Problem 0: [Note: this question is not related to the material this week, it’s something I should have asked last week] Suppose that $X$ is a discrete random variable with probability mass function defined by $p_X(k) = c/k^2$ for each (positive and negative) $k \in \mathbb{Z}$, for some constant $c$. What is the probability mass function of $X^2$? (You do not need to compute the value of $c$).

Problem 1: Given the names, it is tempting to think of expected value as being “the approximate value that we expect to see”, and standard deviation as being “the typical deviation from the expected value”. Which of the following statements are true? Briefly explain your answers.

(a) If $\mathbb{E}X = 0$, then $\mathbb{P}(-100 \leq X \leq 100) \geq 9/10$

(b) If $\mathbb{E}X = 0$, then $\mathbb{P}(-100 \leq X \leq 100) \geq 1/1000$

(c) If $X$ has standard deviation at least 100, then $\mathbb{P}(|X - \mathbb{E}X| \leq 1) \leq 1/10$.

(d) If $X$ has standard deviation at least 100, then $\mathbb{P}(|X - \mathbb{E}X| \leq 1) \leq 999/1000$.

(e) If $X$ has standard deviation at most 1, then $\mathbb{P}(|X - \mathbb{E}X| \geq 100) \leq 1/10$.

(f) If $X$ has standard deviation at most 1, then $\mathbb{P}(|X - \mathbb{E}X| \geq 100) \leq 999/1000$.

Problem 2: Suppose that you flip a coin until you see 10 heads in a row. Let $X$ be the number of coin flips this process takes. By defining a geometrically-distributed random variable on the same probability space as $X$ (or otherwise), prove a finite upper bound on $\mathbb{E}X$.

Problem 3: If $X$ is discrete or continuous and satisfies $\mathbb{E}X = 1$ and $\text{Var}(X) = 5$, find

(a) $\mathbb{E}[(2 + X)^2]$;

(g) $\text{Var}(4 + 3X)$. 
**Problem 4:** Let $X$ be a discrete or continuous random variable such that $E X^2 < \infty$. Prove that the unique minimum of $E[(X - c)^2]$ occurs when $c = E X$.

**Problem 5:** A *mode* of a discrete random variable $X$ is a value $k \in \mathbb{R}$ such that $p_X(k)$ is maximised. A *median* of $X$ is a value $k \in \mathbb{R}$ such that $P(X \leq k)$ and $P(X \geq k)$ are both at least $1/2$. These are both alternatives to $E X$ measuring the typical order of magnitude of $X$, but are generally harder to compute (and may not be uniquely defined).

(a) Prove that every discrete random variable has at least one mode.

(b) Prove that every continuous random variable has at least one median. (This is actually true for all random variables, but a bit fiddly to prove).

(c) [Bonus question, not for credit]: For a continuous random variable $X$, prove that $E|X - c|$ is minimised when $c$ is a median of $X$.

(d) Suppose that $X$ has the Poisson distribution Po($\lambda$). Prove that all modes of $X$ differ by at most 1 from $\lambda$.

(e) Suppose that $X$ has the binomial distribution Bin($n, p$). Prove that all modes of $X$ differ by at most 1 from $(n + 1)p$.

(f) Give a median of $X \sim$ Bin($n, 1/2$).

**Problem 6:** Recall that Markov’s inequality gives us an upper bound on the probability that a nonnegative random variable is large compared to its expected value. Prove that for any nonnegative discrete random variable $X$ with finite variance, and any $\alpha \in [0, 1]$, we have

$$P(X \leq \alpha E X) \leq \frac{\text{Var}(X)}{(1 - \alpha)^2 (E X)^2}.$$  

This is an upper bound on the probability that $X$ is small compared to its expected value.

**Problem 7:** Suppose $X$ has the Poisson distribution Po($\lambda$). In this exercise we will compute $P(X$ is even).

(a) Let $S$ be a finite set of integers, and let $X_n \sim$ Bin($n, 1/n$). Briefly explain why $P(X \in S) = \lim_{n \to \infty} P(X_n \in S)$.

(b) Let $E$ be the set of even numbers and let $E_k$ be the set of even numbers at most $k$. Use Markov’s inequality to give upper bounds on $P(X \in E \setminus E_k)$ and $P(X_n \in E \setminus E_k)$.  

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(c) Use the result of problem 7 on the second homework to compute $\lim_{n \to \infty} \mathbb{P}(X_n \in E)$.

(d) Find the value of $\mathbb{P}(X$ is even) and prove it.

**Problem 8:** Consider the uniform probability space on a square of area 1 (meaning that the probability of falling in a given region is the area of that region). For any point $\omega$ in the square, consider the line from the top left corner of the square to $\omega$. Let $X(\omega)$ be the angle between this line and the leftmost boundary of the square, which we view as a random variable. (Think of $X$ as measuring the angle a laser needs to aim to hit a mosquito in a random position in a square room).

(a) What is the distribution function of $X$?

(b) Give a density function for $X$.

**Problem 9:** Consider the function $f^{(c)} : \mathbb{R} \to \mathbb{R}$ defined by

$$f^{(c)}(x) = \begin{cases} 
  c(1 - x^2) & \text{for } -1 < x < 1 \\
  0 & \text{otherwise.}
\end{cases}$$

(a) For which value(s) of $c$ is $f^{(c)}$ the density function of a random variable $X$?

(b) For a random variable $X$ as in (a), compute $F_X$.

(c) Compute $E_X$.

(d) Give a density function for $|X|$, or explain why one does not exist.

**Problem 10:** Consider a nuclear reaction with the property that for any time period of $t$ seconds, the number of particles that are excited during that time has a Poisson distribution $\text{Po}(2t)$. Starting from time zero, describe the distribution of the time until the first excited particle.

**Problem 11:**

(a) Give an example of a distribution function of a random variable that is neither discrete nor continuous.

(b) Give an example of a random variable which is integrable (meaning $E_X$ is finite) but for which $\text{Var} \, X = \infty$. 

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Problem 12: [Bonus question, not for credit]: We defined the support of a discrete random variable to be the set of values that occur with positive probability. For a continuous random variable $X$ with a density $f_X$, it would make sense to define the support of $X$ to be the set of values $x \in \mathbb{R}$ such that $f_X(x) > 0$. However, this is not well-defined: if we change the value of $f_X$ at a single point it is still a density function for $X$.

The correct definition is a bit cumbersome: we say a set $S$ of real numbers is closed if all convergent sequences in $S$ converge to a number in $S$. For example, $[0, 1]$ is closed, and $(0, 1]$ is not closed. Then, for any random variable $X$, we define $\text{supp}(X)$ to be the intersection of all closed sets $E$ such that $P(X \in E) = 1$. Intuitively, this means we take the smallest set containing all the probability mass, and include the boundary points.

(a) What is the support of the exponential distribution $\text{Exp}(\lambda)$?

(b) What is the support of the random variables in problems 8 and 9?