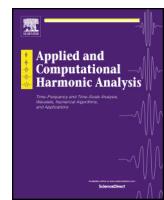




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Letter to the Editor

A note on spike localization for line spectrum estimation [☆]Haoya Li ^{*}, Hongkang Ni, Lexing Ying*Stanford University, Stanford, CA 94305, United States of America*

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ABSTRACT

This note considers the problem of approximating the locations of dominant spikes for a probability measure from noisy spectrum measurements under the condition of residue signal, significant noise level, and no minimum spectrum separation. We show that the simple procedure of thresholding the smoothed inverse Fourier transform allows for approximating the spike locations rather accurately.

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1. Introduction

In this note, we consider the problem of recovering the support of dominant spikes in an unknown probability measure f_* on \mathbb{R} . Let f_* be a probability measure of the form

$$f_*(x) = \sum_{s=1}^S w_s \delta_{x_s}(x) + r(x) \quad (1)$$

where the following conditions hold:

- S is the total number of dominant spikes, x_s are the spike locations, and all weights w_s are bounded by $\beta > 0$ from below, i.e., $w_s \geq \beta > 0$, $1 \leq s \leq S$;

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- r is a residue where its total variation is upper bounded by a constant ω less than β , i.e., $r(\mathbb{R}) = \int_{\mathbb{R}} dr(x) \leq \omega < \beta$.

The observed data is a signal $y(k)$ for $k \in [-K, K]$ that satisfies $|y(k) - \hat{f}_*(k)| \leq \epsilon$, where $\hat{f}_*(k) = \int_{\mathbb{R}} e^{-2\pi i x k} f_*(x) dx$ is the Fourier transform of f_* . The goal is to recover the support $X_* \equiv \{x_s\}$ of the spikes with a certain accuracy. We are particularly interested in the following question: for what ϵ can we estimate the spike locations x_s within precision $O(1/K)$? In this note, we show that when $\epsilon = O(\beta - \omega)$, a simple routine estimates the spike locations within distance $O(K^{-1} \log(1/(\beta - \omega)))$.

In practice, it is more relevant to consider a periodic version of the problem, in which case f_* is a probability measure on the periodic interval $\mathbb{T} \equiv [-\frac{1}{2}, \frac{1}{2}]$ that satisfies the conditions listed above, except that the residue r satisfies $r(\mathbb{T}) \leq \omega < \beta$ instead. The observed data is a vector $(y(k))_{k=-K}^K$ such that $|y(k) - \hat{f}_*(k)| \leq \epsilon$ for $k \in \{-K, \dots, K\}$, where $\hat{f}_* = \int_{\mathbb{T}} e^{-2\pi i x k} f_*(x) dx$ is the Fourier series coefficients of f_* . We show that a similar result can be obtained: when $\epsilon = O(\beta - \omega)$ we can estimate the spike locations within distance $O(K^{-1} \log(1/(\beta - \omega)))$ as long as $K = \Omega(\log(1/(\beta - \omega)))$.

This periodic setting is of more practical importance in signal processing. For example, estimating the low-lying eigenvalues $\{\lambda_s\}_{s=1}^S$ of a Hamiltonian H is an important problem in quantum computing [36,41], where one applies $e^{2\pi i H t}$ to $|\psi\rangle = \sum_{s=1}^S c_s |\psi_s\rangle + c_{\text{res}} |\psi_{\text{res}}\rangle$ that is an imperfect superposition of eigenstates $|\psi_s\rangle$, and the residue $c_{\text{res}} |\psi_{\text{res}}\rangle$ is orthogonal to all $|\psi_s\rangle$'s. The data obtained is the noisy Fourier transform at frequency k of $f_*(x) \equiv \sum_{s=1}^S |c_s|^2 \delta_{\lambda_s}(x) + r(x)$, where $r(x)$ corresponds to the residue $c_{\text{res}} |\psi_{\text{res}}\rangle$ in $|\psi\rangle$. The recovered support then becomes an estimation of the eigenvalues $\{\lambda_s\}_{s=1}^S$.

