] \\ = w_\pi^\top \left(-\frac{1}{2} \epsilon^\top \tau D^2 h_\pi \epsilon \right) - w_\pi^\top Z_\epsilon Z_\pi^{-1} (r_\epsilon - \tau D h_\pi \epsilon - Z_\epsilon Z_\pi^{-1} (r_\pi - \tau h_\pi)) + O(\|\epsilon\|^3) \\ = \frac{1}{2} \epsilon^\top \Lambda(\pi) \epsilon + O(\|\epsilon\|^2 \|\pi - \pi^*\|) + O(\|\epsilon\|^3). \end{aligned}$$

Hence we have $D^2 E(\pi) - \Lambda(\pi) = O(\|\pi - \pi^*\|)$. Using this expansion, one can derive an approximation for the second-order derivatives:

$$\frac{\partial^2 E}{\partial \pi_s^a \partial \pi_t^b} \approx \Lambda_{(sa), (tb)} = -\tau \delta_{\{(sa), (tb)\}} \frac{(w_\pi)_s \phi''(\pi_s^a / \mu_s^a)}{\mu_s^a},$$

which proves (2.9) and shows that Λ is diagonal. The approximate Newton flow thus becomes

$$\frac{d\pi_s^a}{dt} = \mu_s^a (\phi''(\pi_s^a/\mu_s^a))^{-1} (r_s^a - \tau \phi'(\pi_s^a/\mu_s^a) - [(I - \gamma P^a)v_\pi]_s + c_s(\pi))/\tau,$$

which proves (2.10), or equivalently,

$$(5.13) \quad \frac{d(\phi'(\pi_s^a/\mu_s^a))}{dt} = (r_s^a - \tau \phi'(\pi_s^a/\mu_s^a) - [(I - \gamma P^a)v_\pi]_s + c_s(\pi))/\tau.$$

Let $\theta_s^a = \phi'(\pi_s^a/\mu_s^a)$; then

$$\frac{d\theta_s^a}{dt} = (r_s^a - \tau \theta_s^a - [(I - \gamma P^a)v_\pi]_s + c_s(\pi))/\tau.$$

With a learning rate η , this becomes

$$\theta_s^a \leftarrow \eta(r_s^a - [(I - \gamma P^a)v_\pi]_s + c_s(\pi))/\tau + (1 - \eta)\theta_s^a,$$

which proves (2.11). \square

5.3. Proof of Lemma 2.4.

Proof. Let

$$g(x) = \mu_1 \psi(x + x_1) + \cdots + \mu_k \psi(x + x_k).$$

Since $\psi : (L, +\infty) \rightarrow (0, +\infty)$ is decreasing, $g(x)$ is positive and decreasing on $(L - \min_{1 \leq i \leq k} x_i, +\infty)$. When $x \rightarrow -\min_{1 \leq i \leq k} x_i$ from the right, $g(x) \rightarrow +\infty$ since at least one of the terms goes to $+\infty$. If $\min_{1 \leq i \leq k} \{\psi^{-1}(\frac{1}{k\mu_i}) - x_i\} \geq L - \min_{1 \leq i \leq k} x_i$, when $x = \min_{1 \leq i \leq k} \{\psi^{-1}(\frac{1}{k\mu_i}) - x_i\}$

$$\begin{aligned} g(x) &= \sum_{i=1}^k \mu_i \psi \left(\min_{1 \leq j \leq k} \left\{ \psi^{-1} \left(\frac{1}{k\mu_j} \right) - x_j \right\} + x_i \right) \\ &= \sum_{i=1}^k \mu_i \psi \left(\min_{1 \leq j \leq k} \left\{ \psi^{-1} \left(\frac{1}{k\mu_j} \right) - x_j \right\} + x_i - \psi^{-1} \left(\frac{1}{k\mu_i} \right) + \psi^{-1} \left(\frac{1}{k\mu_i} \right) \right) \\ &\geq \sum_{i=1}^k \mu_i \psi \left(\psi^{-1} \left(\frac{1}{k\mu_i} \right) \right) = \sum_{i=1}^k \mu_i \times \frac{1}{k\mu_i} = k \times \frac{1}{k} = 1. \end{aligned}$$

