Matching with Contracts

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We develop a model of matching with contracts which incorporates, as special cases, the college admissions problem, the Kelso-Crawford labor market matching model, and ascending package auctions. We introduce a new "law of aggregate demand" for the case of discrete heterogeneous workers and show that, when workers are substitutes, this law is satisfied by profit-maximizing firms. When workers are substitutes and the law is satisfied, truthful reporting is a dominant strategy for workers in a worker-offering auction/matching algorithm. We also parameterize a large class of preferences satisfying the two conditions. (*JEL* C78, D44)

Since the pioneering U.S. spectrum auctions of 1994 and 1995, related ascending multi-item auctions have been used with much fanfare on six continents for sales of radio spectrum and electricity supply contracts.[1] Package bidding, in which bidders can place bids not just for individual lots but also for bundles of lots ("packages"), has found increasing use in procurement applications (Milgrom, 2004; Peter Cramton et al., 2005). Recent proposals in the United States to allow package bidding for spectrum licenses and for airport landing rights incorporate ideas suggested by Lawrence Ausubel and Milgrom (2002) and by David Porter et al. (2003) (see Ausubel et al., 2005).

Matching algorithms based on economic theory also have important practical applications. Alvin E. Roth and Elliott Peranson (1999) explain how a certain two-sided matching procedure, which is similar to the college admissions algorithm introduced by David Gale and Lloyd Shapley (1962), has been adapted to match 20,000 doctors per year to medical residency programs. After Atila Abdulkadiroğlu and Tayfun Sönmez (2003) advocated a variation of the same algorithm for use by school choice programs, a similar centralized match was adopted by the New York City schools (Abdulkadiroğlu et al., 2005b) and another is being evaluated by the Boston schools (Abdulkadiroğlu et al., 2005a).

This paper identifies and explores certain similarities among all of these auction and matching mechanisms. To illustrate one similarity, consider the labor market auction model of Alexander Kelso and Vincent Crawford (1982), in which firms bid for workers in simultaneous ascending auctions. The Kelso-Crawford model assumes that workers have preferences over firm-wage pairs and that all wage offers are drawn from a prespecified finite set. If that set includes only one wage, then all that is left for the auction to determine is the match of workers to firms, so the auction is effectively transformed into a matching algorithm. The auction algorithm begins with each firm proposing employment to its most preferred set of workers at the one possible wage. When some workers turn it down, the firm makes offers to other workers

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[1] For example, a *New York Times* article about a spectrum auction in the United States was headlined "The Greatest Auction Ever" (*New York Times,* March 16, 1995, p. A17). The scientific community has also been enthusiastic. In its fiftieth anniversary self-review, the U.S. National Science Foundation reported that "from a financial standpoint, the big payoff for NSF's longstanding support [of auction theory research] came in 1995 ... [when t]he Federal Communications Commission established a system for using auctions." See Milgrom (2004).

to fill its remaining openings. This procedure is precisely the hospital-offering version of the Gale-Shapley matching algorithm. Hence, the Gale-Shapley matching algorithm is a special case of the Kelso-Crawford procedure.

The possibility of extending the National Resident Matching Program (the "Match") to permit wage competition is an important consideration in assessing public policy toward the Match, particularly because work by Jeremy Bulow and Jonathan Levin (2003) lends some theoretical support for the position that the Match may compress and reduce doctors' wages relative to a perfectly competitive standard. The practical possibility of such an extension depends on many details, including, importantly, the form in which doctors and hospitals would have to report their preferences for use in the Match. In its current incarnation, the Match can accommodate hospital preferences that encompass affirmative action constraints and a subtle relationship between internal medicine and its subspecialties, so it will be important for any replacement algorithm to allow similar preferences to be expressed. In Section IV, we introduce a parameterized family of valuations that accomplishes that.

A second important similarity is between the Gale-Shapley doctor-offering algorithm and the Ausubel-Milgrom proxy auction. Explaining this relationship requires restating the algorithm in a different form from the one used for the preceding comparison. We show that if the hospitals in the Match consider doctors to be substitutes, then the doctor-offering algorithm is equivalent to a certain *cumulative offer process* in which the hospitals at each round can choose from all the offers they have received at any round, current or past. In a different environment, where there is but a single "hospital" or auctioneer with unrestricted preferences and general contract terms, this cumulative offer process coincides exactly with the Ausubel-Milgrom proxy auction.

Despite the close connections among these mechanisms, previous analyses have treated them separately. In particular, analyses of auctions typically assume that bidders' payoffs are quasi-linear. No corresponding assumption is made in analyzing the medical match or the college admissions problem; indeed, the very possibility of monetary transfers is excluded from those formulations. As discussed below, the quasi-linearity assumption combines with the substitutes assumption in a subtle and restrictive way.

This paper presents a new model that subsumes, unifies, and extends the models cited above. The basic unit of analysis in the new model is the *contract.* To reproduce the Gale-Shapley college admissions problem, we specify that a contract is fully identified by the student and college; other terms of the relationship depend only on the parties' identities. To reproduce the Kelso-Crawford model of firms bidding for workers, we specify that a contract is fully identified by the firm, the worker, and the wage. Finally, to reproduce the Ausubel-Milgrom model of package bidding, we specify that a contract is identified by the bidder, the package of items that the bidder will acquire, and the price to be paid for that package. Several additional variations can be encompassed by the model. For example, Roth (1984, 1985b) allows that a contract might specify the particular responsibilities that a worker will have within the firm.

Our analysis of matching models emphasizes two conditions that restrict the preferences of the firms/hospitals/colleges: a *substitutes* condition and an *law of aggregate demand* condition. We find that these two conditions are implied by the assumptions of earlier analyses, so our unified treatment implies the central results of those theories as special cases.

In the tradition of demand theory, we define *substitutes* by a comparative statics condition. In demand theory, the exogenous parameter change is a price decrease, so the challenge is to extend the definition to models in which there may be no price that is allowed to change. In our contracts model, a price reduction corresponds formally to expanding the firm's opportunity set, i.e., to making the set of feasible contracts larger. Our substitutes condition asserts that when the firm chooses from an expanded set of contracts, the set of contracts it rejects also expands (weakly). As we will show, this abstract substitutes condition coincides exactly with the demand theory condition for standard models with prices. It also coincides exactly with the "substitutable preferences" condition for the college admissions problem (Roth and Marilda Sotomayor, 1990).

The *law of aggregate demand,* which is new in this paper, is similarly defined by a comparative static. It is the condition that when a college or firm chooses from an expanded set, it admits at least as many students or hires at least as many workers.[2]

The term "law of aggregate demand" is motivated by the relation of this condition to the *law of demand* in producer theory. According to producer theory, a profit-maximizing firm demands (weakly) more of any input as its price falls. For the matching model with prices, the law of *aggregate* demand requires that when any input price falls, the aggregate quantity demanded, which includes the quantities demanded of that input and all of its substitutes, rises (weakly). Notice that it is tricky even to state such a law in producer theory with divisible inputs, because there is no general aggregate quantity measure when divisible inputs are diverse. In the present model with indivisible workers, we measure the aggregate quantity of workers demanded or hired by the total number of such workers.

A key step in our analysis is to prove a new result in demand theory: *if workers are substitutes, then a profit-maximizing firm's employment choices satisfy the law of aggregate demand.* Since firms are profit maximizers and regard workers as substitutes in the Kelso-Crawford model, it follows that the law of aggregate demand holds for that model. Thus, one implication of the standard quasi-linearity assumption of auction theory is that the bidders' preferences satisfy the law of aggregate demand. We find that when preferences are *responsive* as originally and still commonly assumed in matching theory analyses, they satisfy the law of aggregate demand.[3] We also prove some new results for the class of auction and matching models that satisfy this law.

The paper is organized as follows. Section I introduces the matching-with-contracts notation, treats an allocation as a set of contracts, and characterizes the *stable allocations* in terms of the solution of a certain system of two equations.

Section II introduces the substitutes condition and uses it to prove that the set of stable allocations is a nonempty lattice, and that a certain generalization of the Gale-Shapley algorithm identifies its maximum and minimum elements. These two extreme points are characterized as a doctor-optimal/hospital-pessimal point, which is a point that is the unanimously most preferred stable allocation for the doctors and the unanimously least preferred stable allocation for the hospitals; and a hospital-optimal/doctor-pessimal point with the reverse attributes.

Section II also proves several related results. First, if there are at least two hospitals and if some hospital has preferences that do not satisfy the substitutes condition, then even if all other hospitals have just a single opening, there exists a profile of preferences for the students and colleges such that no stable allocation exists. This result is important for the construction of matching algorithms. It means that any matching procedure that permits students and colleges to report preferences that do not satisfy the substitutes condition cannot be guaranteed always to select a stable allocation with respect to the reported preferences.

Another result concerns vacancy chain dynamics, which traces the dynamic adjustment of the labor market when a worker retires or a new worker enters the market and the dynamics are represented by the operator we have described. The analysis extends the results reported by Yosef Blum et al. (1997) and foreshadowed by Kelso and Crawford (1982). We find that, starting from a stable allocation, the vacancy adjustment process converges to a new stable allocation.

Section III contains the most novel results of the paper. It introduces the law of aggregate

[2] In their study of a model of "schedule matching," Ahmet Alkan and Gale (2003) independently introduced a similar notion, which they call "size monotonicity."

[3] The original matching formulation of Gale and Shapley (1962) assumed that colleges simply have a rank order listing of students. Roth (1985a) dubbed this "responsiveness" and sought to relax this condition. Similarly, in a model of matching with wages, Crawford and E. M. Knoer (1981) assumed that firm preferences over workers were separable. That restriction was relaxed by Kelso and Crawford (1982).

demand, verifies that it holds for a profit-maximizing firm when inputs are substitutes, and explores its consequences. When both the substitutes and the law of aggregate demand conditions are satisfied, then (a) the set of workers employed and the set of jobs filled is the same at every stable collection of contracts; and (b) it is a dominant strategy for doctors (or workers or students) to report their preferences truthfully in the doctor-offering version of the extended Gale-Shapley algorithm. We also demonstrate the necessity of a weaker version of the law of aggregate demand for these conclusions.

Our conclusion about this dominant strategy property substantially extends earlier findings about incentives in matching. The first such results, due to Lester Dubins and David Freedman (1981) and Roth (1982), established the dominant strategy property for the *marriage problem,* which is a one-to-one matching problem that is a special case of the college admissions problem. Similarly, Gabrielle Demange and Gale (1985) establish the dominant strategy property for the worker-firm matching problem in which each firm has singleton preferences. These results generalize to the case of *responsive preferences,* that is, to the case where each hospital (or college or firm) behaves just the same as a collection of smaller hospitals with one opening each. For the college admissions problem, Abdulkadiroğlu (2003) has shown that the dominant strategy property also holds when colleges have *responsive preferences with capacity constraints,* where the constraints limit the number of workers of a particular type that can be hired. All of these models with a dominant strategy property satisfy our substitutes and law of aggregate demand conditions, so the earlier dominant strategy results are all subsumed by our new result.