1.1. Related work

The problem considered here is a special version of the line spectrum estimation problem. Many algorithms have been proposed for this important task in signal processing, dating back to Prony's method [37]. Prony's method recovers the spectrum exactly using noiseless signals, but it is known to be unstable in the presence of noise. More robust algorithms have been developed, including the matrix pencil method [22], MUSIC [40], ESPRIT [38] and other subspace methods [6]. The modern convex relaxation approach, optimizing the ℓ_1 , total variation, and atomic norms, has been extensively developed in [5,7,8,29,42,43], to name a few.

The most active area of theoretical analysis in line spectrum estimation is super-resolution [1–4,7–9, 11,12,14–20,23–28,30–32,34,35,39,42], where the goal is to recover the spectrum when the minimal separation is smaller than the Rayleigh distance $O(1/K)$. The first work on super-resolution stability was [14], where Donoho introduced the concept of the Rayleigh index and demonstrated the connection between the Rayleigh index, super-resolution factor (SRF), and the allowed perturbation size. In [7,8], Candes and Fernandez-Granda demonstrated that the convex relaxation method could recover the spike locations in both noiseless and noisy situations if the spikes are separated by $2/K$. The optimal separation result ($\approx 1/K$) was obtained by Moitra in [33]. The convex relaxation approach is shown in [35] to achieve near-optimal worst-case performance. A lower bound for reconstruction error is given in [1], and the minimax error rates for reconstruction have been obtained in [4]. The application of MUSIC and ESPRIT in this setting has been investigated in [27,32]. The recent literature on super-resolution is vast, and we refer interested readers to review papers and tutorials such as [10].

Among these works, several papers have discussed the recovery of positive spikes. For example, [35], and [34] analyzed individual spike recovery errors in terms of the super-resolution factor under Rayleigh regularity assumptions. The result in [39] shows that the spectrum can be exactly recovered without assuming spectral gaps if the observed signal is a noiseless superposition of certain point spread functions, and in particular, Gaussian point spread functions. The BLASSO algorithm is shown in [13] to be able to recover positive sources if the noise is of the order $O(\Delta^{2S-1})$, where S is the number of spikes and Δ is the spectral gap.

1.2. Comparison

The setting of this note is somewhat different from the setup considered in the majority of works mentioned above: we do not assume any separation in the support X_* or any known structure of the noise. In addition, the total mass of the residue $r(x)$ can approach β . These conditions make our problem more difficult, thus preventing the direct application of many existing algorithms based on sufficient support separation. As a result, the recovery criteria pursued in this note are somewhat weaker than the ones in the related works above: we ask the estimated support X_e to be close to X_* , i.e., $\max_{x \in X_e} \text{dist}(x, X_*)$ and $\max_{x \in X_*} \text{dist}(x, X_e)$ are required to be small.

We would like to point out that the conditions for (1) are crucial to the tractability of the problem considered in this note. First, it is important that the measure f_* is positive. Otherwise, it would be impossible to identify nearby spikes with opposite amplitudes since they would cancel under noise and become invisible in the signal. It is also key to ensure that w_s has a positive lower bound β since small spikes are impossible to detect from the residue and the noise. Finally, rather than asking for the individual spike locations $\{x_s\}$, one can only seek an estimation of the whole support X_* since it is impossible to distinguish between arbitrarily close (even positive) spikes under noise.

The rest of the paper is organized as follows. Section 2 considers the problem of recovering a measure on \mathbb{R} . In Section 3, we deal with the recovery in the interval case.

2. Real line case

This section considers the real line case. We consider a simple method that utilizes Gaussian smoothing and inverse Fourier transform. Denote by ϕ and $\hat{\phi}$ the Gaussian density function and its Fourier transform:

$$\phi(x) = \frac{1}{\sigma/K} \exp\left(-\pi \frac{x^2}{(\sigma/K)^2}\right), \quad \hat{\phi}(k) = \exp(-\pi(k\sigma/K)^2) \quad (2)$$

where $\sigma = \sqrt{\frac{1}{\pi} \log \frac{6}{\beta-\omega}}$. Let us introduce

$$X_e = \left\{ x : \left| \int_{|k| \leq K} y(k) \hat{\phi}(k) e^{2\pi i k x} dk \right| > \frac{(2\beta + \omega)K}{3\sigma} \right\}. \quad (3)$$

The following theorem states that X_* and X_e are close.

