Choose any four out of the five problems.

**Problem 1.** In the proof of the Combinatorial Nullstellensatz we used the following lemma:

Let $F$ be a field, and consider non-empty subsets $S_1, \ldots, S_n \subseteq F$. Let $d_1, \ldots, d_n \geq 0$ be integers such that $d_i < |S_i|$ for $i = 1, \ldots , n$, and let $P \in F[x_1, \ldots, x_n]$ be a polynomial such that for each $i = 1, \ldots, n$ the degree of $P$ in the variable $x_i$ is at most $d_i$. If $P(x_1, \ldots, x_n) = 0$ for all $(x_1, \ldots, x_n) \in S_1 \times \cdots \times S_n$, then $P$ is the zero polynomial.

Prove this lemma (without using the Combinatorial Nullstellensatz).

**Problem 2.** (International Mathematical Olympiad 2007, Problem 6)

Let $n$ be a positive integer. Consider $S = \{(x, y, z) | x, y, z \in \{0, 1, \ldots, n\}, x + y + z > 0\}$ as a set of $(n+1)^3 - 1$ points in three-dimensional space. Determine the smallest possible number of planes, the union of which contains $S$ but does not include $(0, 0, 0)$.

**Problem 3.** Let $p \geq 3$ be a prime, and let $\Gamma_p = \min_{0 < t \leq 1} (1 + t + \cdots + t^{p-1})/(t^{(p-1)/3})$. Recall that Ellenberg and Gijswijt proved the following result:

If $A \subseteq \mathbb{F}_p^n$ is a subset of $\mathbb{F}_p^n$ without a (non-trivial) three-term arithmetic progression, then $|A| \leq \Gamma_p^n$.

Use this to prove the following statement for any integer $k \geq 2$:

Suppose that $A \subseteq \mathbb{F}_p^n$ is a subset of $\mathbb{F}_p^n$ that does not contain $k$ different (non-trivial) three-term arithmetic progression with the same middle term (which are disjoint apart from the common middle term). Then $|A| \leq 10\sqrt{k}\cdot \Gamma_p^n$.

**Problem 4.** In order to prove the result of Ellenberg and Gijswijt stated in the previous problem, in class we first proved the following result (following Tao’s slice rank reformulation of the proof of Ellenberg and Gijswijt):

Let $p \geq 3$ be a prime. If $A \subseteq \mathbb{F}_p^n$ does not contain a (non-trivial) three-term arithmetic progression, then

$$|A| \leq 3 \cdot \left\{ (a_1, \ldots, a_n) \in \{0, \ldots, p-1\}^n \mid a_1 + \cdots + a_n \leq \frac{(p-1)n}{3} \right\}.$$  

Prove the following generalization of this statement:

Let $p \geq 3$ be a prime, and let $S_1, \ldots, S_n \subseteq \mathbb{F}_p$. If $A \subseteq S_1 \times \cdots \times S_n \subseteq \mathbb{F}_p^n$ does not contain a (non-trivial) three-term arithmetic progression, then

$$|A| \leq 3 \cdot \left\{ (a_1, \ldots, a_n) \in \{0, \ldots, p-1\}^n \mid a_1 + \cdots + a_n \leq \frac{(p-1)n}{3}, a_i < |S_i| \text{ for } i = 1, \ldots, n \right\}.$$  

Please turn the page.
Problem 5. (Behrend’s construction)
The goal of this problem is to show that there is a constant $c$ such that for every sufficiently large integer $N$, there is a subset $A \subseteq \{1, \ldots, N\}$ of size at least $e^{-c \sqrt{\ln N}} N$ without a (non-trivial) three-term arithmetic progression.

a) For any positive integers $M$ and $d$, consider the $d$-dimensional grid $\{1, \ldots, M\}^d \subseteq \mathbb{R}^d$. Prove that one can choose a subset $B \subseteq \{1, \ldots, M\}^d$ of size at least $M^d/(dM^2)$ without a (non-trivial) arithmetic progression in $\mathbb{R}^d$.

b) Consider a subset $B \subseteq \{1, \ldots, M\}^d$ as in a), and let

$$A = \{b_1(2M+1)^{d-1} + b_2(2M+1)^{d-2} + \cdots + b_d \mid (b_1, \ldots, b_d) \in B\}.$$ 

In other words, for every $(b_1, \ldots, b_d) \in B$, we interpret $b_1 \ldots b_d$ as a number in base $(2M+1)$. Prove that $A \subseteq \{1, \ldots, (2M+1)^d\}$ is a subset of size $|A| \geq M^d/(dM^2)$ and that $A$ does not contain a (non-trivial) three-term arithmetic progression.

c) For a large integer $N$, let

$$M = \left\lfloor \frac{1}{2} e^{\sqrt{\ln N}} \right\rfloor - 1 \quad \text{and} \quad d = \lfloor \sqrt{\ln N} \rfloor.$$ 

Deduce from b) that there is a subset $A \subseteq \{1, \ldots, N\}$ of size at least $e^{-c \sqrt{\ln N}} N$ without a (non-trivial) three-term arithmetic progression, where $c$ is some absolute constant.

[Hint for a): Consider the intersection of the grid $\{1, \ldots, M\}^d$ with a sphere of radius $r$ around the origin, and note that this intersection is a convex set.]

Reminder: You only need to submit four out of the five problems (see above).