In Section IV, we introduce a new parameterized family of preferences—the *endowed assignment preferences*—and show that they subsume certain previously identified classes and satisfy both the substitutes and law of aggregate demand conditions.

Section V introduces *cumulative offer processes* as an alternative auction/matching algorithm and shows that when contracts are substitutes, it coincides with the doctor-offering algorithm of the previous sections. Since the Ausubel-Milgrom proxy auction is also a cumulative offer process, our dominant strategy conclusion of Section III implies an extension of the Ausubel-Milgrom dominant strategy result. When contracts are not substitutes, the cumulative offer process can converge to an infeasible allocation. We show that if there is a single hospital/auctioneer, however, the cumulative offer process converges to a feasible, stable allocation, without restrictions on the hospital's preferences. Section VI concludes. All proofs are in the Appendix.

I. Stable Collections of Contracts

The matching model without transfers has many applications, of which the best known among economists is the match of doctors to hospital residency programs in the United States. For the remainder of the paper, we describe the match participants as "doctors" and "hospitals." These groups play the same respective roles as students and colleges in the college admissions problem and workers and firms in the Kelso-Crawford labor market model.

A. *Notation*

The sets of doctors and hospitals are denoted by D and H, respectively, and the set of contracts is denoted by X. We assume only that each contract $x \in X$ is bilateral, so that it is associated with one "doctor" $x_D \in D$ and one "hospital" $x_H \in H$. When all terms of employment are fixed and exogenous, the set of contracts is just the set of doctor-hospital pairs: $X \equiv D \times H$. For the Kelso-Crawford model, a contract specifies a firm, a worker, and a wage, $X \equiv D \times H \times W$.

Each doctor d can sign only one contract. Her preferences over possible contracts, including the null contract $\emptyset$, are described by the total order $>_d$. The null contract represents unemployment, and contracts are *acceptable* or *unacceptable* according to whether they are more preferred than $\emptyset$. When we write preferences as $P_d : x >_d y >_d z$, we mean that P_d names the preference order of d and that the listed contracts (in this case, x, y, and z) are the only acceptable ones.

Given a set of contracts $X' \subset X$ offered in the market, doctor d's *chosen set* $C_d(X')$ is either the null set, if no acceptable contracts are offered, or the singleton set consisting of the most preferred contract. We formalize this as follows:

$$(1) \quad C_d(X') = \begin{cases} \emptyset \text{ if } \{x \in X' | x_D = d, x >_d \emptyset\} = \emptyset \\ \{\max_{>_d}\{x \in X' | x_D = d\}\} \text{ otherwise.} \end{cases}$$

The choices of a hospital h are more complicated, because it has preferences $>_h$ over sets of doctors. Its *chosen set* is a subset of the contracts that name it, that is, $C_h(X') \subset \{x \in X' | x_H = h\}$. In addition, a hospital can sign only one contract with any given doctor:

$$(2) \quad (\forall h \in H)(\forall X' \subset X)(\forall x, x' \in C_h(X'))x \neq x' \Rightarrow x_D \neq x'_D.$$

Let $C_D(X') = \cup_{d \in D} C_d(X')$ denote the set of contracts chosen by some doctor from set X'. The remaining offers in X' are in the *rejected set:* $R_D(X') = X' - C_D(X')$. Similarly, the hospitals' chosen and rejected sets are denoted by $C_H(X') = \cup_{h \in H} C_h(X')$ and $R_H(X') = X' - C_H(X')$.

B. *Stable Allocations: Stable Sets of Contracts*

In our model, an *allocation* is a collection of contracts, as that determines the payoffs to the participants. We study allocations such that there is no alternative allocation that is strictly preferred by some hospital and weakly preferred by all of the doctors that it hires, and such that no doctor strictly prefers to reject his contract. It is a standard observation in matching theory that such an allocation is a *core* allocation in the sense that no coalition of hospitals and doctors can find another allocation, feasible for them, that all weakly prefer and some strictly prefer. This allocation is also a *stable* allocation in the sense that there is no coalition that can deviate profitably, even if the deviating coalition assumes that outsiders will remain willing to accept the same contracts.

We formalize the notion of stable allocations as follows:

DEFINITION: *A set of contracts $X' \subset X$ is a stable allocation if*

(*i*) $C_D(X') = C_H(X') = X'$ *and*
(*ii*) *there exists no hospital h and set of contracts $X'' \neq C_h(X')$ such that*

$$X'' = C_h(X' \cup X'') \subset C_D(X' \cup X'').$$

If condition (i) fails, then some doctor or hospital prefers to reject some contract; that doctor or hospital then *blocks* the allocation. If condition (ii) fails, then there is an alternative set of contracts that a hospital strictly prefers and that its corresponding doctors weakly prefer.

The first theorem states that a set of contracts is stable if any alternative contract would be rejected by some doctor or some hospital from its suitably defined opportunity set. In the formulas below, think of the doctors' opportunity set as X_D and the hospitals' opportunity set as X_H. If X' is the corresponding stable set, then X_D must include, in addition to X', all contracts that would not be rejected by the hospitals, and X_H must similarly include X' and all contracts that would not be rejected by the doctors. If X' is stable, then every alternative contract is rejected by somebody, so $X = X_H \cup X_D$. This logic is summarized in the first theorem.

THEOREM 1: *If $(X_D, X_H) \subset X^2$ is a solution to the system of equations*

$$(3) \qquad X_D = X - R_H(X_H)$$

and

$$X_H = X - R_D(X_D),$$

then $X_H \cap X_D$ is a stable set of contracts and $X_H \cap X_D = C_D(X_D) = C_H(X_H)$. Conversely, for any stable collection of contracts X', there exists some pair (X_D, X_H) satisfying (3) *such that $X' = X_H \cap X_D$.*

Theorem 1 is formulated to apply to general sets of contracts. It is the basis of our analysis of stable sets of contracts in the entire set of models treated in this paper.

II. Substitutes

In this section, we introduce our first restriction on hospital preferences, which is the restriction that contracts are *substitutes.* We use the restriction to prove the existence of a stable set of contracts and to study an algorithm that identifies those contracts.

Our substitutes condition generalizes the Roth-Sotomayor substitutable preferences condition to preferences over contracts. In words, the substitutable preferences condition states that if a contract is chosen by a hospital from some set of available contracts, then that contract will still be chosen from any smaller set that includes it. Our substitutes condition is similarly defined as follows:

DEFINITION: *Elements of X are substitutes for hospital h if for all subsets $X' \subset X'' \subset X$ we have $R_h(X') \subset R_h(X'')$.*[4]

In the language of lattice theory, which we use below, elements of X are substitutes for hospital h exactly when the function R_h is *isotone.*

In demand theory, substitutes is defined by a comparative static that uses prices. It says that, limiting attention to the domain of wage vectors at which there is a unique optimum for the hospital, the hospital's demand for any doctor d is nondecreasing in the wage of each other doctor d'.

Our next result verifies that for resource allocation problems involving prices, our definition of substitutes coincides with the standard demand theory definition. For simplicity, we focus on a single hospital and suppress its identifier h from our notation. Thus, imagine that a hospital chooses doctors' contracts from a subset of $X = D \times W$, where D is a finite set of doctors and $W = \{\underline{w}, \ldots, \bar{w}\}$ is a finite set of possible wages. Assume that the set of wages W is such that the hospitals' preferences are always strict. Suppose that $\bar{w} = \max W$ is a prohibitively high wage, so that no hospital ever hires a doctor at wage $\bar{w}$.

In demand theory, it is standard to represent the hospital's market opportunities by a vector $\mathbf{w} \in W^D$ that specifies a wage w_d at which each doctor can be hired. We can extend the domain of the choice function C to allow market opportunities to be expressed by wage vectors, as follows:

$$\text{(4)} \qquad \mathbf{w} \in W^D \Rightarrow C(\mathbf{w}) \equiv C(\{(d, w_d) | d \in D\}).$$

Formula (4) associates with any wage vector $\mathbf{w}$ the set of contracts $\{(d, w_d) | d \in D\}$ and defines $C(\mathbf{w})$ to be the choice from that set.

With the choice function extended this way, we can now describe the traditional demand theory substitutes condition. The condition asserts that increasing the wage of doctor d from w_d to w'_d (weakly) increases demand for any other doctor d'.

DEFINITION: *C satisfies the* demand-theory substitutes *condition if (i) $d \neq d'$, (ii) $(d', w_{d'}) \in C(\mathbf{w})$ and (iii) $w'_d > w_d$ imply that $(d', w_{d'}) \in C(w'_d, \mathbf{w}_{-d})$.*

To compare the two conditions, we need to be able to assign a vector of wages to each set of contracts X'. It is possible that, in X', some doctor is unavailable at any wage or is available at several different wages. For a profit-maximizing hospital, the doctor's relevant wage is the lowest wage, if any, at which she is available. Moreover, such a hospital does not distinguish between a doctor who is unavailable and one who is available only at a prohibitively high wage. Thus, from the perspective of a profit-maximizing hospital, having contracts X' available is equivalent to facing a wage vector $\hat{W}(X')$ specified as follows:

$$\text{(5)} \qquad \hat{W}_d(X') = \min\{s | s = \bar{w} \text{ or } (d, s) \in X'\}.$$

In view of the preceding discussion, a profit-maximizing hospital's choices must obey the following identity:

[4] This condition is identical in form to the "condition alpha" used in the study of social choice, for example by Amartya Sen (1970), but the meanings of the choice sets are different. In Sen's model, the choice set is a collection of mutually exclusive choices, whereas in the matching problems, the set describes the choice itself.

$$C(X') = C(\hat{W}(X')). \tag{6}$$

THEOREM 2: *Suppose that $X = D \times W$ is a finite set of doctor-wage pairs and that* (6) *holds. Then C satisfies the* demand theory substitutes *condition if and only if its contracts are substitutes.*

In particular, this shows that the Kelso-Crawford "gross substitutes" condition is subsumed by our substitutes condition.

A. *A Generalized Gale-Shapley Algorithm*

We now introduce a monotonic algorithm that will be shown to coincide with the Gale-Shapley algorithm on its original domain. To describe the monotonicity that is found in the algorithm, let us define an order on $X \times X$ as follows:

$$((X_D, X_H) \geq (X'_D, X'_H)) \Leftrightarrow (X_D \supset X'_D \text{ and } X_H \subset X'_H). \tag{7}$$

With this definition, $(X \times X, \geq)$ is a finite lattice.

The generalized Gale-Shapley algorithm is defined as the iterated applications of a certain function $F : X \times X \rightarrow X \times X$, as defined below.

$$\begin{aligned} F_1(X') &= X - R_H(X') \\ F_2(X') &= X - R_D(X') \\ F(X_D, X_H) &= (F_1(X_H), F_2(F_1(X_H))). \end{aligned} \tag{8}$$

As we have previously observed, since the doctors' choices are singletons, a revealed preference argument establishes that the function $R_D : X \rightarrow X$ is isotone. If the contracts are substitutes for the hospitals, then the function $R_H : X \rightarrow X$ is isotone as well. When both are isotone, the function $F : (X \times X, \geq) \rightarrow (X \times X, \geq)$ is also isotone, that is, it satisfies $((X_D, X_H) \geq (X'_D, X'_H)) \Rightarrow (F(X_D, X_H) \geq F(X'_D, X'_H))$.

