Problem 1: Suppose we have seven bins and seven balls. With each ball, we choose a bin uniformly at random and put the ball in that bin. (We do this independently for each ball). Let $X$ be the eventual number of balls in the first bin and let $Y$ be the eventual number of balls in the second bin. What is the joint probability mass function of $(X, Y)$?

Problem 2: Suppose we repeatedly flip a fair coin. Let $X$ be the number of trials until we see heads for the first time, and let $Y$ be the number of trials until we see tails.

(a) Describe the joint probability mass function of $X$ and $Y$.

(b) Describe the probability mass function of $X + Y$.

(c) What is the marginal distribution of $X$?

Problem 3: Suppose the joint density of $X$ and $Y$ is given by

$$f_{X,Y}(x, y) = \begin{cases} xe^{-(x+y)} & \text{if } x, y > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Are $X$ and $Y$ independent? Why or why not?

Problem 4: Let $X, Y, Z$ be independent geometric random variables with the same geometric distribution $\text{Geom}(p)$.

(a) What is $\mathbb{P}(X = Y)$?

(b) What is $\mathbb{P}(X \geq 2Y)$?

(c) What is $\mathbb{P}(X + Y \leq Z)$?

Problem 5: As in the last question, let $X \sim \text{Geom}(p)$ and $Y \sim \text{Geom}(p)$ be independent. Let $U = \min\{X, Y\}$ and let $V = |X - Y|$. Show that $U$ and $V$ are independent. [Hint: splif nioc fo ecnuqes a aiv snoitubirtsid eseht esilaeR]
Problem 6: Let $X_1, \ldots, X_n$ be mutually independent random variables each having the discrete uniform distribution on the set $\{1, \ldots, n\}$. What is the probability mass function of $\max\{X_1, \ldots, X_n\}$?

Problem 7: The joint probability density function $f_{X,Y}$ of $X$ and $Y$ is given by

$$f_{X,Y}(x, y) = \begin{cases} c(y^2 - x^2)e^{-y} & \text{for } -y \leq x \leq y \text{ and } 0 \leq y \\ 0 & \text{otherwise} \end{cases}$$

(a) Find $c$.

(b) Find the marginal densities of $X$ and $Y$.

Problem 8: Suppose that $X$, $Y$ and $Z$ are independent random variables that are uniform on $[0, 1]$.

(a) Find the density function of $XY$.

(b) Find the density function of $(XY)^Z$.

(c) [Bonus question, not for credit]: Can you give an intuitive explanation for the answer to (b)?

Problem 9: Three points $X_1, X_2, X_3$ are chosen uniformly and independently at random on a line segment.

(a) What is the probability that $X_2$ lies between $X_1$ and $X_3$?

(b) [Bonus question, not for credit]: Can you give an intuitive explanation for the answer to (a)?

Problem 10: The Lévy concentration function $Q_X : [0, \infty) \rightarrow [0, 1]$ of a random variable $X$ is defined by $Q_X(t) = \sup_{x \in \mathbb{R}} P(x \leq X \leq x + t)$. (That is, it gives the maximum probability that $X$ falls in an interval of length $t$). Prove that if $X$ and $Y$ are independent discrete random variables, then $Q_{X+Y}(t) \leq \min\{Q_X(t), Q_Y(t)\}$ for any $t \geq 0$. [Hint: Y fo semoctuo no noitidnoC]