Since $\psi^{-1}(\frac{1}{k\mu_i}) \geq L$, we have $\max_{1 \leq i \leq k} \{\psi^{-1}(\frac{1}{k\mu_i}) - x_i\} \geq \max_{1 \leq i \leq k} \{L - x_i\} = L - \min_{1 \leq i \leq k} x_i$. Then when $x = \max_{1 \leq i \leq k} \{\psi^{-1}(\frac{1}{k\mu_i}) - x_i\}$,

$$g(x) = \sum_{i=1}^k \mu_i \psi \left(\max_{1 \leq j \leq k} \left\{ \psi^{-1} \left(\frac{1}{k\mu_j} \right) - x_j \right\} + x_i \right) \leq \sum_{i=1}^k \mu_i \psi \left(\psi^{-1} \left(\frac{1}{k\mu_i} \right) \right) = 1.$$

By the continuity of g , there exists a solution x to (2.17) on

$$\left[\max \left\{ L - \min_{1 \leq i \leq k} x_i, \min_{1 \leq i \leq k} \left\{ \psi^{-1} \left(\frac{1}{k\mu_i} \right) - x_i \right\} \right\}, \max_{1 \leq i \leq k} \left\{ \psi^{-1} \left(\frac{1}{k\mu_i} \right) - x_i \right\} \right],$$

and the solution is unique by the strict monotonicity of g on $(L - \min_{1 \leq i \leq k} x_i, \infty)$. \square

5.4. Proof of Proposition 2.5.

Proof. By Lemma 2.4 there is a unique solution \tilde{c}_s to the equation $\sum_{a \in A} \mu_s^a \psi(\tilde{c}_s + x_a) = 1$, where x_a is defined as in (2.16). Now update the policy by

$$\pi_s^a \leftarrow \mu_s^a \psi(\tilde{c}_s + x_a) = \sum_{a \in A} \mu_s^a \psi \left(-\frac{\eta}{\tau} c_s(\pi) - (1 - \eta) \phi'(\pi_s^a / \mu_s^a) - \frac{\eta}{\tau} (r_s^a - [(I - \gamma P^a) v_\pi]_s) \right);$$

one ensures that $\pi_s^a \geq 0$ and $\sum_{a \in A} \pi_s^a = 1$, and that the multiplier $c_s(\pi)$ with this property is unique. \square

5.5. Proof of Theorem 2.6.

Proof. In subsection 2.3, we have proved that the approximate Newton flow

$$\frac{d\pi_s^a}{dt} = \mu_s^a (\phi''(\pi_s^a / \mu_s^a))^{-1} (r_s^a - \tau \phi'(\pi_s^a / \mu_s^a) - [(I - \gamma P^a) v_\pi]_s + c_s(\pi)) / \tau$$

converges globally, so it suffices to show that the limiting policy is optimal. Denote the limiting policy by π^\diamond . Since $\mu_s^a > 0$ and $(\phi''((\pi^\diamond)_s^a / \mu_s^a))^{-1} > 0$, we have

$$(5.14) \quad r_s^a - \tau \phi'((\pi^\diamond)_s^a / \mu_s^a) - [(I - \gamma P^a) v_{\pi^\diamond}]_s + c_s(\pi^\diamond) = 0,$$

and $c_s(\pi^\diamond)$ is a multiplier that ensures $\sum_{a \in A} (\pi^\diamond)_s^a = 1$. From the theory of regularized MDP (cf. [9]), we know that the optimal policy π^* is the unique solution to the Bellman maximal equation:

$$(5.15) \quad v = \max_{\pi} r_{\pi} + \gamma P_{\pi} v - \tau h_{\pi}.$$

Since $v_{\pi^\diamond} = (I - \gamma P_{\pi^\diamond})^{-1} (r_{\pi^\diamond} - \tau h_{\pi^\diamond})$, we have $v_{\pi^\diamond} - \gamma P_{\pi^\diamond} v_{\pi^\diamond} = r_{\pi^\diamond} - \tau h_{\pi^\diamond}$, or equivalently

$$v_{\pi^\diamond} = r_{\pi^\diamond} + \gamma P_{\pi^\diamond} v_{\pi^\diamond} - \tau h_{\pi^\diamond}.$$

Thus it now suffices to show that π^\diamond is the optimizer of the constrained maximization problem $\max_{\pi} r_{\pi} + \gamma P_{\pi} v_{\pi^\diamond} - \tau h_{\pi}$, or in the component form

$$(5.16) \quad \max_{\pi} \sum_{a \in A} (r_s^a + \gamma (P^a v_{\pi^\diamond})_s) \pi_s^a - \tau \sum_{a \in A} \mu_s^a \phi(\pi_s^a / \mu_s^a).$$