Thus, $F : X \times X \rightarrow X \times X$ is an isotone function from a finite lattice into itself. Using fixed point theory for finite lattices, the set of fixed points is a nonempty lattice, and iterated applications of F, starting from the minimum and maximum points of $X \times X$, converge monotonically to a fixed point of F.[5] We summarize the particular application here with the following theorem.[6]

THEOREM 3: *Suppose contracts are substitutes for the hospitals. Then:*

(*a*) *The set of fixed points of F on $X \times X$ is a nonempty finite lattice, and in particular includes a smallest element $(\underline{X}_D, \underline{X}_H)$ and a largest element $(\bar{X}_D, \bar{X}_H)$;*

(*b*) *Starting at $(X_D, X_H) = (X, \emptyset)$, the generalized Gale-Shapley algorithm converges monotonically to the highest fixed point*

$$(\bar{X}_D, \bar{X}_H) = \sup\{(X', X'') | F(X', X'') \geq (X', X'')\}; \text{ and}$$

(*c*) *Starting at $(X_D, X_H) = (\emptyset, X)$, the generalized Gale-Shapley algorithm converges monotonically to the lowest fixed point*

$$(\underline{X}_D, \underline{X}_H) = \inf\{(X', X'') | F(X', X'') \leq (X', X'')\}.$$

The facts that $(\bar{X}_D, \bar{X}_H)$ is the highest fixed point of F and that $(\underline{X}_D, \underline{X}_H)$ is the lowest in the specified order mean that for any other fixed point (X_D, X_H), $\underline{X}_D \subset X_D \subset \bar{X}_D$ and $\bar{X}_H \subset X_H \subset \underline{X}_H$. Because doctors are better off when they can choose from a larger set of contracts, it

[5] This special case of Tarski's fixed point theorem can be simply proved as follows: Let Z be a finite lattice with maximum point $\bar{z}$. Let $z_0 = \bar{z}$, $z_1 = F(z_0)$, ... , $z_n = F(z_{n-1})$. Plainly, $z_1 \leq z_0$ and, since F is isotone, $z_2 = F(z_1) \leq F(z_0) = z_1$ and similarly $z_{n+1} \leq z_n$ for all n. So, the decreasing sequence $\{z_n\}$ converges in a finite number of steps to a point $\hat{z}$ with $F(\hat{z}) = \hat{z}$. Moreover, for any fixed point $\tilde{z}$, since $\tilde{z} \leq \bar{z}$, $\tilde{z} = F^n(\tilde{z}) \leq F^n(\bar{z}) = \hat{z}$ for n large, so $\hat{z}$ is the maximum fixed point. A similar argument applies for the minimum fixed point.

[6] Hiroyuki Adachi (2000) and Federico Echenique and Jorge Oviedo (2004) have also characterized the Gale-Shapley algorithm for the marriage problem and the college admissions problem, respectively, as iterated applications of a certain operator on a lattice. See also Tamás Fleiner (2003).

follows that the doctors unanimously weakly prefer $C_D(\bar{X}_D)$ to $C_D(X_D)$ to $C_D(\underline{X}_D)$ and similarly that the hospitals unanimously prefer $C_H(\underline{X}_H)$ to $C_H(X_H)$ to $C_H(\bar{X}_H)$. Notice, by Theorem 1, that $C_D(\bar{X}_D) = C_H(\bar{X}_H) = \bar{X}_D \cap \bar{X}_H$ and $C_D(\underline{X}_D) = C_H(\underline{X}_H) = \underline{X}_D \cap \underline{X}_H$, so we have the following welfare conclusion.

THEOREM 4: *Suppose contracts are substitutes for the hospitals. Then, the stable set of contracts $\bar{X}_D \cap \bar{X}_H$ is the unanimously most preferred stable set for the doctors and the unanimously least preferred stable set for the hospitals. Similarly, the stable set $\underline{X}_D \cap \underline{X}_H$ is the unanimously most preferred stable set for the hospitals and the unanimously least preferred stable set for the doctors.*

Theorems 3 and 4 duplicate and extend familiar conclusions about stable matches in the Gale-Shapley matching problem and a similar conclusion about equilibrium prices in the Kelso-Crawford labor market model. These new theorems encompass both these older models, and additional ones with general contract terms.

To see how the Gale-Shapley algorithm is encompassed, consider the doctor-offering algorithm. As in the original formulation, we suppose that hospitals have a ranking of doctors that is independent of the other doctors they will hire, so hospital h just chooses its n_h most preferred doctors (from among those who are acceptable and have proposed to hospital h).

Let $X_H(t)$ be the cumulative set of contracts offered by the doctors to the hospitals through iteration t, and let $X_D(t)$ be the set of contracts that have not yet been rejected by the hospitals through iteration t. Then, the contracts "held" at the end of the iteration are precisely those that have been offered but not rejected, which are those in $X_D(t) \cap X_H(t)$. The process initiates with no offers having been made or rejected, so $X_D(0) = X$ and $X_H(0) = \emptyset$.

Iterated applications of the operator F described above define a monotonic process, in which the set of doctors makes an ever-larger (accumulated) set of offers and the set of unrejected offers grows smaller round by round. Using the specification of F and starting from the extreme point $(X, \emptyset)$, we have

$$(9) \qquad X_D(t) = X - R_H(X_H(t-1))$$
$$X_H(t) = X - R_D(X_D(t)).$$

After offers have been made in iteration $t - 1$, the hospital's cumulative set of offers is $X_H(t - 1)$. Each hospital h holds onto the n_h best offers it has received at any iteration, provided that many acceptable offers have been made; otherwise, it holds all acceptable offers that have been made. Thus, the accumulated set of rejected offers is $R_H(X_H(t - 1))$ and the unrejected offers are those in $X - R_H(X_H(t - 1)) = X_D(t)$. At round t, if a doctor's contract is being held, then the last offer the doctor made was its best contract in $X_D(t)$. If a doctor's last offer was rejected, then its new offer is its best contract in $X_D(t)$. The contracts that doctors have not offered at this round or any earlier one are therefore those in $R_D(X_D(t))$. So, the accumulated set of offers doctors have made are those in $X - R_D(X_D(t)) = X_H(t)$. Once a fixed point of this process is found, we have, by Theorem 1, a stable set of contracts $X_H(t) \cap X_D(t)$.

To illustrate the algorithm, consider a simple example with two doctors and two hospitals, where $X \equiv D \times H$ and agents have the following preferences:

$$(10) \qquad P_{d_1} : h_1 > h_2 \qquad\qquad P_{h_1} : \{d_1\} > \{d_2\} > \emptyset$$
$$P_{d_2} : h_1 > h_2 \qquad\qquad P_{h_2} : \{d_1, d_2\} > \{d_1\} > \{d_2\} > \emptyset.$$

The algorithm is initialized with $X_D(0) = X = \{(d_1, h_1), (d_1, h_2), (d_2, h_1), (d_2, h_2)\}$ and $X_H(0) = \emptyset$. Table 1 applies the operator given in (9) to illustrate the algorithm.

For $t = 1$, starting from $X_D(1) = X$, both doctors choose to work for the hospital h_1, so they reject their contracts with hospital h_2: thus, $R_D(X_D(1)) = \{(d_1, h_2), (d_2, h_2)\}$. Next, $X_H(1)$ is calculated as the complement of $R_D(X_D(1))$, so $X_H(1) = \{(d_1, h_1), (d_2, h_1)\}$. From $X_H(1)$, the hospitals choose (d_1, h_1) and reject (d_2, h_1), completing the row for $t = 1$.

TABLE 1—ILLUSTRATION OF THE ALGORITHM

	$X_D(t)$	$R_D(X_D(t))$	$X_H(t)$	$R_H(X_H(t))$
$t = 0$	X	$\{(d_1, h_2), (d_2, h_2)\}$	$\varnothing$	$\varnothing$
$t = 1$	X	$\{(d_1, h_2), (d_2, h_2)\}$	$\{(d_1, h_1), (d_2, h_1)\}$	$\{(d_2, h_1)\}$
$t = 2$	$\{(d_1, h_1), (d_1, h_2), (d_2, h_2)\}$	$\{(d_1, h_2)\}$	$\{(d_1, h_1), (d_2, h_1), (d_2, h_2)\}$	$\{(d_2, h_1)\}$
$t = 3$	$\{(d_1, h_1), (d_1, h_2), (d_2, h_2)\}$	$\{(d_1, h_2)\}$	$\{(d_1, h_1), (d_2, h_1), (d_2, h_2)\}$	$\{(d_2, h_1)\}$

For $t = 2$, we first compute $X_D(2)$ as the complement of $R_H(X_H(1))$. The doctors choose $\{(d_1, h_1), (d_2, h_2)\}$, so they reject $\{(d_1, h_2)\}$. The hospitals must then choose from the complement $X_H(2) = X - \{(d_1, h_2)\}$. The hospitals choose (d_1, h_1) and (d_2, h_2) and reject (d_2, h_1). In the third round, $X_D(3) = X_D(2)$ and the process has reached a fixed point.

The example illustrates the monotonicity of the algorithm: $X_H(t)$ grows larger step by step, while $X_D(t)$ grows smaller. At termination of the algorithm, the intersection of the choice sets is the stable set of contracts $X_H(3) \cap X_D(3) = \{(d_1, h_1), (d_2, h_2)\}$. Moreover, when $X_D(t)$ and $X_H(t)$ are interpreted as suggested above, the process $\{X_D(t), X_H(t)\}$ described by (9) with the initial conditions $X_D(0) = X$ and $X_H(0) = \varnothing$ coincides with the doctor-offering Gale-Shapley algorithm.

For the hospital-offering algorithm, a similar analysis applies, but with a different interpretation of the sets and a different initial condition. Let $X_D(t)$ be the cumulative set of contracts offered by the hospitals to the doctors *before* iteration t, and let $X_H(t)$ be the set of contracts that have not yet been rejected by the hospitals *up to and including* iteration t. Then, the contracts "held" at the end of iteration t are precisely those that have been offered but not rejected, which are those in $X_D(t + 1) \cap X_H(t)$. With this interpretation, the analysis is identical to the one above. The Gale-Shapley hospital offering algorithm is characterized by (9) and the initial conditions $X_D(0) = \varnothing$ and $X_H(0) = X$.

The same logic applies to the Kelso-Crawford model, provided one extends their original treatment to include a version in which the workers make offers in addition to the treatment in which firms make offers. The words of the preceding paragraphs apply exactly, but a contract offer now includes a wage, so, for example, a hospital whose contract offer is rejected by a doctor may find that its next most preferred contract to the same doctor is at a higher wage.

