Problem 1. Let $f \in \mathbb{Q}[x]$ be a polynomial in $x$. Prove that there exists some non-zero polynomial $g \in \mathbb{Q}[x]$ such that in the product $f \cdot g$ all monomials have prime exponents (i.e. the product $f \cdot g$ can be written as $\sum_{p \in P} a_p x^p$ for some finite set $P$ of prime numbers and $a_p \in \mathbb{Q}$ for all $p \in P$).

Problem 2. Given a positive integer $n$, what is the maximum possible number $m$ such that there are distinct subsets $C_1, \ldots, C_m \subseteq \{1, \ldots, n\}$ satisfying both of the following conditions?

(i) $|C_i|$ is odd for all $i = 1, \ldots, m$.
(ii) $|C_i \cap C_j|$ is odd for all $1 \leq i < j \leq m$.

[Recall that in class we discussed the three other variants of this problem, where the word “odd” is replaced by “even” in one or both of the conditions (i) and (ii).]

Problem 3. Given a positive integer $d$, show that there cannot be more than $d + 1$ points in $\mathbb{R}^d$ such that the distances between any two of these points are equal.

Problem 4. In class we proved the following theorem: If $a, b > 0$ and $p_1, \ldots, p_n \in \mathbb{R}^d$ are distinct points such that the distance between any two of these points is equal to $a$ or equal to $b$, then $n \leq \frac{1}{2}(d^2 + 5d + 4)$.

We proved this theorem by considering the functions $f_i(x) = (\|x - p_i\|^2 - a^2) \cdot (\|x - p_i\|^2 - b^2)$ for $i = 1, \ldots, n$ (which are polynomials in $x_1, \ldots, x_d$), and showing that these functions are linearly independent.

a) Prove that the functions $f_i(x)$ for $i = 1, \ldots, n$ together with the additional $d + 1$ polynomials $1, x_1, \ldots, x_d$ are still linearly independent.

b) Deduce the stronger upper bound $n \leq \frac{1}{2}(d^2 + 3d + 2) = \binom{d+2}{2}$ for the theorem.

[The details of the proof we discussed in class can also be found in Miniature 15 in Matoušek’s book, see the syllabus for more details.]