Problem 1: Roll two dice. Let $X$ be the maximum of the two numbers and let $Y$ be the minimum of the two numbers

(a) Find the joint probability mass function of $X$ and $Y$

(b) What is the marginal probability mass function $p_Y(y)$?

(b) What is the conditional probability mass function $p_{X|Y}(\cdot|y)$, for $y \in \{1, \ldots, 6\}$?

Problem 2: Let $X, Y \sim \text{Bin}(n, p)$ be two independent binomial random variables.

(a) What is the conditional probability mass function $p_{X|X+Y}(\cdot|m)$, for $m \in \{0, \ldots, 2n\}$?

(b) [Bonus question, not for credit]: Can you give an intuitive explanation for the answer to (a)?

Problem 3: Consider a uniformly random point on the unit sphere $\{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1 \}$ (meaning that the probability of falling in a region is proportional to the surface area of that region). Let $\Phi$ and $\Theta$ be the latitude–longitude coordinates of the point (meaning that the $x$-coordinate is $\cos \Phi \cos \Theta$, the $y$-coordinate is $\cos \Phi \sin \Theta$, and the $z$-coordinate is $\sin \Phi$, with $0 \leq \Theta \leq 2\pi$ and $-\pi/2 \leq \Phi \leq \pi/2$).

(a) Find the joint density of $\Phi$ and $\Theta$

(b) Compute the conditional density $f_{\Phi|\Theta}(\phi|0)$

(c) Compute the conditional density $f_{\Theta|\Phi}(\theta|0)$

(d) [Bonus question, not for credit]: Can you give an intuitive explanation comparing the answers to (b) and (c)?
**Problem 4:** Suppose \( \vec{X} = (X_1, X_2) \) has a multivariate normal distribution, meaning that 
\[
f_{\vec{X}}(\vec{x}) = \det(2\pi \Sigma)^{-1/2} \exp\left(-\frac{1}{2} (\vec{x} - \vec{\mu})^T \Sigma^{-1} (\vec{x} - \vec{\mu})\right)
\] for some \( \vec{b} \in \mathbb{R}^2 \) and \( 2 \times 2 \) matrix \( \Sigma \). Suppose also that \( \text{Cov}(X_1, X_2) = 0 \). Prove that \( X_1 \) and \( X_2 \) are independent.

**Problem 5:** A coin having probability \( p \) of coming up heads is continually flipped until both heads and tails have appeared. Find

(a) the expected number of flips;

(b) the probability that the last flip lands on heads.

**Problem 6:** The number of people who enter an elevator on the ground floor is a Poisson random variable with mean 10. If there are \( n \) floors above the ground floor, and if each person is equally likely to get off at any one of these \( n \) floors, independently of where the others get off, compute the expected number of stops that the elevator will make before discharging all of its passangers.

**Problem 7:** Let \( X \sim \text{Bin}(n, p) \). In this exercise we will compute \( \mathbb{E}[1/(X + 1)] \) in an unusual way. Imagine a game show featuring you and \( n \) other participants. Each participant (including you) wins with probability \( p \), independently, then a prize pool of \( \$1 \) is shared between all the winners (for example, if 3 people win then those people get \( 1/3 \) of a dollar, and if nobody wins no prize is paid out).

(a) Let \( Z \) be the total amount that the game show company has to pay out. What is \( \mathbb{E}Z \)?

(b) Let \( Y \) be the amount of money you receive personally. What is \( \mathbb{E}Y \)?

(c) Explain why \( \mathbb{E}[Y|Y > 0] = \mathbb{E}[1/(X + 1)] \).

(d) Deduce the value of \( \mathbb{E}[1/(X + 1)] \).

**Problem 8:** toss a coin \( N \) times, where \( N \) has a Poisson distribution with mean \( \lambda \). Let \( X \) be the number of heads. What is \( \text{Var} \ [X] \)?

**Problem 9:**

(a) Let \( X \in \text{Geom}(p) \) have a geometric distribution. Compute the moment-generating function of \( X \).

(b) Let \( U \) be uniform on \( [a, b] \). Compute the moment-generating function of \( U \).