B. *When Contracts Are* Not *Substitutes*

It is clear from the preceding analysis that that definition of substitutes is just sufficient to allow our mathematical tools to be applied. In this section, we establish more. We show that unless contracts are substitutes for every hospital, the very existence of a stable set of contracts cannot be guaranteed.[7]

THEOREM 5: *Suppose H contains at least two hospitals, which we denote by h and h'. Further suppose that R_h is not isotone, that is, contracts are not substitutes for h. Then there exist preference orderings for the doctors in set D, a preference ordering for a hospital h' with a single job opening such that, regardless of the preferences of the other hospitals, no stable set of contracts exists.*

Together, Theorems 3 and 5 characterize the set of preferences that can be allowed as inputs into a matching algorithm if we wish to guarantee that the outcome of the algorithm is a stable set of contracts with respect to the reported preferences. According to Theorem 3, we can allow all preferences that satisfy substitutes and still reach an outcome that is a stable collection of contracts. According to Theorem 5, if we allow any preference that does not satisfy the substitutes condition, then there is some profile of singleton preferences for the other parties such that no stable collection of contracts exists.

[7] Kelso and Crawford (1982) provide a particular example in which preferences do not satisfy substitutes and there does not exist a stable allocation.

This theory also reaffirms and extends the close connection between the substitutes condition and other concepts that has been established in the recent auctions literature with quasi-linear preferences. Milgrom (2000) studies an auction model with discrete goods in which the set of possible bidder values is required to include all the additive functions.[8] He shows that if goods are substitutes, then a competitive equilibrium exists. If, however, there are at least three bidders and if there is any allowed value such that the goods are not all substitutes, then there is some profile of values such that no competitive equilibrium exists. Faruk Gul and Ennio Stacchetti (1999) establish the same positive existence result. They also show that if preferences include all values in which a bidder wants only one particular good as well as any one for which goods are not all substitutes, and if the number of bidders is sufficiently large, then there is some profile of preferences for which no competitive equilibrium exists. Ausubel and Milgrom (2002) establish that if (a) there is some bidder for whom preferences are not demand theory substitutes, (b) values may be any additive function, and (c) there are at least three bidders in total, then there is some profile of preferences such that the Vickrey outcome is not stable and the core imputations do not form a lattice. Conversely, if all bidders have preferences that are demand theory substitutes, then the Vickrey outcome is in the core and the core imputations do form a lattice. Taken together, these results establish a close connection between the substitutes condition, the cooperative concept of the core, the noncooperative concepts of Vickrey outcomes, and competitive equilibrium.

C. "*Vacancy Chain*" *Dynamics*

Suppose that a labor market has reached equilibrium, with all interested doctors placed at hospitals in a stable match. Then suppose one doctor retires. Imagine a process in which a hospital seeks to replace its retired doctor by raiding other hospitals to hire additional doctors. If the hospital makes an offer that would succeed in hiring a doctor away from another hospital, the affected hospital has three options: it may make an offer to another doctor (or several), improve the terms for its current doctor, or leave the position vacant. Suppose it makes whatever contract offer would best serve its purposes.

In models with a fixed number of positions and no contracts, this process in which doctors and vacancies move from one hospital to another has been called *vacancy chain dynamics* (Blum et al., 1997). Kelso and Crawford (1982) consider similar dynamics in the context of their model.

The formal results for our extended model are similar to those of the older theories. Starting with a stable collection of contracts X', let X'_H be the set of contracts that some doctor weakly prefers to her current contract in X', and let $X'_D = X' \cup (X - X'_H)$. As in the proof of Theorem 1, we have $F(X'_D, X'_H) = (X'_D, X'_H)$ and $X' = X'_D \cap X'_H$.

To study the dynamic adjustment that results from the retirement of doctor d, we suppose the process starts from the initial state $(X_D(0), X_H(0)) = (X'_D, X'_H)$. This means that the employees start by considering only offers that are at least as good as their current positions and that hospitals remember which employees have rejected them in the past. The doctors' rejection function is changed by the retirement of doctor d to $\hat{R}_D$, where $\hat{R}_D(X'') = R_D(X'') \cup \{x \in X'' | x_D = d\}$; that is, in addition to the old rejections, all contract offers addressed to the retired doctor are rejected.

To synchronize the timing with our earlier notation, let us imagine that hospitals make offers at round $t - 1$ and doctors accept or reject them at round t. Hospitals consider as potentially available the doctors in $X_H(t - 1) = X - \hat{R}_D(X_D(t - 1))$ and the doctors then reject all but the best offers, so the cumulative set of offers received is $X_D(t) = X - R_H(X_H(t - 1))$. Define

$$(11)\quad \hat{F}(X_D, X_H) = (X - R_H(X_H), X - \hat{R}_D(X_D)).$$

[8] A valuation function is additive if the value of any set of items is the sum of the separate values of the elements. Such a function v is also described as "modular." Additivity is equivalent to the requirement that for all sets A and B, $v(A \cup B) + v(A \cap B) = v(A) + v(B)$.

If contracts are substitutes for the hospitals, then $\hat{F}$ is isotone and, since $F(X'_D, X'_H) = (X'_D, X'_H)$, it follows that $\hat{F}(X'_D, X'_H) \geq (X'_D, X'_H)$. Then, since $(X_D(0), X_H(0)) = (X'_D, X'_H)$, we have $(X_D(1), X_H(1)) = \hat{F}(X_D(0), X_H(0)) \geq (X_D(0), X_H(0))$. Iterating, $(X_D(n), X_H(n)) = \hat{F}(X_D(n-1), X_H(n-1)) \geq (X_D(n-1), X_H(n-1))$: the doctors accumulate offers and the contracts that are potentially available to the hospitals shrink. A fixed point is reached and, by Theorem 1, it corresponds to a stable collection of contracts.

THEOREM 6: *Suppose that contracts are substitutes and that (X'_D, X'_H) is a stable set of contracts. Suppose that a doctor retires and that the ensuing adjustment process is described by $(X_D(0), X_H(0)) = (X'_D, X'_H)$ and $(X_D(t), X_H(t)) = \hat{F}(X_D(t-1), X_H(t-1))$. Then, the sequence $\{(X_D(t), X_H(t))\}$ converges to a stable collection of contracts at which all the unretired doctors are weakly better off and all the hospitals are weakly worse off than at the initial state (X'_D, X'_H).*[9]

The sequence of contract offers and job moves described by iterated applications of $\hat{F}$ includes possibly complex adjustments. Hospitals that lose a doctor may seek several replacements. Hospitals whose doctors receive contract offers may retain those doctors by offering better terms or may hire a different doctor and later rehire the original doctor at a new contract. All along the way, the doctors find themselves choosing from more and better options, and the hospitals find themselves marching down their preference lists by offering costlier terms, paying higher wages, or making offers to other doctors whom they had earlier rejected.

III. Law of Aggregate Demand

We now introduce a second restriction on preferences that allows us to prove the next two results about the structure of the set of stable matches. We call this restriction the law of aggregate demand. Roughly, this law states that as the price falls, agents should demand more of a good. Here, price falls correspond to more contracts being available, and more demand corresponds to taking on (weakly) more contracts. We formalize this intuition with the following definition.

DEFINITION: *The preferences of hospital $h \in H$ satisfy the* law of aggregate demand *if for all $X' \subset X''$, $|C_h(X')| \leq |C_h(X'')|$.*

According to this definition, if the set of possible contracts expands (analogous to a decrease in some doctors' wages), then the total number of contracts chosen by hospital h either rises or stays the same. The corresponding property for doctor preferences is implied by revealed preference, because each doctor chooses at most one contract. Just as for the substitutes condition, when wages are endogenous, we interpret the definition as applying to the domain of wage vectors for which the hospital's optimum is unique.

The next theorem shows the important relationship between profit maximization, substitutes, and the law of aggregate demand.

THEOREM 7: *If hospital h's preferences are quasi-linear and satisfy the substitutes condition, then they satisfy the law of aggregate demand.*

Below, we use the law of aggregate demand to characterize both necessary and sufficient conditions for the rural hospitals' property and ensure that truthful revelation is a dominant strategy for the doctors. Previously, in matching models without money, the dominant strategy result was known only for responsive preferences with capacity constraints (Abdulkadiroğlu, 2003). We subsume that result with our theorem.

A. *Rural Hospitals Theorem*

In the match between doctors and hospitals, certain rural hospitals often had trouble filling all their positions, raising the question of whether there are other core matches at which the rural hospitals might do better. Roth (1986) analyzed this question for the case of $X = D \times$

[9] The theorem does not claim, and it is not generally true, that this new point must be the new doctor-best stable set of contracts.

H and responsive preferences, and found that the answer is no: every hospital that has unfilled positions at some stable match is assigned exactly the same doctors at every stable match. In particular, every hospital hires the same number of doctors at every stable match.

In this section, we show by an example that this last conclusion does not generalize to the full set of environments in which contracts are substitutes.[10] We then prove that if preferences satisfy the law of aggregate demand and substitutes, then the last conclusion of Roth's theorem holds: every hospital signs exactly the same *number* of contracts at every point in the core, although the doctors assigned and the terms of employment can vary. Finally, we show that any violation of the law of aggregate demand implies that preferences exist such that the above conclusion does not hold.

Suppose that $H = \{h_1, h_2\}$ and $D = \{d_1, d_2, d_3\}$. For hospital h_1, suppose its choices maximize $>$, where $\{d_3\} > \{d_1, d_2\} > \{d_1\} > \{d_2\} > \emptyset > \{d_1, d_3\} > \{d_2, d_3\}$. This preference satisfies substitutes.[11] Suppose h_2 has one position, with its preferences given by $\{d_1\} > \{d_2\} > \{d_3\} > \emptyset$. Finally, suppose d_1 and d_2 prefer h_1 to h_2 while d_3 has the reverse preference. Then, the matches $X' = \{(h_1, d_3), (h_2, d_1)\}$ and $X'' = \{(h_1, d_1), (h_1, d_2), (h_2, d_3)\}$ are both stable but hospital h_1 employs a different number of doctors and the set of doctors assigned differs between the two matches.

This example involves a failure of the law of aggregate demand, because as the set of contracts available to h_1 expands by the addition of (h_1, d_3) to the set $\{(h_1, d_1), (h_1, d_2)\}$, the number of doctors demanded declines from two to one. When the law of aggregate demand holds, however, we have the following result.

THEOREM 8: *If hospital preferences satisfy substitutes and the law of aggregate demand, then for every stable allocation* (X_D, X_H) *and every* $d \in D$ *and* $h \in H$, $|C_d(X_D)| = |C_d(\bar{X}_D)|$ *and* $|C_h(X_H)| = |C_h(\bar{X}_H)|$. *That is, every doctor and hospital signs the same number of contracts at every stable collection of contracts.*

The next theorem verifies that the counterexamples developed above can always be generalized whenever any hospital's preferences violate the law of aggregate demand.

