Problem 1: Let \((R, \Theta)\) be a point uniformly chosen in the circle of radius 1 centered at the origin, given in polar coordinates (so the \(x\)-coordinate of our random point is \(R \cos \Theta\) and the \(y\)-coordinate is \(R \sin \Theta\), with \(R \geq 0\) and \(0 \leq \Theta \leq 2\pi\)). Find the joint density of \((R, \Theta)\).

Problem 2: Let \(X\) and \(Y\) be random variables uniformly distributed on \([0, 1]\). Which pairs of the following events are independent? (You do not need proofs, just the answers).

(a) \(X > 1/3\)
(b) \(Y > 2/3\)
(c) \(X > Y\)
(d) \(X + Y < 1\)

Problem 3: A machine is able to function as long as 3 of its 5 components are functioning. If each component independently functions for a random amount of time with density function \(f(x) = xe^{-x}\) (for \(x > 0\)), compute the density function of the length of time that the machine functions.

Problem 4: \(X\) and \(Y\) have joint density function given by

\[
f_{X,Y}(x, y) = \begin{cases} 
\frac{1}{xy^2} & \text{if } x, y \geq 1 \\
0 & \text{otherwise}
\end{cases}
\]

(a) Compute the joint density function of \(U = XY\) and \(V = X/Y\).
(b) What are the marginal densities of \(U\) and \(V\)?

Problem 5: Let \(X\) and \(Y\) be independent standard normal random variables.

(a) What is the joint density of \(U = |X - Y|\) and \(Y\)?
(b) [Bonus question, not for credit]: Do $U = |X - Y|$ and $V = X - Y$ have a joint density?

**Problem 6:** $X_1, \ldots, X_4$ are independent random variables each with variance $\sigma^2$. What is the covariance between $X_1 + X_2 + X_3 + X_4$ and $2X_1 - 3X_2 + 6X_3$?

**Problem 7:** Consider two random variables $X$ and $Y$ on the same probability space. Suppose that we would like to approximate $X$ by a linear function $aY + b$ of $Y$, where $a, b$ are constant. Prove that the mean-squared error $\mathbb{E}[(X - (aY+b))^2]$ is minimised when $a = \text{Cov}(X,Y)/\text{Var}(Y)$ and $b = \mathbb{E}X - \mathbb{E}Y \text{Cov}(X,Y)/\text{Var}(Y)$.

(This process of approximating one random variable as a linear function of the other is called *linear regression.*)

**Problem 8:**

(a) Give examples of random variables $X, Y$ on the same probability space which have $\text{Cov}(X,Y) = 0$ but are not independent

(b) Give examples of random variables $X, Y, Z$ such that $\text{Cov}(X,Y) > 0$ and $\text{Cov}(Y,Z) > 0$ but $\text{Cov}(X,Z) < 0$

**Problem 9:** The *correlation* between two random variables on the same probability space is defined as $\text{Corr}(X,Y) = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}$ (that is, we normalise the covariance by the standard deviations of $X$ and $Y$). This is a ”scale-free” measure of the linear relationship between $X$ and $Y$: if $X$ is height and $Y$ is age, it doesn’t matter whether we measure $X$ in inches or centimetres.

(a) Give examples of pairs of random variables with correlation 1 and correlation $-1$.

(b) [Bonus question, not for credit]: Prove that for any $X$ and $Y$ on the same probability space (each having finite variance) we have $-1 \leq \text{Corr}(X,Y) \leq 1$. 

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