Since ϕ is convex and μ is positive, $\tau \sum_{a \in A} \mu_s^a \phi(\pi_s^a / \mu_s^a)$ is also a convex function in π_s . By the theory of convex optimization (cf. [5, chapter 5]), the Karush–Kuhn–Tucker (KKT) condition is sufficient for the optimality when the objective function is convex, and the KKT condition for the problem (5.16) is

$$\begin{aligned} r_s^a + \gamma (P^a v_{\pi^\diamond})_s - \tau \phi'(\pi_s^a / \mu_s^a) + \lambda_s &= 0, \\ \sum_{a \in A} \pi_s^a &= 1, \\ \pi_s^a &\geq 0, \end{aligned}$$

where λ_s is the Lagrange multiplier. Now let $\pi = \pi^\diamond$ and $\lambda_s = c_s(\pi^\diamond) - (v_{\pi^\diamond})_s$. From the first-order condition (5.14), one can directly observe that the KKT condition above is satisfied, which makes π^\diamond the optimizer for (5.16) and v_{π^\diamond} the solution to the Bellman equation (5.15). Thus v_{π^\diamond} and π^\diamond are indeed the optimal value function and the optimal policy, which closes the proof. \square

5.6. Proof of Theorem 3.1.

Proof. The proof is divided into three steps. First, we present some results needed to prove the local convergence. Using induction in the second step, we then demonstrate the local convergence of $\pi^{(k)}$ to π^* . Finally, we prove that the convergence rate is quadratic.

Step 1. Preparation. From the scheme

$$(5.17) \quad \nabla\Phi(\pi^{(k+1)}) - \nabla\Phi(\pi^{(k)}) = - \left(f(\pi^{(k)}) + \nabla\Phi(\pi^{(k)}) - B^\top c(\pi^{(k)}) \right),$$

one can obtain the inequality

$$(5.18) \quad \begin{aligned} \|f(\pi^{(k+1)}) - f(\pi^{(k)})\| &\geq \frac{(f(\pi^{(k)}) - f(\pi^{(k+1)}))^\top (\pi^{(k+2)} - \pi^{(k+1)})}{\|\pi^{(k+2)} - \pi^{(k+1)}\|} \\ &= \frac{-(f(\pi^{(k+1)}) + \nabla\Phi(\pi^{(k+1)}) - B^\top c(\pi^{(k)}))^\top (\pi^{(k+2)} - \pi^{(k+1)})}{\|\pi^{(k+2)} - \pi^{(k+1)}\|} \\ &= \frac{-(f(\pi^{(k+1)}) + \nabla\Phi(\pi^{(k+1)}) - B^\top c(\pi^{(k+1)}))^\top (\pi^{(k+2)} - \pi^{(k+1)})}{\|\pi^{(k+2)} - \pi^{(k+1)}\|} \\ &= \frac{(\nabla\Phi(\pi^{(k+2)}) - \nabla\Phi(\pi^{(k+1)}))^\top (\pi^{(k+2)} - \pi^{(k+1)})}{\|\pi^{(k+2)} - \pi^{(k+1)}\|}, \end{aligned}$$

where we use the constraint $B\pi^{(k+1)} = B\pi^{(k+2)} = \mathbf{1}_{|S|}$. By directly calculating $\nabla^2\Phi$ from the definition of Φ , we can see that $\nabla^2\Phi$ is diagonal and Φ is strongly convex since ϕ is strongly convex. As a result, there is some constant $\omega > 0$ such that

$$(5.19) \quad (\nabla\Phi(\pi) - \nabla\Phi(\tilde{\pi}))^\top (\pi - \tilde{\pi}) \geq \omega \|\pi - \tilde{\pi}\|^2$$

for any π and $\tilde{\pi}$. Thus from (5.18) one can deduce that

$$(5.20) \quad \begin{aligned} \|f(\pi^{(k+1)}) - f(\pi^{(k)})\| &\geq \frac{(\nabla\Phi(\pi^{(k+2)}) - \nabla\Phi(\pi^{(k+1)}))^\top (\pi^{(k+2)} - \pi^{(k+1)})}{\|\pi^{(k+2)} - \pi^{(k+1)}\|} \\ &\geq \frac{\omega \|\pi^{(k+2)} - \pi^{(k+1)}\|^2}{\|\pi^{(k+2)} - \pi^{(k+1)}\|} = \omega \|\pi^{(k+2)} - \pi^{(k+1)}\|. \end{aligned}$$

