Homework 6

Unless noted otherwise, all graphs considered are simple. The solution of every problem should be no longer than one page.

**Problem 1:** Show that a connected plane graph $G$ is bipartite if and only if all its faces have even length.

**Problem 2:** Let $G$ be a graph on $n \geq 3$ vertices and $3n - 6 + k$ edges for some $k > 0$. Show that any drawing of $G$ in the plane contains at least $k$ crossing pairs of edges.

**Problem 3:** Let $G$ be a plane graph in which every face has length 3 (including the outer face) and suppose the vertices are colored arbitrarily with three colors. Prove that there is an even number of faces that get all three colors.

**Problem 4:** Let $G$ be a graph with the property that any two odd cycles in it intersect (they have at least one vertex in common). Prove that $\chi(G) \leq 5$.

**Problem 5:** For a vertex $v$ in a connected graph $G$, let $G_r$ be the subgraph of $G$ induced by the vertices at distance $r$ from $v$. Show that $\chi(G) \leq \max_{0 \leq r \leq n} \chi(G_r) + \chi(G_{r+1})$. 