Theorem 1. Suppose $\epsilon \leq \frac{\beta-\omega}{6}$. Then $X_* \subset X_e$ and for any $x \in X_e$, $\text{dist}(x, X_*) \leq \tau/K$ with $\tau = \frac{1}{\pi} \log \frac{6}{\beta-\omega}$.

Proof. Since $|y(k) - \hat{f}_*(k)| \leq \epsilon$ for $|k| \leq K$, direct calculations show that for any x

$$\begin{aligned} & \left| \int_{|k| \leq K} y(k) \hat{\phi}(k) e^{2\pi i k x} dk - \phi * f_*(x) \right| \\ &= \left| \int_{|k| \leq K} (y(k) - \hat{f}_*(k)) \hat{\phi}(k) e^{2\pi i k x} dk - \int_{|k| > K} \hat{f}_*(k) \hat{\phi}(k) e^{2\pi i k x} dk \right| \\ &\leq \left| \int_{|k| \leq K} (y(k) - \hat{f}_*(k)) \hat{\phi}(k) e^{2\pi i k x} dk \right| + \left| \int_{|k| > K} \hat{f}_*(k) \hat{\phi}(k) e^{2\pi i k x} dk \right| \end{aligned} \quad (4)$$

$$\begin{aligned} &\leq \epsilon \int_{|k| \leq K} |\hat{\phi}(k)| dk + \int_{|k| > K} |\hat{\phi}(k)| dk < \epsilon \int_{-\infty}^{\infty} \exp(-\pi(k\sigma/K)^2) dk + \frac{K}{\sigma} \exp(-\pi\sigma^2) \\ &\leq \epsilon \frac{K}{\sigma} + \frac{K}{\sigma} \exp(-\pi\sigma^2). \end{aligned}$$

Here we have used the following result for the complementary error function:

$$\int_K^{\infty} e^{-\pi(\frac{\sigma}{K}k)^2} dk \leq \frac{K}{2\sigma} e^{-\pi\sigma^2}, \text{ for any } K \geq 0.$$

We derive this bound here for completeness. Let

$$h(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{\frac{2}{\pi}} \int_x^{\infty} e^{-\frac{1}{2}t^2} dt},$$

then $h(0) = 1$, and a direct calculation gives

$$h'(x) = \frac{e^{-\frac{x^2}{2}} (e^{-\frac{x^2}{2}} - x \int_x^{\infty} e^{-\frac{1}{2}t^2} dt)}{\sqrt{\frac{2}{\pi}} (\int_x^{\infty} e^{-\frac{1}{2}t^2} dt)^2} \geq \frac{e^{-\frac{x^2}{2}} (e^{-\frac{x^2}{2}} - \int_x^{\infty} te^{-\frac{1}{2}t^2} dt)}{\sqrt{\frac{2}{\pi}} (\int_x^{\infty} e^{-\frac{1}{2}t^2} dt)^2} = 0.$$

Thus $h(x) \geq 1$ for any $x \geq 0$. Then a change of variable gives

$$\int_K^{\infty} e^{-\frac{1}{2}(\sqrt{2\pi}\frac{\sigma}{K}k)^2} dk \leq \sqrt{\frac{\pi}{2}} \frac{1}{\sqrt{2\pi}\frac{\sigma}{K}} e^{-\frac{(\sqrt{2\pi}\sigma)^2}{2}} = \frac{1}{2} \frac{K}{\sigma} e^{-\pi\sigma^2}.$$