Let K be a closed set contained in $\{\pi : B^\top \pi = \mathbf{1}_{|S|}, \pi_s^a > 0\}$ such that K contains a ball centered at π^* with radius $\delta_0 > 0$, which is guaranteed to exist since $(\pi^*)_s^a > 0$. Define the conjugate function of Φ as

$$(5.21) \quad \Phi^*(x) = \max_{\pi \in \Delta} \left[\sum_{s \in S, a \in A} \pi_s^a x_{sa} - \Phi(\pi) \right],$$

where $\Delta = \{\pi : B^\top \pi = \mathbf{1}_{|S|}, \pi_s^a \geq 0\}$. Since Φ is ω -strongly convex and Δ is a closed convex set, it can be deduced from classical convex analysis results (see [11], for example) that $\nabla\Phi^*$ is $\frac{1}{\omega}$ -Lipschitz continuous, and $\pi = \nabla\Phi^*(\nabla\Phi(\pi))$. Moreover, from the definition of Φ^* one can observe that $\Phi^*(x + B^\top c) = \Phi^*(x) + \mathbf{1}_{|S|}^\top c$, and thus $\nabla\Phi^*(x + B^\top c) = \nabla\Phi^*(x)$. Similar results concerning the conjugate functions have also been used in [18] and [9]. Thanks to the properties of Φ^* , we have the identity

$$(5.22) \quad \pi^{(k+1)} = \nabla\Phi^*(\nabla\Phi(\pi^{(k+1)})) = \nabla\Phi^*(B^\top c(\pi^{(k)}) - f(\pi^{(k)})) = \nabla\Phi^*(-f(\pi^{(k)})),$$

where we have used the update scheme (5.17). Moreover, by the result of Theorem 2.6 we have $\frac{d(\nabla\Phi(\pi))}{dt} = 0$ at $\pi = \pi^*$, so $f(\pi^*) + \nabla\Phi(\pi^*) = B^\top c(\pi^*)$ and

$$(5.23) \quad \pi^* = \nabla\Phi^*(\nabla\Phi(\pi^*)) = \nabla\Phi^*(B^\top c(\pi^*) - f(\pi^*)) = \nabla\Phi^*(-f(\pi^*)).$$

Since $\nabla\Phi^*$ and $-f$ are continuous on K , it can be concluded from (5.22) and (5.23) that there exists $\delta_1 > 0$ such that $\|\pi^{(k+1)} - \pi^*\| < \frac{1}{16} \min\{\frac{\omega}{M}, \delta_0\}$ whenever $\|\pi^{(k)} - \pi^*\| \leq \delta_1$, where $M = \sup_{\pi \in K} |\nabla^2 f(\pi)|$.

Step 2. Prove the convergence by induction. Now let $\delta = \min\{\frac{\omega}{16M}, \frac{\delta_0}{16}, \delta_1\}$. Assuming that $\|\pi^{(0)} - \pi^*\| < \delta$, we proceed to prove that $\|\pi^{(k)} - \pi^*\| \leq \frac{1}{2} \min\{\frac{\omega}{M}, \delta_0\}$ for any k by induction. To this end, we first strengthen the induction hypothesis to

$$(5.24) \quad \begin{aligned} \|\pi^{(k)} - \pi^*\| &\leq \left(\frac{1}{2} - \frac{1}{2^{k+2}}\right) \min\left\{\frac{\omega}{M}, \delta_0\right\}, \quad k = 0, 1, \dots, n, \\ \|\pi^{(k+1)} - \pi^{(k)}\| &\leq \left(\frac{1}{2} - \frac{1}{2^{k+2}}\right) \|\pi^{(k)} - \pi^{(k-1)}\|, \quad k = 1, 2, \dots, n. \end{aligned}$$

