Unless noted otherwise, all graphs considered are simple. The solution of every problem should be no longer than one page.

**Problem 1:** Show that in a tree containing an even number of edges, there is at least one vertex with even degree.

**Problem 2:** Given a graph $G$ and a vertex $v \in V(G)$, $G - v$ denotes the subgraph of $G$ induced by the vertex set $V(G) \setminus \{v\}$. Show that every connected graph $G$ of order at least two contains vertices $x$ and $y$ such that both $G - x$ and $G - y$ are connected.

**Problem 3:** Let $T$ be an $n$-vertex tree with exactly $2k$ odd-degree vertices. Prove that $T$ decomposes into $k$ paths (i.e. its edge-set is the disjoint union of $k$ paths).

**Problem 4:** Prove that a connected graph $G$ is a tree if and only if any family of pairwise (vertex-)intersecting paths $P_1, \ldots, P_k$ in $G$ have a common vertex.

**Problem 5:**

(a) What is the Prüfer code of the following tree?

```
   3   6   1   10
  / \   /   /   /
 4   2 9  5
 /\  / \  /
7  8
```

(b) Which labeled tree has Prüfer code $(5,1,1,7,7,5)$?

(c) Describe which Prüfer codes correspond to stars (i.e. to trees isomorphic to $K_{1,n-1}$).

(d) Describe what trees correspond to Prüfer codes containing exactly 2 different values.

**Problem 6:** Here we prove Cayley’s formula in a different way. Let $T$ be a forest on vertex set $[n]$ with components $T_1, \ldots, T_r$. Prove, by induction on $r$ or otherwise, that the number of labelled trees on the vertex set $1, \ldots, n$ containing $T$ is $n^{r-2} \prod_{i=1}^{r} |T_i|$. Deduce Cayley’s formula.