Step 1. We first show that $X_* \subset X_e$ with the help of (4). For any $x_s \in X_*$,

$$\begin{aligned} \left| \int_{|k| \leq K} y(k) \hat{\phi}(k) e^{2\pi i k x_s} dk \right| &\geq (\phi * f_*)(x_s) - \left| \int_{|k| \leq K} y(k) \hat{\phi}(k) e^{2\pi i k x_s} dk - \phi * f_*(x_s) \right| \\ &\geq \frac{K}{\sigma} \beta - \epsilon \frac{K}{\sigma} - \frac{K}{\sigma} \exp(-\pi\sigma^2) \geq \frac{K}{\sigma} \left(\beta - \frac{\beta - \omega}{6} - \frac{\beta - \omega}{6} \right) \\ &= \frac{K}{\sigma} \left(\frac{2\beta + \omega}{3} \right). \end{aligned}$$

Thus $x_s \in X_e$ by definition (3).

Step 2. Now let us show that for any $x \in X_e$, $\text{dist}(x, X_*) \leq \tau/K$. Using proof by contradiction, let us assume that there exists $x \in X_e$ that violates this. Then

$$\begin{aligned} \frac{(2\beta + \omega)K}{3\sigma} &\leq \left| \int_{|k| \leq K} y(k) \hat{\phi}(k) e^{2\pi i k x} dk \right| \\ &\leq (\phi * f_*)(x) + \left| \sum_{|k| \leq K} y(k) \hat{\phi}(k) e^{2\pi i k x} - \phi * f_*(x) \right| \end{aligned} \tag{5}$$

$$\begin{aligned} &< \frac{K}{\sigma} \exp(-\pi \frac{\tau^2}{\sigma^2}) + \frac{K}{\sigma} \omega + \epsilon \frac{K}{\sigma} + \frac{K}{\sigma} \exp(-\pi \sigma^2) \\ &\leq \frac{K}{\sigma} \left(\frac{\beta - \omega}{6} + \omega + \frac{\beta - \omega}{6} + \frac{\beta - \omega}{6} \right) = \frac{(\beta + \omega)K}{2\sigma}, \end{aligned}$$

which leads to $\frac{\beta+\omega}{2} \geq \frac{2\beta+\omega}{3}$ and contradicts with $\beta > \omega$. Here we have used (4) in the third line of (5). \square

3. Periodic interval case

This section considers the case of the periodic interval \mathbb{T} . Introduce the periodic Gaussian function

$$\phi_p(x) = \sum_{j \in \mathbb{Z}} \phi(x+j) = \sum_{j \in \mathbb{Z}} \frac{1}{\sigma/K} \exp\left(-\pi \frac{(x+j)^2}{(\sigma/K)^2}\right), \quad x \in \mathbb{T}. \quad (6)$$

Notice that $\int_0^1 \phi_p(x) dx = 1$. Its Fourier coefficients are given by

$$\hat{\phi}_p(k) = \exp(-\pi(k\sigma/K)^2), \quad k \in \mathbb{Z}. \quad (7)$$

We provide the derivation here for completeness.

$$\begin{aligned} \exp(-\pi(k\sigma/K)^2) &= \int_{-\infty}^{\infty} \phi(x) e^{-2\pi i k x} dx = \sum_{j \in \mathbb{Z}} \int_j^{j+1} \phi(x) e^{-2\pi i k x} dx \\ &= \sum_{j \in \mathbb{Z}} \int_0^1 \phi(x+j) e^{-2\pi i k x} dx = \int_0^1 \sum_{j \in \mathbb{Z}} \phi(x+j) e^{-2\pi i k x} dx = \hat{\phi}_p(k), \end{aligned}$$

where the dominated convergence theorem is used in the second-last equality. The following lemma bounds ϕ_p in terms of ϕ (defined in (2)).