We first prove (5.24) for $n = 1$. Note that

$$(5.25) \quad \|\pi^{(0)} - \pi^*\| \leq \delta \leq \left(\frac{1}{2} - \frac{1}{2^{0+2}}\right) \min\left\{\frac{\omega}{M}, \delta_0\right\},$$

by the definition of δ , and that

$$(5.26) \quad \|\pi^{(1)} - \pi^*\| \leq \frac{1}{16} \min\left\{\frac{\omega}{M}, \delta_0\right\} \leq \left(\frac{1}{2} - \frac{1}{2^{1+2}}\right) \min\left\{\frac{\omega}{M}, \delta_0\right\},$$

by the definition of δ_1 and the fact that $\|\pi^{(0)} - \pi^*\| \leq \delta_1$. Then

$$(5.27) \quad \|\pi^{(1)} - \pi^{(0)}\| \leq \|\pi^{(1)} - \pi^*\| + \|\pi^{(0)} - \pi^*\| \leq \frac{1}{8} \min\left\{\frac{\omega}{M}, \delta_0\right\}.$$

In addition, from (5.25) and (5.26) we know that $\pi^{(0)} \in K$ and $\pi^{(1)} \in K$. Then by (5.20),

$$(5.28) \quad \begin{aligned} \|\pi^{(2)} - \pi^{(1)}\| &\leq \frac{1}{\omega} \|f(\pi^{(1)}) - f(\pi^{(0)})\| \\ &= \frac{1}{\omega} \|\nabla f(\pi^{(0)} + \zeta((\pi^{(1)} - \pi^{(0)}))) (\pi^{(1)} - \pi^{(0)})\| \\ &= \frac{1}{\omega} \|(\nabla f(\pi^{(0)} + \zeta((\pi^{(1)} - \pi^{(0)}))) - \nabla f(\pi^*)) (\pi^{(1)} - \pi^{(0)})\| \\ &\leq \frac{M}{\omega} \|(\pi^{(0)} + \zeta((\pi^{(1)} - \pi^{(0)}))) - \pi^*\| \|\pi^{(1)} - \pi^{(0)}\| \\ &\leq \frac{M}{\omega} \max\left\{\|\pi^{(1)} - \pi^*\|, \|\pi^{(0)} - \pi^*\|\right\} \|\pi^{(1)} - \pi^{(0)}\|, \end{aligned}$$

where we have used the identity $\nabla f(\pi^*)(\pi^{(1)} - \pi^{(0)}) = 0$ and the fact that $\pi^{(1)}$ and $\pi^{(0)}$ are contained in K . In fact, we can prove that

$$\nabla f(\pi^*)(\pi^{(k+1)} - \pi^{(k)}) = 0 \quad \text{for any } k,$$

as follows. Since $f(\pi)_{sa} = -(r_s^a - ((I - \gamma P^a)v_\pi)_s)$ has a similar form with $E(\pi)$, we can directly obtain $\nabla f(\pi)$:

$$(5.29) \quad (\nabla f(\pi))_{sa, tb} = \lambda_{sa, t}(\pi) (-f(\pi)_{tb} + \tilde{c}(\pi)_t - \nabla\Phi(\pi)_{tb}),$$

where $\lambda_{sa,t}(\pi) = Z_\pi^{-\top} \tilde{\rho}_{sa}$ and $\tilde{\rho}_{sa}$ is the s th row of $I - \gamma P^a$. Since $f(\pi^*) + \nabla \Phi(\pi^*) = B^\top c(\pi^*)$, we have

$$\begin{aligned}
 & (\nabla f(\pi^*)(\pi^{(k+1)} - \pi^{(k)}))_{sa} \\
 &= \sum_{t \in S, b \in A} \lambda_{sa,t}(\pi^*) (-f(\pi^*)_{tb} + \tilde{c}(\pi^*)_t - \nabla \Phi(\pi^*)_{tb}) (\pi_{tb}^{(k+1)} - \pi_{tb}^{(k)}) \\
 (5.30) \quad &= \sum_{t \in S, b \in A} \lambda_{sa,t}(\pi^*) (\tilde{c}(\pi^*)_t - c(\pi^*)_t) (\pi_{tb}^{(k+1)} - \pi_{tb}^{(k)}) \\
 &= \sum_{t \in S} \left[\left(\lambda_{sa,t}(\pi^*) (\tilde{c}(\pi^*)_t - c(\pi^*)_t) \right) \left(\sum_{b \in A} (\pi_{tb}^{(k+1)} - \pi_{tb}^{(k)}) \right) \right] \\
 &= 0,
 \end{aligned}$$

where the last equality results from the fact that $\sum_{b \in A} \pi_{tb}^{(k+1)} = \sum_{b \in A} \pi_{tb}^{(k)} = 1$ for any t . Now from (5.25), (5.26), and (5.28), we obtain

