Problem 1: Suppose that $X$ is a random variable that is uniformly distributed on the interval $[0, 1]$. What is the expected value of $e^X$?

Problem 2: Suppose a random variable $X$ has the Cauchy distribution with density given by

$$f_X(x) = \frac{1}{\pi(1 + x^2)}.$$ 

Is $X$ integrable? Why or why not?

Problem 3: Consider a uniformly random point in a square of unit area. Let $X$ be the distance between the random point and the nearest edge of the square.

(a) What is the distribution function of $X$?

(b) What is the density function of $X$?

(c) compute $E X$.

(d) compute Var $X$.

Problem 4: Alice throws a dart at a circular target with a 20cm radius. Suppose that the position of the dart is a uniformly random point on the target. She gets 10 points if the dart lands within 2cm of the centre, 5 points if it is between 2 and 5 cm of the centre, and 1 point otherwise. What is her expected score?

Problem 5: Let $X$ be a continuous random variable with density $f_X$, which only takes positive values (meaning that $f_X(x) = 0$ for $x < 0$). Prove that $E X^n = \int_0^\infty nx^{n-1}(1 - F_X(x)) \, dx$ for any $n > 0$.

Problem 6: Let $X$ be a random variable whose distribution function $F_X$ is strictly increasing. What is the distribution of $F_X(X)$?
Problem 7: Let $Z$ be a standard normal random variable, and let $g : \mathbb{R} \to \mathbb{R}$ be a differentiable function which does not grow “too fast” in the sense that $\lim_{x \to \infty} (g(x)/e^{x^2/2}) = \lim_{x \to -\infty} (g(x)/e^{x^2/2}) = 0$.

(a) Show that $\mathbb{E}[g'(Z)] = \mathbb{E}[Zg(Z)]$.

(b) Find $\mathbb{E}Z^4$.

Problem 8: Let $Z$ be a standard normal random variable, with distribution function $\Phi$.

(a) Explain why $\mathbb{P}(Z \geq z) = \mathbb{P}(Z \leq -z)$

(b) Prove that $\mathbb{P}(|Z| \geq z) = 2(1 - \Phi(z))$. 