Survival Analysis: Martingale CLT

Lu Tian and Richard Olshen

Stanford University

Martingale

- If $M_1(\cdot)$ and $M_2(\cdot)$ are mean zero martingales defined on the same filtration, and that for every t, $E(M_j^2(t)) < \infty$ for $j = 1$ and 2, then there exists a right-continuous predictable process $< M_1, M_2 > (\cdot)$ such that $M_1(\cdot)M_2(\cdot) - < M_1, M_2 > (\cdot)$ is a zero-mean martingale.

- $< M_1, M_2 >$ tells us about the covariance function of $M_1(\cdot)$ and $M_2(\cdot)$ since

$$cov(M_1(t), M_2(s)) = E(< M_1, M_2 > (min(t, s)))$$

Martingale

- if $< M_1, M_2 > (\cdot) = 0$ a.s., then $M_1(\cdot)M_2(\cdot)$ is a martingale and $M_1(\cdot)$ and $M_2(\cdot)$ are called orthogonal.
- Suppose that $N_i(\cdot), i = 1, \cdots, K$ are counting processes with continuous compensator $A_i(\cdot), i = 1, \cdots, K$ respectively. Then if no two of the counting process can jump at the same time, $< M_i, M_j > (\cdot) = 0$ a.s. for $i \neq j$, where $M_i(\cdot) = N_i(\cdot) - A_i(\cdot)$.

Martingale Integration

- Let $N(\cdot)$ be a counting process with continuous compensator $A(\cdot)$, such that $M(\cdot) = N(\cdot) - A(\cdot)$ is a zero mean martingale. If $H(\cdot)$ is a bounded, predictable process defined on the same filtration, the process $Q(\cdot)$ defined by

$$Q(t) = \int_0^t H(s)dM(s)$$

is also a zero mean martingale.

- Justifications:

$$\begin{aligned} E\left\{\int_s^t H(u)dM(u)|\mathcal{F}_s\right\} =& E\left\{\mathbf{E}\left[\int_{\mathbf{s}}^{\mathbf{t}} \mathbf{H}(\mathbf{u})\mathbf{dM}(\mathbf{u})|\mathcal{F}_{\mathbf{u}^-}\right]|\mathcal{F}_s\right\} \\ =& E\left\{\int_s^t H(u)\mathbf{E}\left[\mathbf{dM}(\mathbf{u})|\mathcal{F}_{\mathbf{u}^-}\right]|\mathcal{F}_s\right\} = 0 \end{aligned}$$

Martingale Integration

- If $E(M(s)^2) < \infty$ and $N(\cdot)$ is bounded, then for all t :

$$< Q, Q > (t) = \int_0^t H^2(s) dA(s), a.s.$$

- $$var[Q(t)] = E(< Q, Q > (t)) = E\left(\int_0^t H^2(s) dA(s)\right).$$

Martingale Integration

- Suppose that $N_i(\cdot), i = 1, 2, \cdots$ are bounded counting processes, $M_i(\cdot), i = 1, 2, \cdots$ are the corresponding zero-mean counting process martingales, each M_i satisfies $E(M_i^2(t)) < \infty$ for any t and that $H_i(\cdot), i = 1, 2, \cdots$ are bounded and predictable processes. Let

$$Q_i(t) = \int_0^t H_i(u) dM_i(u),$$

then$< Q_i, Q_j > (t) = \int_0^t H_i(s) H_j(s) d < M_i, M_j > (s)$ a.s.

- If $M_i(\cdot), i = 1, 2, \cdots$ are orthogonal, then so are $Q_i(\cdot), i = 1, 2, \cdots$ and

$$Var\{\Sigma Q_i(t)\} = \sum_{i=1}^{n} E\left(\int_0^t H_i^2(s) dA_i(s) \right).$$

Martingale Central Limit Theorem

- $N_{in}(\cdot)$ is a counting process with continuous compensator $A_{in}(\cdot)$
- H_{in} is locally bounded and predictable.
- No two of the counting processes can jump at the same time, so that the n martingales $M_{in}(\cdot) = N_{in}(\cdot) - A_{in}(\cdot)$ are orthogonal.