$$\begin{aligned}
 (5.31) \quad \|\pi^{(2)} - \pi^{(1)}\| &\leq \frac{M}{\omega} \max \left\{ \|\pi^{(1)} - \pi^*\|, \|\pi^{(0)} - \pi^*\| \right\} \|\pi^{(1)} - \pi^{(0)}\| \\
 &\leq \frac{M}{\omega} \cdot \frac{1}{16} \min \left\{ \frac{\omega}{M}, \delta_0 \right\} \|\pi^{(1)} - \pi^{(0)}\| \\
 &\leq \left(\frac{1}{2} - \frac{1}{2^{1+2}} \right) \|\pi^{(1)} - \pi^{(0)}\|.
 \end{aligned}$$

Now, assuming that the induction hypothesis (5.24) holds for some $n \geq 1$, we have

$$\begin{aligned}
 (5.32) \quad \|\pi^{(n+1)} - \pi^*\| &\leq \|\pi^{(n+1)} - \pi^{(n)}\| + \|\pi^{(n)} - \pi^*\| \\
 &\leq \left(\prod_{k=1}^n \left(\frac{1}{2} - \frac{1}{2^{k+2}} \right) \right) \|\pi^{(1)} - \pi^{(0)}\| + \|\pi^{(n)} - \pi^*\| \\
 &\leq \frac{1}{2^n} \cdot \frac{1}{8} \min \left\{ \frac{\omega}{M}, \delta_0 \right\} + \left(\frac{1}{2} - \frac{1}{2^{n+2}} \right) \min \left\{ \frac{\omega}{M}, \delta_0 \right\} \\
 &= \left(\frac{1}{2} - \frac{1}{2^{n+2}} + \frac{1}{2^{n+3}} \right) \min \left\{ \frac{\omega}{M}, \delta_0 \right\} \\
 &= \left(\frac{1}{2} - \frac{1}{2^{(n+1)+2}} \right) \min \left\{ \frac{\omega}{M}, \delta_0 \right\},
 \end{aligned}$$

which also implies that $\pi^{(n+1)} \in K$. Now using the same reasoning as (5.28) but with $(\pi^{(0)}, \pi^{(1)}, \pi^{(2)})$ replaced by $(\pi^{(n)}, \pi^{(n+1)}, \pi^{(n+2)})$, one obtains

$$(5.33) \quad \|\pi^{(n+2)} - \pi^{(n+1)}\| \leq \frac{M}{\omega} \max \left\{ \|\pi^{(n+1)} - \pi^*\|, \|\pi^{(n)} - \pi^*\| \right\} \|\pi^{(n+1)} - \pi^{(n)}\|.$$

After plugging (5.32) and the induction hypothesis into this inequality, we get

$$\begin{aligned}
 (5.34) \quad \|\pi^{(n+2)} - \pi^{(n+1)}\| &\leq \frac{M}{\omega} \cdot \left(\frac{1}{2} - \frac{1}{2^{n+3}} \right) \min \left\{ \frac{\omega}{M}, \delta_0 \right\} \|\pi^{(n+1)} - \pi^{(n)}\| \\
 &\leq \left(\frac{1}{2} - \frac{1}{2^{(n+1)+2}} \right) \|\pi^{(n+1)} - \pi^{(n)}\|.
 \end{aligned}$$

With (5.32) and (5.34) we have shown that (5.24) holds with n replaced by $n+1$. As a result, (5.24) holds for any n . From the second inequality in (5.24), it is clear that

$\pi^{(k)}$ converges (at least exponentially fast). Denote the limit of $\pi^{(k)}$ by $\tilde{\pi}$ for now; we obtain from (5.17) that

$$(5.35) \quad f(\tilde{\pi}) + \nabla\Phi(\tilde{\pi}) - B^\top c(\tilde{\pi}) = 0,$$

and thus $\tilde{\pi} = \pi^*$ by Theorem 2.6.

