MATH 151: Introduction to Probability

Homework 9

To be completed by 17 March

Problem 1: Suppose you have a stock of 1000 batteries, and a machine. A battery can run the machine for an amount of time (in hours) that is exponentially distributed with mean 1. You run the machine as long as possible, swapping out batteries when they die.

Suppose that new batteries are only available over a period of two days, starting 40 days from now. Use the central limit theorem to estimate the probability that you run out of batteries during that period, in terms of the standard normal distribution function Φ.

Problem 2: Consider a sequence of random variables $X_1, X_2, \ldots$ where each $X_n$ has distribution function

$$F_{X_n}(x) = \begin{cases} \left(1 - \frac{1}{1+nx}\right)^n & \text{for } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Does $X_n$ converge in distribution? If so, what is the limiting distribution?

Problem 3:

(a) **[Bonus question, not for credit]**: Give an example of a sequence of random variables that converges in probability but not almost surely

(b) Give an example of a sequence of random variables $(X_n)_n$ that converges in distribution to $X$ but for which $\mathbb{E}X_n$ does not converge to $\mathbb{E}X$

Problem 4: Recall the Cauchy distribution with density given by

$$f_X(x) = \frac{1}{\pi(1+x^2)}.$$

Suppose $X_1, \ldots, X_n$ is a sequence of independent random variables each with this distribution, and let $A_n$ be the sample average $(\mathbb{E}X_1 + \cdots + \mathbb{E}X_n)/n$. In this exercise we will prove that $A_n$ does not converge in probability to any constant value. This gives some weight to the fact that if a random variable is not integrable we cannot sensibly assign it an expected value, even if the density function is symmetric (and therefore one might be tempted to say it has expected value zero).
(a) Prove Lagrange’s identity \((a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (ad - bc)^2\)

(b) Using the substitution \(u = x - a / x\), or otherwise, prove that for any \(a, b \geq 0\), we have \[
\int_0^\infty \frac{1}{(x^2 - a)^2 + b^2 x^2} \, dx = \frac{\pi}{2ab}.
\]

(c) Using your answers to (a) and (b), or otherwise, prove that the average of two independent Cauchy-distributed random variables has a Cauchy distribution.

(d) Deduce that if \(n\) is a power of 2 then \(A_n\) has a Cauchy distribution.

(e) Deduce that \(A_n\) does not converge in probability to any constant value.

**Problem 5:** Consider a group of \(n\) people. Each pair of people is friends with probability \(p\), independently.

(a) What is the distribution of the number of pairs of people who are friends?

(b) Suppose \(p = 1/2\), and let \(X_n\) be the number of people with fewer than \(n/4\) friends. Prove that \(X_n \overset{p}{\to} 0\).

(c) Now let \(p = 1/(n - 1)\), and let \(Y_n\) be the number of people with no friends. What do you think is the (appropriately normalised) limiting distribution of \(Y_n\)? (You don’t have to prove it, but please give some kind of brief explanation for your answer.

(d) Now let \(p = \log n / n\) (the log is base \(e\)), and let \(Z_n\) be the number of people with no friends. What do you think is the limiting distribution of \(Z_n\)? Prove it. (You may use the fact that \(\lim_{n \to \infty} n(1 - \log n / n)^n = 1\).)

**Problem 6:** Consider a sequence of random variables \(X_1, X_2, \ldots\). Let \(X\) be a constant random variable that always takes the same value \(C\). Prove that \(X_n \overset{p}{\to} C\) if and only if \(X_n \overset{d}{\to} X\). (We stated this in class but did not prove it).

**Problem 7:** In this exercise we will prove a special case of the central limit theorem with the method of moments. Let \(X_1, \ldots, X_n\) be a sequence of independent random variables each satisfying \(\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = 1/2\) (this is called the *Rademacher* distribution). Let \(S_n = X_1 + \cdots + X_n\).

(a) Using the result of problem 7a in homework 5, or otherwise, give a general formula for the moments of the standard normal distribution.
(b) Prove that for any $k \in \mathbb{N}$ and any $s_1, \ldots, s_k \in \mathbb{N}$ we have

$$\mathbb{E} \left[ \prod_{i=0}^{k} X_i^{s_i} \right] = \begin{cases} 1 & \text{if each } s_i \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

(c) Prove that for even $k$ we have

$$\binom{k}{2, \ldots, 2} \left( \binom{n}{k/2} \right) \leq \mathbb{E} S_n^k \leq \binom{k}{2, \ldots, 2} \frac{n^{k/2}}{(k/2)!}$$

(d) Deduce that $S_n/\sqrt{n} \xrightarrow{d} N(0, 1)$. 