THEOREM 9: *If there exists a hospital* h, *sets* $X' \subset X'' \subset X$ *such that* $|C_h(X')| > |C_h(X'')|$, *and at least one other hospital, then there exist singleton preferences for the other hospitals and doctors such that the number of doctors employed by* h *is different for two stable matches.*

Theorem 9 establishes that the law of aggregate demand is not only a sufficient condition for the rural hospitals result of Theorem 8 but, in a particular sense, a necessary one as well.

B. *Truthful Revelation as a Dominant Strategy*

The main result of this section concerns doctors' incentives to report their preferences truthfully. For the doctor-offering algorithm, if hospital preferences satisfy the law of aggregate demand and the substitutes condition, then it is dominant strategy for doctors to reveal truthfully their preferences over contracts.[12] We further show that both preference conditions play essential roles in the conclusion.

We will show the positive incentive result for the doctor-offering algorithm in two steps which highlight the different roles of the two preference assumptions. First, we show that the substitutes condition, by itself, guarantees that doctors cannot benefit by exaggerating the ranking of an unattainable contract. More precisely, if there exists a preferences list for a doctor d such that d obtains contract x by submitting this list, then d can also obtain x by submitting a preference list that includes only contract x. Second, we will show that adding the law of

[10] A similar example appears in Ruth Martínez et al. (2000).

[11] These preferences, however, do not display the "single improvement property" that Gul and Stacchetti (1999) introduce and show is characteristic of substitutes preferences in models with quasi-linear utility.

[12] It is, of course, not a dominant strategy for hospitals to truthfully reveal; nor would it be so even if we considered the hospital-offering algorithm. For further discussion of this point, see Roth and Sotomayor (1990).

aggregate demand guarantees that a doctor does at least as well as reporting truthfully as by reporting any singleton. Together, these are the dominant strategy result.

To understand why submitting unattained contracts cannot help a doctor d, consider the following. Let x be the most-preferred contract that d can obtain by submitting any preference list (holding all other submitted preferences fixed). Note that all that d accomplishes when reporting that certain contracts are preferred to x is to make it easier for some coalition to block outcomes involving x. Thus, if x is attainable with any report, it is attainable with the report $P_d : x$ that ranks x as the only acceptable contract. This intuition is captured in the following theorem.

THEOREM 10: *Let hospitals' preferences satisfy the substitutes condition and let the matching algorithm produce the doctor-optimal match. Fixing the preferences of hospitals and of doctors besides d, let x be the outcome that d obtains by reporting preferences* $P_d : z_1 >_d z_2 >_d \cdots >_d z_n >_d x$. *Then, the outcome that d obtains by reporting preferences* $P'_d : x$ *is also x.*

Some other doctors may be strictly better off when d submits her shorter preference list; there are fewer collections of contracts that d now objects to, so the core may become larger, and the doctor-optimal point of the enlarged core makes all doctors weakly better off and may make some strictly better off.

Without the law of aggregate demand, however, it may still be in a doctor's interest to conceal her preferences for unattainable positions. To see this, consider the case with two hospitals and three doctors, where contracts are simply elements of $D \times H$, and let preferences be:

$$P_{d_1} : h_1 > h_2 \tag{12}$$

$$P_{h_1} : \{d_3\} > \{d_1, d_2\} > \{d_1\} > \{d_2\}$$

$$P_{d_2} : h_2 > h_1$$

$$P_{h_2} : \{d_1\} > \{d_2\} > \{d_3\}$$

$$P_{d_3} : h_2 > h_1.$$

With these preferences, the only stable match is $\{(d_1, h_2), (d_3, h_1)\}$, which leaves d_2 unemployed. However, if d_2 were to reverse her ranking of the two hospitals, then $\{(d_1, h_1), (d_2, h_1), (d_3, h_2)\}$ would be chosen by the doctor-offering algorithm, leaving d_2 better off. Essentially, by offering a contract to h_2, d_2 has changed the number of positions available. When the preferences of the hospitals satisfy the law of aggregate demand, however, making more offers to the hospitals (weakly) increases the number of contracts the hospitals accept.

THEOREM 11: *Let hospitals' preferences satisfy substitutes and the law of aggregate demand, and let the matching algorithm produce the doctor-optimal match. Then, fixing the preferences of the other doctors and of all the hospitals, let x be the contract that d obtains by submitting the set of preferences* $P_d : z_1 >_d z_2 >_d \cdots >_d z_n >_d x$. *Then the preferences* $P''_d : y_1 > y_2 > \cdots > y_n > x > y_{n+1} > \cdots > y_N$ *obtain a contract that is* P''_d*-preferred or indifferent to x.*

According to this theorem, when a doctor's true preferences are P''_d, the doctor can never do better according to these true preferences than by reporting the preferences truthfully.

In fact, the law of aggregate demand is "almost" a necessary condition as well. The exceptions can arise because certain violations of the law of aggregate demand are unobservable from the choice data of the algorithm, and these cannot affect incentives. Thus, consider an example where terms t are included in the contract, and where a hospital h has preferences $\{(d_1, h, t_1)\} > \{(d_1, h, \tilde{t}_1), (d_2, h, t_2)\} > \{(d_1, h, \tilde{t}_1)\} > \{(d_2, h, t_2)\}$. Although these preferences violate the law of aggregate demand, the algorithm will never "see" the violation, as either $(d_1, h, t_1) >_{d_1} (d_1, h, \tilde{t}_1)$ or $(d_1, h, \tilde{t}_1) >_{d_1} (d_1, h, t_1)$. Thus, whichever terms that d_1 first offers will determine a conditional set of preferences for the hospital that does satisfy the law of aggregate demand. (The hospital will never reject an offer of either (d_1, h, t_1) or $(d_1, h, \tilde{t}_1)$.)

The next theorem says that if some hospital's preferences violate the law of aggregate demand in a way that can even potentially be observed from the hospital's choices, then there exist preferences for the other agents such that it is *not* a dominant strategy for doctors to report truthfully, even when the other assumptions we have used are satisfied.

THEOREM 12: *Let hospital h have preferences such that* $|C_h(X)| > |C_h(X \cup \{x\})|$ *and let there exist two contracts* y, z *such that* $y_D \neq z_D \neq x_D \neq y_D$ *and* $y, z \in R_h(X \cup \{x\}) - R_h(X)$. *Then if another hospital* h' *exists, there exist singleton preferences for the hospitals besides h and preferences for the doctors such that it is not a dominant strategy for all doctors to reveal their preferences truthfully.*

Thus, to the extent that the law of aggregate demand for hospital preferences has observable consequences for the progress of the doctor-offering algorithm, it is an indispensable condition to ensure the dominant strategy property for doctors.

IV. Classes of Conforming Preferences

Although the class of substitutes valuations is quite limited,[13] it is broader than the set of responsive preferences in useful ways. The substitutes valuations accommodate all the affirmative action and subspecialty constraints described above and allow a doctor's marginal product to depend on which other doctors the hospital attracts.

In this section, we introduce a parameterized class of quasi-linear substitutes valuations for hospitals evaluating sets of new hires. The valuations are based on an *endowed assignment model,* according to which hospitals have a set of jobs to fill and an existing endowment of doctors, while doctors' productivities vary among jobs. A hospital values new doctors according to the their incremental value in the assignment problem.

From a general job assignment perspective, the key restrictions of endowed assignment valuations are that each doctor can do one and only one job and that the hospital's output is the sum of outputs of its various jobs. A hospital cannot use one doctor for two jobs, nor can it combine the skills of two doctors in the same job. Also, there is no interaction in the productivity of doctors in different jobs. Given this mathematical structure, if a doctor is lost, it will always be optimal for the hospital to retain all of its other doctors, although it may choose to reassign some of the retained doctors to fill vacated positions.

Another way to describe the set of *endowed assignment valuations,* V_{EA}, is to build it up from three properties, as follows. First, a *singleton valuation* is one of the form $v(S) = \max_{d \in S} \alpha_d$ for some nonnegative vector $\boldsymbol{\alpha}$. Singleton valuations represent the possible valuations by a hospital with just one opening, and V_{EA} includes all singleton valuations. Second, if a hospital is composed of two units, $j = 1, 2$, each of which has a valuation $v_j \in V_{EA}$, then the hospital's maximum value from assigning workers between its units, denoted by $(v_1 \vee v_2)(S) \equiv \max_{R \subset S} v_1(R) + v_2(S - R)$, is also in the set. We call this property *closure under aggregation.* Third, if a hospital's value is derived from a value in V_{EA} by endowing the hospital with a set of doctors T, then the hospital's incremental value for extra doctors $v(S|T) \equiv v(S \cup T) - v(T)$ is also in the set V_{EA}. We call this property *closure under endowment.* Finally, two valuations v_1 and v_2 are called *equivalent* if they differ by an additive constant. For example, for any fixed set of doctors T, $v(\cdot\,|T)$ is equivalent to $v(\cdot \cup T)$.

THEOREM 13: *Let* V_{EA} *denote the smallest family of valuations of sets of doctors that includes all the singleton valuations and is closed under aggregation and endowment. Then, for each* $v \in V_{EA}$, *there exists a set of jobs, J, a set of doctors T, and a* $(|D| + |T|) \times |J|$*-matrix* $[\alpha_{ij}]$, *such that v is equivalent to the following:*

[13] Hans Reijnierse et al. (2002) (see also Yuan-Chuan Lien and Jun Yan, 2003) have characterized the valuation functions that satisfy the substitutes condition as follows. Let $v(S)$ be the hospital's value of doctor set S and let $v_S(T) = v(S \cup T) - v(S)$ denote the incremental value of the set of doctors T. Then, doctors are substitutes if for every set S and three other doctors d_1, d_2 and d_3, there is no unique maximum in $\{v_S(d_1) + v_S(d_2, d_3), v_S(d_2) + v_S(d_1, d_3), v_S(d_3) + v_S(d_1, d_2)\}$.

$$(13)\quad v(S|J, \boldsymbol{\alpha}, T) \equiv \max_{z} \sum_{d \in S} \alpha_{dj} z_{dj} \text{ subject to}$$

$$\sum_{d \in S} z_{dj} \leq 1 \text{ for } j \in J$$

$$\sum_{j \in J} z_{dj} \leq \begin{cases} 1 \text{ for } d \in S \cup T \\ 0 \text{ for } d \notin S \cup T \end{cases}$$

$$z_{dj} \in \{0, 1\} \text{ for all } d, j.$$

Conversely, for every such J, T, *and* $\boldsymbol{\alpha}$, $v(\,\cdot\,|J, \boldsymbol{\alpha}, T) \in V_{\mathrm{EA}}$.

We dub this family of valuations the *endowed assignment valuations* because of the form of optimization (13). The family V_{EA} is an extension of the assignment valuations derived by Lloyd Shapley (1962), used by Roth (1984, 1985b) in models of matching with wages and explicit job assignments, and emphasized in a study of package bidding by Lehmann et al. (2001). The endowed assignment valuations are quite useful for applications. They also fit nicely with the theory reported in previous sections for the following reason.