Step 3. Prove the convergence rate is quadratic. Since $\pi^{(k)}$ converges to π^* and ∇f is Lipschitz continuous on K , we have

$$(5.36) \quad \lim_{k \rightarrow \infty} \frac{f(\pi^{(k+1)}) - f(\pi^{(k)}) - \nabla f(\pi^*) (\pi^{(k+1)} - \pi^{(k)})}{\|\pi^{(k+1)} - \pi^{(k)}\|} = 0.$$

On the other hand, we have

$$(5.37) \quad \begin{aligned} & f(\pi^{(k+1)}) - f(\pi^{(k)}) - \nabla f(\pi^*) (\pi^{(k+1)} - \pi^{(k)}) \\ &= f(\pi^{(k+1)}) + \nabla\Phi(\pi^{(k+1)}) - B^\top c(\pi^{(k)}) - \nabla f(\pi^*) (\pi^{(k+1)} - \pi^{(k)}) \\ &= f(\pi^{(k+1)}) + \nabla\Phi(\pi^{(k+1)}) - B^\top c(\pi^{(k)}), \end{aligned}$$

where we have used (5.30). Combining with (5.36), we arrive at

$$(5.38) \quad \lim_{k \rightarrow \infty} \frac{f(\pi^{(k+1)}) + \nabla\Phi(\pi^{(k+1)}) - B^\top c(\pi^{(k)})}{\|\pi^{(k+1)} - \pi^{(k)}\|} = 0.$$

With the last three lines of (5.18), we obtain

$$(5.39) \quad \lim_{k \rightarrow \infty} \frac{(\nabla\Phi(\pi^{(k+2)}) - \nabla\Phi(\pi^{(k+1)}))^\top (\pi^{(k+2)} - \pi^{(k+1)})}{\|\pi^{(k+1)} - \pi^{(k)}\| \|\pi^{(k+2)} - \pi^{(k+1)}\|} = 0,$$

by multiplying the unit vector $\frac{\pi^{(k+2)} - \pi^{(k+1)}}{\|\pi^{(k+2)} - \pi^{(k+1)}\|}$ to the fraction in (5.38). Then by (5.19) we get

$$(5.40) \quad 0 = \lim_{k \rightarrow \infty} \frac{\|\pi^{(k+2)} - \pi^{(k+1)}\|^2}{\|\pi^{(k+1)} - \pi^{(k)}\| \|\pi^{(k+2)} - \pi^{(k+1)}\|} = \lim_{k \rightarrow \infty} \frac{\|\pi^{(k+2)} - \pi^{(k+1)}\|}{\|\pi^{(k+1)} - \pi^{(k)}\|},$$

from which we can conclude that $\pi^{(k)}$ converges to π^* superlinearly, i.e.,

$$(5.41) \quad \lim_{k \rightarrow \infty} \frac{\|\pi^{(k+1)} - \pi^*\|}{\|\pi^{(k)} - \pi^*\|} = 0.$$

In fact, for any ϵ (assume $\epsilon < 1/2$ without loss of generality), there is some $k(\epsilon)$ such that for any $k > k(\epsilon)$, $\frac{\|\pi^{(k+2)} - \pi^{(k+1)}\|}{\|\pi^{(k+1)} - \pi^{(k)}\|} < \epsilon$; then for any $k > k(\epsilon)$

$$\begin{aligned} \|\pi^{(k+1)} - \pi^*\| &\leq \sum_{n=k+1}^{\infty} \|\pi^{(n+1)} - \pi^{(n)}\| \leq \sum_{n=k+1}^{\infty} \epsilon^{n-k} \|\pi^{(k+1)} - \pi^{(k)}\| \\ &\leq \frac{\epsilon}{1-\epsilon} \|\pi^{(k+1)} - \pi^{(k)}\| \leq 2\epsilon \|\pi^{(k+1)} - \pi^{(k)}\|. \end{aligned}$$

Then

$$\begin{aligned} \|\pi^{(k)} - \pi^*\| &\geq \|\pi^{(k+1)} - \pi^{(k)}\| - \|\pi^{(k+1)} - \pi^*\| \\ &\geq \left(\frac{1}{2\epsilon} - 1\right) \|\pi^{(k+1)} - \pi^*\|. \end{aligned}$$