Lemma 2. *If $K \geq 3\sigma$, then $\phi_p(x)$ is increasing on $[-\frac{1}{2}, 0]$ and decreasing on $[0, \frac{1}{2}]$. For $x \in [-\frac{1}{3}, \frac{1}{3}]$,*

$$\phi(x) \leq \phi_p(x) \leq (1 + 10^{-4})\phi(x). \quad (8)$$

In particular,

$$\sum_{k \in \mathbb{Z}} \hat{\phi}_p(k) = \phi_p(0) \leq (1 + 10^{-4})\phi(0) \leq (1 + 10^{-4})\frac{K}{\sigma}. \quad (9)$$

Proof. Without loss of generality, we assume $x \geq 0$ since ϕ_p is an even function. Notice that the Jacobi theta function $\Theta(x | i\sigma^2/K^2) = \sum_{k \in \mathbb{Z}} \exp(-\pi(k\sigma/K)^2 + 2\pi i k x)$ is just the Fourier series of $\phi_p(x)$, and we can use its product form [21]

$$\phi_p(x) = \Theta(x | i\sigma^2/K^2) = \prod_{n=1}^{\infty} (1 - q^{2n}) (1 + 2q^{2n-1} \cos(2\pi x) + q^{4n-2}),$$

where $q = e^{-\pi\sigma^2/K^2}$. Therefore, $\phi_p(x)$ is decreasing on $[0, \frac{1}{2}]$ since each product term is positive and decreasing in x .

When $x \in [0, \frac{1}{3}]$, $\phi(x) \leq \phi_p(x)$ is trivial from the definition of ϕ_p . The following calculation can establish the other direction of (8).

$$\begin{aligned}
\frac{\phi_p(x)}{\phi(x)} &= \sum_{j \in \mathbb{Z}} \frac{\phi(x+j)}{\phi(x)} = \sum_{j \in \mathbb{Z}} \exp(-\pi \frac{K^2}{\sigma^2} (j^2 + 2xj)) \leq \sum_{j \in \mathbb{Z}} \exp(-9\pi(j^2 + 2xj)) \\
&= 1 + \exp(-9\pi(1 - 2x)) + \sum_{j=1}^{\infty} \exp(-9\pi(j^2 + 2xj)) + \sum_{j=2}^{\infty} \exp(-9\pi(j^2 - 2xj)) \\
&\leq 1 + \exp(-9\pi(1 - 2/3)) + \sum_{j=1}^{\infty} \exp(-9\pi j) + \sum_{j=2}^{\infty} \exp(-9\pi j) \\
&\leq 1 + 10^{-4},
\end{aligned}$$

which completes the proof. \square

Let us introduce

$$X_e = \left\{ x : \left| \sum_{|k| \leq K} y(k) \hat{\phi}_p(k) e^{2\pi i k x} \right| > \frac{6\beta + 5\omega}{11} \phi_p(0) \right\}, \quad (10)$$

where $\sigma = \sqrt{\frac{1}{\pi} \log \frac{12}{\beta - \omega}}$. The following theorem states that X_* and X_e are close in the periodic case.

Theorem 3. Suppose $\omega < \beta$ and $\epsilon \leq \frac{\beta - \omega}{3}$. Then $X_* \subset X_e$ and for any $x \in X_e$, $\text{dist}(x, X_*) \leq \tau/K$ with $\tau = \frac{1}{\pi} \log \frac{12}{\beta - \omega}$ for any $K \geq 3\tau$.

Proof. Since $|y(k) - \hat{f}_*(k)| \leq \epsilon$ for $|k| \leq K$, direct calculations show that for any x

$$\begin{aligned}
&\left| \sum_{|k| \leq K} y(k) \hat{\phi}_p(k) e^{2\pi i k x} - (\phi_p * f_*)(x) \right| \\
&= \left| \sum_{|k| \leq K} (y(k) - \hat{f}_*(k)) \hat{\phi}_p(k) e^{2\pi i k x} - \sum_{|k| > K} \hat{f}_*(k) \hat{\phi}_p(k) e^{2\pi i k x} \right| \\
&\leq \left| \sum_{|k| \leq K} (y(k) - \hat{f}_*(k)) \hat{\phi}_p(k) e^{2\pi i k x} \right| + \left| \sum_{|k| > K} \hat{f}_*(k) \hat{\phi}_p(k) e^{2\pi i k x} \right| \\
&\leq \epsilon \sum_{|k| \leq K} |\hat{\phi}_p(k)| + \sum_{|k| > K} |\hat{\phi}_p(k)| < \epsilon \sum_{k \in \mathbb{Z}} \hat{\phi}_p(k) + \int_{|k| > K} \exp(-\pi(k\sigma/K)^2) dk \\
&\leq \epsilon \phi_p(0) + \frac{K}{\sigma} \exp(-\pi\sigma^2) < (\epsilon + \exp(-\pi\sigma^2)) \phi_p(0),
\end{aligned} \quad (11)$$