THEOREM 14: *Every endowed assignment valuation* $v(\,\cdot\,|J, \boldsymbol{\alpha}, T) \in V_{\mathrm{EA}}$ *is a substitutes valuation.*

To the best of our knowledge, all of the substitutes valuations that have been used or proposed for practical applications are included among the endowed assignment valuations. Indeed, the question of whether all substitutes valuations are endowed assignment valuations is an open one. Here, we review some of the most popular valuations to show how they fit into the class.

The most commonly studied class of preferences for matching problems like the Match, for which employment terms are exogenously fixed, are the *responsive* preferences. These specify that each hospital h has a fixed number of openings n_h, a set of acceptable doctors $D_h^A \subset D$, and a strict ordering $>_h$ of the acceptable doctors. When a doctor is unacceptable, that means that hiring the doctor is always worse than leaving her position unfilled. Given any set of available doctors, the hospital hires its n_h most preferred acceptable doctors, if that many are available, and otherwise hires all of the acceptable doctors.

To map responsive preferences into the assignment problem framework, define a utility function $u_h : D \rightarrow \mathbb{R}$ to represent $>_h$ and set the minimum acceptable utility to zero. Formally, (1) if $d, d' \in D_h^A$, then $d >_h d' \Leftrightarrow u_h(d) > u_h(d')$ and (2) $d \in D_h^A \Leftrightarrow u_h(d) > 0$. Using this utility, we specify an assignment problem as follows: $J = \{1, \ldots, n_h\}$, and $\alpha_{dj} = u_h(d) + 1$, and fix the wage at 1. Using a positive wage is a device to ensure that the hospital strictly prefers not to hire a doctor it plans not to assign to any job. It is evident that with this valuation preference, the hospital most prefers to hire the set of available, acceptable doctors with the highest ranking according to $>_h$.

Among the extensions of responsive preferences that have been most important in practice are ones to accommodate quotas of various kinds. For example, one use of quotas is to manage subspecialties of internal medicine, in which a certain number of positions are to be filled by doctors planning that subspecialty if enough acceptable doctors are available. One can reserve m_s jobs for subspecialty s by providing that there are m_s jobs in which doctors in that subspecialty have their productivity increased by a large constant M. Then, if m_s doctors in subspecialty s are available, at least that many will be demanded at the optimal solution to (13). If that many qualified doctors are not available, the jobs will be filled by other doctors, according to their productivities.

Another use of quotas is for an affirmative action policy that absolutely reserves jobs for members of certain target groups. For each affirmative action group G, define a corresponding set of jobs J_G and let $J = \cup_G J_G$. Specify that for any doctor $d \in G$ and job $j \in J_G$, $\alpha_{dj} = u_h(d)$ and otherwise $\alpha_{dj} = 0$. One can also introduce an unrestricted category of jobs j' that any doctor can fill with productivity $\alpha_{dj'} = u_h(d)$. This obviously ensures that a certain number of jobs are reserved for each target group.

A recent treatment of affirmative action by Roth (1991), extended by Abdulkadiroğlu (2003), uses the class of *responsive preferences with capacity constraints.* This class requires that the set of doctors be partitioned into a finite number of affirmative action groups and that for each group G the hospital has a *capacity* m_G that limits the number of doctors who can be hired from that group. In addition, at most n_h doctors in total can be hired.

To represent such preferences using endowed assignment valuations, let us specify that there are m_G jobs of type G and $\Sigma_G m_G$ jobs in total. The hospital is endowed with $n_h - \Sigma_G m_G$ doctors. The endowed doctors have productivity $\alpha_{ij} = M$ in every job, where M is a large number. This ensures that at most n_h new doctors will be hired at any optimum. For the nonendowed doctors, we calculate productivities $u_h(d)$ in the same fashion as for the responsive preferences and set $\alpha_{dj} = u_h(d)$ for jobs j of type G, and $\alpha_{dj} = 0$ for all other jobs. This ensures that at most m_G doctors will be hired from group G. By inspection, this specification represents Abdulkadiroğlu's affirmative action preferences.

Unlike the specifications used for the current National Resident Matching Program algorithm, the endowed assignment valuations permit overlapping affirmative action categories. For example, suppose that a small hospital has a target of hiring one female and one minority doctor. In the endowed assignment structure, the hospital may assign a minority female doctor to fill either a minority slot or a female slot, but not both. This is necessary to retain the substitutes demand structure, for otherwise making a minority female doctor available could lead the hospital to hire a nonminority male doctor whom it would otherwise reject.

Endowed assignment valuations thus provide a flexible, parameterized way for hospitals to represent their preferences for matching with or without wages using a class of quasi-linear preferences that satisfies the substitutes condition and the law of aggregate demand.

V. Cumulative Offer Processes and Auctions

The algorithms described by the system (9) with different starting points have the property that they can terminate only at a stable set of contracts. Nevertheless, unless preferences satisfy the substitutes condition, the system is not guaranteed to converge at all, even when a fixed point exists. In this section, we offer a different characterization of the Gale-Shapley doctor-offering algorithm that will prove especially well suited to situations in which contracts may not be substitutes, but in which there is just one "hospital": the *auctioneer.* For now, we allow the possibility that there are several hospitals.

The alternative representation is constructed by replacing the system of equations (9) by the following system:

$$(14)\qquad X_D(t) = X - R_H(X_H(t-1))$$
$$X_H(t) = X_H(t-1) \cup C_D(X_D(t)).$$

We call the algorithm that begins with $X_D(0) = X$ and $X_H(0) = \emptyset$ and obeys (14) a *cumulative offer process,* because the formalism captures the idea that hospitals accumulate offers from doctors in the set $X_H(t)$ and hold their best choices $C_H(X_H(t))$ from the accumulated set. There is no assumption of consistency imposed on the algorithm, so it is possible that several hospitals are "holding" contract offers from the same doctor. The corresponding allocation is, of course, infeasible, since each doctor can ultimately accept just one contract.

At each round t of the cumulative offer process, all doctors make their best offers from the set of not-yet-rejected choices $X_D(t)$, but any doctor d for whom a contract is being held simply repeats one of its earlier offers. Thus, new offers are made only by doctors who have been rejected.

To see why this is so, let $x \in C_H(X_H(t))$ be a contract that is being held, and consider the corresponding doctor $x_D = d$. By revealed preference, doctor d strictly prefers x to any contract that she has not yet offered, since those contracts were available to offer at the time that x was offered. Since d's most preferred contract in $X_d(t)$ at the current time must be weakly preferred to x, it must be coincide with one of d's earlier offers.

The second equation of system (14) is the one that distinguishes the cumulative offer process from the system in (9). In the cumulative offer process, without any assumptions about hospitals' preferences, $X_H(t)$ grows monotonically from round to round, so the sequence of sets converges. In contrast, the earlier process was guaranteed to converge only when contracts are substitutes for the hospitals.

When contracts are substitutes, the two systems of equations are equivalent.

THEOREM 15: *Suppose that contracts are substitutes for the hospitals and that* $X_D(0) = X$ *and* $X_H(0) = \emptyset$. *Then, the sequences of pairs* $\{(X_D(t), X_H(t))\}$ *generated by the two laws of motion (9) and (14) are identical.*

Even with the initial condition $X_D(0) = X$ and $X_H(0) = \emptyset$, the algorithms described by (9) and (14) may differ when contracts are not substitutes. In that case, by inspection of the system (14), the cumulative offer process still converges, because $X_H(t)$ is bounded by the finite set X and grows monotonically from round to round. What is at issue is whether the hospital's choice from its final search set in the cumulative offer process is a feasible and stable set of contracts.

We will find below that when there is a single hospital, the outcome is indeed a feasible and stable set of contracts. In that case, the cumulative offer process coincides with the *generalized proxy auction* of Ausubel and Milgrom (2002). Those authors analyze in detail the case when a bid consists of a price and a subset of the set of goods that the bidder wishes to buy. At each round, the seller "holds" the collection of bids that maximizes its total revenues, subject to the constraint that each good can be sold only once. The generalized proxy auction, however, is not limited to the sale of goods and, in fact, is identical in scope to our present model of matching with contracts. In particular, the auctioneer may impose a variety of constraints on the feasible collections of bids and may weigh nonprice factors, either exclusively or in combination with prices, to decide which collection of bids to hold. Bidders, for their part, may make bids that include factors besides price, and may not include price at all.

To illustrate the role of general contracts in this auction setting, consider the auction design suggested by Paul Brewer and Charles Plott (1996), in which bidders seek to buy access to a railroad track. In that application, a bid specifies a train's direction of travel, departure and arrival times, and price offered. It is assumed that trains travel at a uniform speed along the track. In this setting, the contract terms must include the direction and the two times, and the seller is constrained to hold only combinations of bids such that trains maintain safe distances from one another at all times.

A second example of the generalized proxy process is a procurement auction in which the buyer scores suppliers on the basis of such factors as quality, excess capacity, credit rating, and historical reliability, as well as price, and in which the buyer prefers to set aside some amount of its purchase for minority contractors or to maintain geographic diversity of supply to reduce the chance of supply disruptions. In an asset sale, the seller may weigh the probability that the sale will be completed, for example due to financing contingencies or because a union or antitrust regulators must approve the sale.

The cumulative offer process model with general contracts accommodates all of these possibilities. The auctioneer in the model corresponds to a single "hospital"—hereafter the *auctioneer*—with a choice function, C_H, that selects her most preferred collection of contract proposals. We have the following result (which is first stated using different notation than in the Ausubel-Milgrom paper):

THEOREM 16: *When the doctor-offering cumulative offer process with a single hospital terminates at time t with outcome* $(X_D(t), X_H(t))$, *the hospital's choice* $C_H(X_H(t))$ *is a stable collection of contracts.*

Cumulative offer processes connect the theory of matching with contracts to the emerging theory of package auctions and auctions with complex constraints.

VI. Conclusion

We have introduced a general model of matching with bilateral contracts that encompasses and extends two-sided matching models with and without money and certain auction models. The new formulation allows some contract terms to be exogenously fixed and others to be endogenous, in any combinations. In this very general framework, we characterize stable collections of contracts in terms of the solution to a certain system of equations.

The key to the analysis is to extend two concepts of demand theory to models with or without prices. The first concept to be extended is the notion of *substitutes.* Our definition applies essentially the Roth-Sotomayor substitutable preferences condition to a more general class of contracts: contracts are substitutes if, whenever the set of feasible bilateral contracts expands, the set of contracts that the firm rejects also expands. We show that (a) our definition coincides with the usual demand theory condition when both apply, (b) when contracts are substitutes, a stable collection of contracts exists, and (c) if any hospital or firm has preferences that are not substitutes, then there are preferences with single openings for each other firm such that no stable allocation exists. We further show that when the substitute condition applies, (a) both the doctor-offering and hospital-offering Gale-Shapley algorithms can be represented as iterated operations of the same operator (starting from different initial conditions), and (b) starting at a stable allocation from which a doctor retires, a natural market dynamic mimics the Gale-Shapley process to find a new stable allocation.