Define

- $U_{in}(t) \stackrel{def}{=} \int_0^t H_{in}(s) dM_{in}(s)$
- $U_{in,\epsilon}(t) \stackrel{def}{=} \int_0^t H_{in}(s) 1[|H_{in}(s)| \geq \epsilon] dM_{in}(s)$.

Martingale Central Limit Theorem

If

(a) $< \Sigma_{i=1}^{n} U_{in}, \Sigma_{i=1}^{n} U_{in} > (t) \xrightarrow{p} \alpha(t)$

(b) $< \Sigma_{i=1}^{n} U_{in,\epsilon}, \Sigma_{i=1}^{n} U_{in,\epsilon} > (t) \xrightarrow{p} 0 \quad \forall \epsilon > 0$

as $n \to \infty$, then as $n \to \infty$

$$\Sigma_{i=1}^{n} U_{in}(\cdot) \to U(\cdot)$$

weakly, where $U(\cdot)$ is a zero-mean Gaussian process with independent increments and variance function $\alpha(\cdot)$. The Gaussian process $U(t) = \int_0^t f(s)dW(s)$, where $\int_0^t f(s)^2 ds = \alpha(t)$ and $W(\cdot)$ is a Weiner Process.

Weiner Process

- **Definition:** If $W(\cdot)$ is a Gaussian process satisfying $W(0) = 0$, $E(W(t)) = 0$ for all t, and Cov$(W(s), W(t)) = \min(s,t)$ for all t,s, then $W(\cdot)$ is a Wiener process.

1. $W(\cdot)$ is a zero-mean martingale.
2. The predictable quadratic variation process for $W(\cdot)$ satisfies

$$< W, W > (t) = t.$$

3. $Q(t) \stackrel{def}{=} \int_0^t f(s)dW(s)$ is a zero-mean Gaussian process with $Q(0) = 0$, independent increments, and variance function

$$\text{var}\ (Q(t)) = \int_0^t f^2(s)ds.$$

Weak Convergence in Stochastic Process

- $X_n(\cdot)$ converges weakly to $X(\cdot)$ if for all bounded, continuous (with respect to the Skorohod topology), real value functions f,

$$E\{f(X_n)\} \to E\{f(X)\} \quad \text{as} \quad n \to \infty.$$

- Example of $f(\cdot)$:

1. $f(X_n(\cdot)) = \sup_{t\in[0,\tau]} |X_n(t)|$
2. $f(X_n(\cdot)) = \int_0^\tau X_n(t)dt.$
3. $f(X_n(\cdot)) = X_n(t_0)$

Weak Convergence in Stochastic Process

- A sequence of random processes $X_n(\cdot) \in D[0,\tau]$ converges weakly to $X(\cdot)$ if
  1. The finite dimensional distribution converge:
     $(X_n(t_1), \cdots, X_n(t_K)) \to (X(t_1), \cdots, X_n(t_K))$ in distribution.
  2. $X_n(\cdot)$ is tight which can be implied by the condition:

$$\lim_{\delta \to 0} \limsup_{n \to \infty} P \left[\sup_{|s-t|<\delta} |X_n(t) - X_n(s)| > \epsilon \right] = 0$$

- The tightness condition is difficult to verify in general. In the MCLT, the tightness condition is guaranteed via condition (b).

Martingale CLT

- From MCLT

$$\Sigma_{i=1}^n U_{in}(\cdot) \to U(\cdot) \quad \text{weakly.}$$

-

$$\sup_{t\in[0,\tau]} |\Sigma_{i=1}^n U_{in}(t)| \to \sup_{t\in[0,\tau]} |U(t)|$$

  in distribution.