For any $G > 0$, take $\epsilon = 1/(2G + 2)$; then for any $k > k(\epsilon)$,

(5.42)

$$\left\| \pi^{(k)} - \pi^* \right\| \geq \left(\frac{1}{2\epsilon} - 1 \right) \left\| \pi^{(k+1)} - \pi^* \right\| = (G + 1) \left\| \pi^{(k+1)} - \pi^* \right\| > G \left\| \pi^{(k+1)} - \pi^* \right\|,$$

which shows that $\lim_{k \rightarrow \infty} \frac{\left\| \pi^{(k)} - \pi^* \right\|}{\left\| \pi^{(k+1)} - \pi^* \right\|} = +\infty$ and thus (5.41) holds. Now, from (5.17) and (5.30) we have

$$\begin{aligned} & \left\| f(\pi^{(k+1)}) + \nabla \Phi(\pi^{(k+1)}) - B^\top c(\pi^{(k)}) \right\| \\ &= \left\| f(\pi^{(k+1)}) - f(\pi^{(k)}) - \nabla f(\pi^*) \left(\pi^{(k+1)} - \pi^{(k)} \right) \right\| \\ &= \left\| \left(\int_0^1 \left[\nabla f(\pi^{(k)} + t(\pi^{(k+1)} - \pi^{(k)})) - \nabla f(\pi^*) \right] dt \right) \left(\pi^{(k+1)} - \pi^{(k)} \right) \right\| \\ &\leq \tilde{C} \left\| \pi^{(k)} - \pi^* \right\| \left\| \pi^{(k+1)} - \pi^{(k)} \right\| \end{aligned}$$

for some constant \tilde{C} , where we used (5.41) and the Lipschitz continuity of ∇f in the last equality. Multiplying both sides by $\pi^{(k+2)} - \pi^{(k+1)}$, and by (5.19) and the last three lines of (5.18), we have

$$\begin{aligned} & \omega \left\| \pi^{(k+2)} - \pi^{(k+1)} \right\|^2 \\ &\leq \left(\nabla \Phi(\pi^{(k+2)}) - \nabla \Phi(\pi^{(k+1)}) \right)^\top \left(\pi^{(k+2)} - \pi^{(k+1)} \right) \\ &= \left(f(\pi^{(k+1)}) + \nabla \Phi(\pi^{(k+1)}) - B^\top c(\pi^{(k)}) \right)^\top \left(\pi^{(k+2)} - \pi^{(k+1)} \right) \\ &\leq \tilde{C} \left\| \pi^{(k)} - \pi^* \right\| \left\| \pi^{(k+1)} - \pi^{(k)} \right\| \left\| \pi^{(k+2)} - \pi^{(k+1)} \right\|, \end{aligned}$$

which implies that

$$(5.43) \quad \left\| \pi^{(k+2)} - \pi^{(k+1)} \right\| \leq \tilde{C} \left\| \pi^{(k)} - \pi^* \right\| \left\| \pi^{(k+1)} - \pi^{(k)} \right\|,$$

with some constant \tilde{C} . From (5.41), we have

$$(5.44) \quad \lim_{k \rightarrow \infty} \frac{\left\| \pi^{(k)} - \pi^{(k+1)} \right\|}{\left\| \pi^{(k)} - \pi^* \right\|} = \lim_{k \rightarrow \infty} \frac{\left\| \pi^{(k+1)} - \pi^{(k+2)} \right\|}{\left\| \pi^{(k+1)} - \pi^* \right\|} = 1.$$

Combining this with (5.43) leads to

$$(5.45) \quad \left\| \pi^{(k+1)} - \pi^* \right\| \leq C \left\| \pi^{(k)} - \pi^* \right\|^2$$

for some constant C , which closes the proof. \square

6. Conclusion and discussion. This paper presents a fast approximate Newton method for the policy gradient algorithm with provable quadratic convergence. The proposed method gives a systematic theory that includes the well-known natural policy gradient algorithm as a particular case and naturally extends to other regularizers such as the reverse KL divergence, the Hellinger divergence, and the α -divergence.

With a relatively simple proof, we show the local quadratic convergence of the proposed approximate Newton method and the global convergence of the approximate

Newton gradient flow to the optimal solution. The quadratic convergence is confirmed numerically on both medium and large sparse models. In contrast with mirror descent type first-order methods (e.g., [40]) that take up to tens of thousands of iterations even with a manually tuned learning rate, the proposed approximate Newton algorithms typically converge in fewer than 10 iterations, despite the significant discount rate (≈ 1) and small regularization coefficient (≈ 0).

For future work, we plan to adapt the technique used here to other gradient-based algorithms for solving the MDP problems. Other forms of f -divergence can also be included. An interesting direction is to apply different types of numerical schemes for ordinary differential equations to the approximate Newton gradient flow presented in subsection 2.3, which can help obtain a good initial policy such that the discrete approximate Newton method can achieve fast quadratic convergence. Another direction is considering continuous-state-space MDP problems by leveraging function approximation, effective spatial discretization, or model reduction.

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