The second relevant demand theory concept is the law of demand, which we extend both to include heterogeneous inputs and to encompass models with or without prices. The *law of aggregate demand* condition holds that when the set of feasible contracts expands, the *number* of contracts that the firm chooses to sign weakly increases. In terms of traditional demand theory, this means that, for example, when the wages of some of a heterogeneous group of workers falls, if the workers are substitutes, then the total number of workers employed rises. We show that (a) when inputs are substitutes, the choices of a profit-maximizing firm/hospital satisfy the law of aggregate demand. Moreover, when the choices of every hospital/firm satisfy the law of aggregate demand and the substitutes condition, then (b) the set of workers/doctors employed is the same at every stable allocation, (c) the number employed by each firm/hospital is also the same, and (d) truthful reporting is a dominant strategy for doctors in the doctor-offering algorithm. Moreover, we prove (e) that if the law of aggregate demand fails in any potentially observable way, then the preceding dominant strategy property does not hold.

For these results to be useful for practical mechanism design, one needs to account for how preferences, especially hospital preferences, are to be reported to the mechanism. There needs to be a convenient way for hospitals to express a rich array of preferences, and one needs to know whether the preferences being reported actually satisfy the conditions of the various theorems. Toward that end, we introduce a parametric form that we call extended assignment valuations, which strictly generalizes several existing specifications and which always satisfies both the substitutes and law of aggregate demand conditions.

Finally, we introduce an alternative treatment of the doctor-offering algorithm: the *cumulative offer process.* We show that when contracts are substitutes, the previously characterized doctor-offering algorithm coincides exactly with a cumulative offer process. When contracts are not substitutes, but there is just one hospital (the "auctioneer"), the cumulative offer process coincides with the Ausubel-Milgrom ascending proxy auction. This identity clarifies the connection between these algorithms and, combined with the dominant strategy theorem reported above, generalizes the Ausubel-Milgrom dominant strategy theorem for the proxy auctions.

Our new approach reveals deep similarities among several of the most successful auction and matching designs in current use and among the environmental conditions in which, theoretically, the mechanisms should perform at their best. Understanding these similarities can help us to understand the limitations of these mechanisms, paving the way for new designs.

Appendix

PROOF OF THEOREM 1:

Let $(X_D, X_H) \subset X^2$ be a solution to (3). Then, $X_D \cap X_H = X_D - R_D(X_D) = C_D(X_D)$ and similarly $X_H \cap X_D = X_H - R_H(X_H) = C_H(X_H)$, so $X_H \cap X_D = C_H(X_H) = C_D(X_D)$.

Next, we show that $X' \equiv X_H \cap X_D$ is a stable set of contracts. Since $X' = C_H(X_H) = C_D(X_D)$, it follows by revealed preference that $X' =$

$C_H(X') = C_D(X')$, so condition (i) of the definition of stability is satisfied.

Consider any hospital h and set of contracts $X'' \subset C_D(X' \cup X'')$ naming hospital h. Since $X' = C_D(X_D)$, revealed preference of the doctors implies that $X'' \cap X_D \subset C_D(X_D)$ and hence that $X'' \cap R_D(X_D) = X'' \cap X_D \cap R_D(X_D) \subset C_D(X_D) \cap R_D(X_D) = \emptyset$. Thus, $X'' \subset X - R_D(X_D) = X_H$ by (3). So if $X'' \neq C_h(X')$, then by the revealed preference of hospital h, $X'' \prec_h C_h(X_H) = C_h(X')$. Hence, $X'' \neq C_h(X' \cup X'')$, so condition (ii) is satisfied. So, the set of contracts X' is unblocked.

For the converse, suppose that X' is a stable collection of contracts. Then, $C_D(X') = X'$. Let X_H be the set of contracts that the doctors weakly prefer to X' and X_D the set of contracts that doctors weakly disprefer. By construction, $\{X', X_H - X', X_D - X'\}$ is a partition of X.

To obtain (3), observe first that if there is some h such that $C_h(X_H) \neq X'_h$, then $X'' = C_h(X_H)$ violates stability condition (ii). So, $C_h(X_H) = X'_h$ for all h, that is, $C_H(X_H) = X'$. It follows that $X - R_H(X_H) = X - (X_H - X') = X - (X - X_D) = X_D$ and that $X - R_D(X_D) = X - (X_D - C_D(X_D)) = X - (X_D - X') = X_H$.

Hence, (X_D, X_H) satisfies (3) and $X_H \cap X_D = X'$.

PROOF OF THEOREM 2:

Let $i \neq j$, $(j, w_j) \in C(\mathbf{w})$ and $w'_i > w_i$. Define $Z(\mathbf{w}) \equiv \{(j, \tilde{w}_j) | \tilde{w}_j \geq w_j\}$. Then, $Z(w'_i, \mathbf{w}_{-i}) \subset Z(\mathbf{w})$. If contracts are substitutes, then $R(Z(w'_i, \mathbf{w}_{-i})) \subset R(Z(\mathbf{w}))$. By (6), since $(j, w_j) \in C(\mathbf{w})$, it follows that $(j, w_j) \in C(Z(\mathbf{w}))$, so $(j, w_j) \notin R(Z(\mathbf{w}))$. Hence, $(j, w_j) \notin R(Z(w'_i, \mathbf{w}_{-i}))$. So, $(j, w_j) \in Z(w'_i, \mathbf{w}_{-i}) - R(Z(w'_i, \mathbf{w}_{-i})) = C(Z(w'_i, \mathbf{w}_{-i}))$ and thus $(j, w_j) \in C(w'_i, \mathbf{w}_{-i})$ by assumption (6). Thus, C satisfies demand theory substitutes.

Conversely, suppose contracts are not substitutes. Then, there exists a set X', an element $(i, w_i) \notin X'$, and $(j, w_j) \in R(X')$ such that $(j, w_j) \notin R(X'')$, where $X'' = X' \cup \{(i, w_i)\}$. Using (6), $w_i < \hat{W}_i(X')$. Let $\mathbf{w}'' = \hat{W}(X'')$ and $w'_i = \hat{W}_i(X')$. Then, $w'_i > w''_i$ and $(j, w_j) \in C(\mathbf{w}'')$, but $(j, w_j) \notin C(\mathbf{w}''_{-i}, w'_i)$, so C does not satisfy the demand theory substitutes condition.

PROOF OF THEOREM 5:

We may limit attention to the case with exactly two hospitals by specifying that the doctors find the other hospitals to be unacceptable.

Suppose R_h is not isotone. Then, there exists some $x, y \in X$ and $X' \subset X$ such that for all $x \in X'$, $x_H = h$ and such that $x \in R_h(X') - R_h(X' \cup \{y\})$. By construction, since $x, y \in C_h(X' \cup \{y\})$, contracts x and y specify different doctors, say, $d_1 \equiv x_D \neq y_D \equiv d_2$. Let x' and y' denote the corresponding contracts for doctors d_1 and d_2 in which hospital h' is substituted for h.

We specify preferences as follows: first, for hospital h', we take $\{x'\} \succ_{h'} \{y'\} \succ_{h'} \emptyset$ and all other contracts are unacceptable. Second, doctors in $x_D(C_H(X') \cup C_H(X' \cup \{y\})) - \{d_1, d_2\}$ prefer their elements of $C_H(X') \cup C_H(X' \cup \{y\})$ to any other contract. Third, d_1 has $\{x\} \succ_{d_1} \{x'\}$ and ranks all other contracts lower. Fourth, d_2 has $\{y'\} \succ_{d_2} \{y\}$ and ranks all other contracts lower. Finally, the remaining doctors find all contracts from hospitals h and h' to be unacceptable.

Consider a feasible, acceptable allocation X'' such that $y' \in X''$. Since h' and d_2 can have only one contract in X'', $x', y \notin X''$. Then, h's contracts in X'' form a subset of X', so x is not included and d_1 has a contract less preferred than x'. Then, the deviation by (d_1, h') to x' blocks X''.

Consider a feasible, acceptable allocation X'' such that $y' \notin X''$. Then, either $x, y \in X''$ or X'' is blocked by a coalition including h, d_1 and d_2 using the contracts x and y. However, if $x, y \in X''$, then a deviation by (d_2, h') to contract y' blocks X''.

Since all feasible allocations are blocked, there exists no stable set of contracts.

PROOF OF THEOREM 7:

Suppose $X = D \times H \times W$ and let $Z(\mathbf{w}) \equiv \{(j, \tilde{w}_j) | \tilde{w}_j \geq w_j\}$. Then, the law of aggregate demand is the statement that for any wage vectors $\mathbf{w}, \hat{\mathbf{w}}$ satisfying $\mathbf{w} \geq \hat{\mathbf{w}}$ such that the choices sets are singletons, $|C(Z(\mathbf{w}))| \leq |C(Z(\hat{\mathbf{w}}))|$. The proof is by contradiction. Suppose the law of aggregate demand does not hold. Then there exists a wage vector $\mathbf{w}$ and a doctor d such that for some (and hence all) $\varepsilon > 0$, $|C(Z(w_d + \varepsilon, \mathbf{w}_{-d}))| > |C(Z(w_d - \varepsilon, \mathbf{w}_{-d}))|$. Since h's preferences are quasi-linear, changing doctor d's wage can affect the hiring of other doctors only if it affects the hiring of doctor d. It follows that there are exactly two optimal choices for the hospital at wage vector $\mathbf{w}$; these are $C(Z(w_d - \varepsilon, \mathbf{w}_{-d}))$ at which doctor d is hired and $C(Z(w_d + \varepsilon, \mathbf{w}_{-d}))$ at which d is not hired but such that two other doctors are hired, that is, there exist doctors d', $d'' \in C(Z(w_d + \varepsilon, \mathbf{w}_{-d})) - C(Z(w_d - \varepsilon, \mathbf{w}_{-d}))$. Let the corresponding payoff for the hospital (when faced with wage vector $\mathbf{w}$) be π.

Consider the wage vector $\mathbf{w}' = (w_d - \varepsilon, w_{d'} - 2\varepsilon, \mathbf{w}_{-\{d,d'\}})$. For ε positive and sufficiently small, the hospital's payoff at wage vector $\mathbf{w}'$ is $\pi + 2\varepsilon$ if it chooses $C(Z(w_d + \varepsilon, \mathbf{w}_{-d}))$ and it is $\pi + \varepsilon$ if it chooses $C(Z(w_d - \varepsilon, \mathbf{w}_{-d}))$, and one of these choices must be optimal. So, $C(\mathbf{w}') = C(Z(w_d + \varepsilon, \mathbf{w}_{-d}))$. But then, raising the wage of doctor d' from $w_{d'} - 2\varepsilon$ to $w_{d'}$ while holding the other wages at $\mathbf{w}'_{-d'}$ reduces the demand for doctor d'' from one to zero, in violation of the demand theory substitutes condition.