where the last step uses (9).

Step 1. We first show that $X_* \subset X_e$ with the help of (11). For any $x_s \in X_*$,

$$\begin{aligned}
\left| \sum_{|k| \leq K} y(k) \hat{\phi}_p(k) e^{2\pi i k x_s} \right| &\geq (\phi_p * f_*)(x_s) - \left| \sum_{|k| \leq K} y(k) \hat{\phi}_p(k) e^{2\pi i k x_s} - (\phi_p * f_*)(x_s) \right| \\
&\geq \beta \phi_p(0) - (\epsilon + \exp(-\pi\sigma^2)) \phi_p(0) \\
&\geq \left(\beta - \frac{\beta - \omega}{3} - \frac{\beta - \omega}{12} \right) \phi_p(0) > \left(\frac{6\beta + 5\omega}{11} \right) \phi_p(0),
\end{aligned}$$

thus $x_s \in X_e$ by definition (10).

Step 2. Now we show that for any $x \in X_e$, $\text{dist}(x, X_*) \leq \tau/K$. We proceed with proof by contradiction. Assume that there exists $x \in X_e$ violating this. Together with $\tau/K \leq \frac{1}{3}$ and Lemma 2, we have

$$\begin{aligned} (\phi_p * f_*)(x) &= \sum_{s=1}^S w_s \phi_p(x - x_s) + (\phi_p * r)(x) \leq \phi_p(\tau/K) + \omega \phi_p(0) \\ &\leq (1 + 10^{-4}) \frac{K}{\sigma} \exp(-\pi \frac{\tau^2}{\sigma^2}) + \omega \phi_p(0) \leq (1 + 10^{-4}) \phi_p(0) \exp(-\pi \frac{\tau^2}{\sigma^2}) + \omega \phi_p(0). \end{aligned}$$

Therefore

$$\begin{aligned} \frac{6\beta + 5\omega}{11} \phi_p(0) &\leq \left| \sum_{|k| \leq K} y(k) \hat{\phi}_p(k) e^{2\pi i k x} \right| \leq (\phi_p * f_*)(x) + \left| \sum_{|k| \leq K} (k) \hat{\phi}_p(k) e^{2\pi i k x} - (\phi_p * f_*)(x) \right| \\ &< (1 + 10^{-4}) \phi_p(0) \exp(-\pi \frac{\tau^2}{\sigma^2}) + \omega \phi_p(0) + (\epsilon + \exp(-\pi \sigma^2)) \phi_p(0) \\ &\leq \left((1 + 10^{-4}) \frac{\beta - \omega}{12} + \omega + \frac{\beta - \omega}{3} + \frac{\beta - \omega}{12} \right) \phi_p(0) \\ &< \frac{6\beta + 5\omega}{11} \phi_p(0), \end{aligned} \tag{12}$$

which is a contradiction. \square

Remark 4. In this work, we focus primarily on the one-dimensional case. However, for higher-dimensional cases, where the spike set $X_* \subset \mathbb{T}^n$ (or \mathbb{R}^n) for some $n \geq 2$, we can recover a neighborhood X_e of X_* using a similar approach. Specifically, we can multiply the signal by a multidimensional Gaussian function and perform the inverse Fourier transform. Then we can obtain X_e by a thresholding method similar to the one used in this note, with some necessary adjustments of the constants.

Data availability

No data was used for the research described in the article.

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