PROOF OF THEOREM 8:

By definition, $X_D \subset \bar{X}_D$, so by revealed preference, $|C_d(\bar{X}_D)| \geq |C_d(X_D)|$. Also, $\bar{X}_H \subset X_H$, so by the law of aggregate demand, $|C_h(\bar{X}_H)| \leq |C_h(X_H)|$. By Theorem 1, $C_D(X_D) = C_H(X_H)$ and $C_D(\bar{X}_D) = C_H(\bar{X}_H)$, so $\Sigma_{d\in D} |C_d(X_D)| = \Sigma_{h\in H} |C_h(X_H)|$ and $\Sigma_{d\in D} |C_d(\bar{X}_D)| = \Sigma_{h\in H} |C_h(\bar{X}_H)|$. Combining these leads to $\Sigma_{d\in D} |C_d(\bar{X}_D)| \geq \Sigma_{d\in D} |C_d(X_D)| = \Sigma_{h\in H} |C_h(X_H)| \geq \Sigma_{h\in H} |C_h(\bar{X}_H)| = \Sigma_{d\in D} |C_d(\bar{X}_D)|$, which begins and ends with the same sum. Hence, none of the inequalities can be strict.

PROOF OF THEOREM 9:

Since $|C_h(X')| > |C_h(X'')|$, there exists some set Y, $X' \subset Y \subset X''$ and contract x such that $|C_h(Y)| > |C_h(Y \cup \{x\})|$. Since $x \in C_h(Y \cup \{x\})$ (as otherwise $C_h(Y) = C_h(Y \cup \{x\})$ for the preferences to be rationalizable) there must exist two contracts $y, z \in R_h(Y \cup \{x\}) - R_h(Y)$, such that $y \neq x \neq z \neq y$. Moreover, since $y, z \in C_h(Y)$, $y_D \neq z_D$.

Denoting by h' the second hospital whose existence is hypothesized by the theorem, we specify preferences as follows. Let all the doctors with contracts in Y have those contracts be their most favored, and let all other doctors find any contract with h unacceptable. Let all doctors find any contract not involving hospital h or h' to be unacceptable.

In principle, there are three cases.

If $x_D = y_D$, then let $P_{x_D} : y > x$ and $P_{z_D} : z$. Then, there exist two stable matches, $C_h(Y \cup \{x\})$ and $C_h(Y)$, with z_D employed in the first match but not in the second.

The case $x_D = z_D$ is symmetric.

Finally, if $y_D \neq x_D \neq z_D$, then let x', y', z' denote contracts with hospital h' where the doctors (and any other terms) are the same as in x, y, z, respectively. Specify the remaining preferences by $P_{x_D} = x' > x$, $P_{y_D} = y > y'$, $P_{z_D} = z > z'$, and $P_{h'} = \{y'\} > \{x'\} > \varnothing > \{z'\}$. Then, there exist two different stable matches, $\{x'\} \cup C_h(Y)$ and $\{y'\} \cup C_h(Y \cup \{x\})$, with z_D employed in the first match but not in the second.

PROOF OF THEOREM 10:

Let X' denote the collection of contracts chosen by the algorithm when doctor d submits preference P_d. If this collection, which is stable under the reported preferences, is not stable under P'_d, then there exists a blocking coalition. This blocking coalition must contain d, as no other doctor's preferences have changed, but that is impossible, since x is d's favorite contract according to the preferences P'_d. Since X' is stable under P'_d, the doctor-optimal stable match under P'_d (the existence of which is guaranteed by Theorem 2) must make every doctor (weakly) better off than at X'. In particular, doctor d must obtain x.

PROOF OF THEOREM 11:

From Theorem 10, $P'_d : x$ also obtains x. Hence, by the rural hospitals theorem, d is employed at every point in the core when $P'_d : x$ is submitted. So, every allocation X' at which d is unemployed is blocked by some coalition and set of contracts when d submits P'_d. Consequently, if d submits the preferences $P^*_d : y_1 > y_2 > \cdots > y_n > x$, then every allocation at which d is unemployed is still blocked, by the same coalition and set of contracts. Since the doctor-best stable allocation is one at which d gets a P^*_d-acceptable contract, that allocation is weakly P^*_d-preferred to x. Finally, the doctor-optimal match when P^*_d is submitted is still the doctor-optimal match when P''_d is submitted, as d's preferences over contracts less preferred than x cannot be used to block a match where she receives a contract weakly preferred to x.

PROOF OF THEOREM 12:

Consider contracts x', y', and z' such that $x_D = x'_D$, $y_D = y'_D$, $z_D = z'_D$, and $x'_H = y'_H = z'_H = h'$. Let the preferences of the three identified be $P_{x_D} : x' > x$, $P_{y_D} : y > y'$, and $P_{z_D} : z' > z$, and let those of h' be $P_{h'} : \{y'\} > \{z'\} > \{x'\}$. For the other contracts $\hat{x} \in Y \equiv C_h(X)$, let $\hat{x}$ be $\hat{x}_D$'s most favored contract. For the remaining doctors, let any contract with h or h' be unacceptable.

With the preceding preferences, the only stable allocation includes contracts x and y', leaving doctor z_D unemployed. If, however, z_D misrepresents her preferences and reports $P'_{z_D} : z > z'$, then the doctor-optimal stable match includes z, leaving doctor z_D better off.

PROOF OF THEOREM 13:

Setting $|J| = 1$ and $T = \emptyset$, it is clear by inspection that this family includes all singleton valuations.

The family is closed under endowment, because given any such valuation $v(\cdot\, |J, \boldsymbol{\alpha}, T)$, endowing a set of doctors T' simply creates the valuation $v(\cdot\, |J, \boldsymbol{\alpha}, T \cup T'))$.

To verify that the family is closed under aggregation, consider any two valuations $v(\cdot\, |J, \boldsymbol{\alpha}, T)$ and $v(\cdot\, |J', \boldsymbol{\alpha}', T')$. Since J and T are just index sets, we may assume without loss of generality that $T \cap T' = J \cap J' = \emptyset$. Define $T'' = T \cup T'$ and $J'' = J \cup J'$. Let

$$(15) \qquad \alpha''_{ij} \equiv \begin{cases} \alpha_{ij} & \text{if } i \in D \cup T, j \in J \\ \alpha'_{ij} & \text{if } i \in D \cup T', j \in J' \\ 0 & \text{otherwise.} \end{cases}$$

By inspection of the optimization problems, $v(\cdot\, |J, \boldsymbol{\alpha}, T) \vee v(\cdot\, |J', \boldsymbol{\alpha}', T') = v(\cdot\, |J'', \boldsymbol{\alpha}'', T'')$.

For the converse, consider the valuation $v(\cdot\, |J, \boldsymbol{\alpha}, T)$ given by (13). We derive it constructively from the singleton valuations, aggregation, and endowment, as follows. Let v_j be the singleton valuation associated with row j in matrix $[\alpha_{ij}]$ for $j = 1, \ldots, |J|$. Since V_{EA} is closed under aggregation, $v_1 \vee \cdots \vee v_{|J|} = v(\cdot\, |J, \boldsymbol{\alpha}, \emptyset) \in V_{\text{EA}}$. Since V_{EA} is also closed under endowment, we may endow the doctors in T to obtain $v(\cdot\, |J, \boldsymbol{\alpha}, T) \in V_{\text{EA}}$.

PROOF OF THEOREM 14:

L. Shapley (1962) (see also Lehmann et al., 2001) had already proved that every assignment valuation without endowments $v(\cdot\, |J, \boldsymbol{\alpha}, \emptyset)$ is a substitutes valuation. Ausubel and Milgrom (2002) prove that a valuation v is a substitutes valuation if and only if the corresponding indirect profit function $(\pi(p) = v(S) - \sum_{d \in S} p_d)$ is submodular. Let $\hat{\pi}$ be the indirect profit function corresponding to $v(\cdot\, |J, \boldsymbol{\alpha}, \emptyset)$. Then the indirect profit function corresponding to $v(\cdot\, |J, \boldsymbol{\alpha}, T)$ is $\pi(p_1, \ldots, p_N) = \hat{\pi}(p_1, \ldots, p_N, 0, \ldots, 0)$. Since $\hat{\pi}$ is the profit function of the substitutes valuation $\hat{v}_h$, it is submodular, and hence π is submodular as well.

PROOF OF THEOREM 15:

Suppose that contracts are substitutes for the hospitals. We proceed by induction. The initial condition specifies that the sequences are identical through time $t = 0$. Denote the sequence corresponding to the cumulative offer process by a superscript C and denote the alternative process defined by (9) with no superscript. Assume the inductive hypothesis that the sequences are the same up to round $t - 1$ and suppress the corresponding superscripts for the values at that round. Then, $X_D(t) = X - R_H(X_H(t-1)) = X^C_D(t)$. This also implies that

$$(16) \qquad X = X_H(t-1) \cup X_D(t).$$

To complete the proof, we must show that $X^C_H(t) = X_H(t)$ or, equivalently, that $X_H(t-1) \cup C_D(X_D(t)) = X - R_D(X_D(t))$.

For $t \geq 2$, $X^C_H(t-2) \subset X^C_H(t-1)$ by construction, so by the inductive hypothesis $X_H(t-2) \subset X_H(t-1)$. Since contracts are substitutes, R_H is isotone, so $X - R_H(X_H(t-1)) \subset X - R_H(X_H(t-2))$ and hence $X_D(t) \subset X_D(t-1)$. For $t = 1$, the inclusion $X_D(t) \subset X_D(t-1)$ is implied by the initial condition $X_D(0) = X$.

Recall that, by revealed preference, R_D is isotone. It follows that $R_D(X_D(t)) \subset R_D(X_D(t-1))$ and hence, using (9), that $X_H(t-1) = X - R_D(X_D(t-1)) \subset X - R_D(X_D(t)) = X_H(t)$. Thus, $X_H(t-1) = X_H(t-1) \cap (X - R_D(X_D(t))) = X_H(t-1) - R_D(X_D(t))$. So, $X_H(t-1) \cup C_D(X_D(t)) = X_H(t-1) \cup (X_D(t) - R_D(X_D(t))) = (X_H(t-1) - R_D(X_D(t))) \cup (X_D(t) - R_D(X_D(t))) = (X_H(t-1) \cup X_D(t)) - R_D(X_D(t)) = X - R_D(X_D(t))$, where the last step equality follows from (16).

PROOF OF THEOREM 16:

By construction, any contract not in $X_H(t)$ is less preferred by some doctor d than every corresponding contract in $X_H(t)$ (because doctor d offers her most preferred contracts in sequence). So, any collection of contracts that includes some not in $X_H(t)$ must be strictly less preferred by one of the doctors. Any profitable coalitional deviation must use only contracts in $X_H(t)$.

By construction, the one hospital/auctioneer must be part of any deviating coalition, and $C_H(X_H(t))$ is its strictly most preferred collection of contracts in $X_H(t)$, so there is no profitable coalitional deviation using just contracts in $X_H(t)$.

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