Prop
\( \dim V = n \). Let \( \varphi, \psi \in \mathcal{L}(U, V) \)

(i) Let \( \mathbf{v}_1, \ldots, \mathbf{v}_n = B \) be a basis for \( V \).
Then \( \det \varphi = \det M(\varphi; B, B) \)

(iii) \( \det (\psi \circ \varphi) = \det \psi \cdot \det \varphi \)

(iv) \( \det (AB) = \det A \det B \) for \( n \times n \) matrices.

(iii) Let \( \varphi \in \mathcal{L}(U, U) \). Then \( \varphi \) is an isomorphism if and only if \( \det \varphi \neq 0 \).

Proof
By \( \det \), if \( M(\varphi; B, B) = (M_{ij}) = M \)
\[ \phi(U_j) = \bigwedge_{i=1}^n M_{i,j} U_i \]

Hence
\[ \chi(\phi(U_1), \ldots, \phi(U_n)) = \sum_{c \in S_n} \text{sgn}(c) \bigwedge_{k=1}^n M_{c(k),k} \chi(U_{c(1)}, \ldots, U_{c(n)}) \]

Suppose \( \phi \) is an isomorphism.
\[ \det(\psi \circ \rho) = \chi(\phi \circ \rho(U_1), \ldots, \phi \circ \rho(U_n)) / \chi(U_1, \ldots, U_n) \]
\[
\det \psi \det \phi \quad \text{since} \quad \phi \circ \psi
\]

\[
\Rightarrow \psi(\varphi(\omega_i)), \ldots, \psi(\varphi(\omega_n)) \quad \text{is a basis, and the formula for } \det \text{ of a linear operator is independent of the choice of basis. If } \phi \text{ is not an iso, then } \psi \circ \phi \text{ is also not an iso (exercise)}
\]

\[
\Rightarrow \det \phi = 0 = \det (\psi \circ \phi) \quad \text{as read (iii) below)
\]
(iii) $\phi$ is an iso if $\Rightarrow \phi(v_i) \ldots \phi(v_n) = 0$ is a basis $\Rightarrow v(\phi(v_i), \ldots, \phi(v_n)) = 0$ for any $v \in \mathbb{K}^n$.

\[ \Rightarrow \det \phi \neq 0. \]

**Invariant Subspaces**

**Definition.** A linear map $\phi : V \rightarrow V$ for a vector space to itself is called a linear operator.

Consider a linear operator $\phi \in L(V, V)$.

A subspace $U \subseteq V$ is called **invariant** under $\phi$ if $u \in U \Rightarrow \phi(u) \in U$.

If $U \subseteq V$ is invariant under $\phi$, then $\phi : U \rightarrow U$ and $u \mapsto \phi(u)$.
is an operator on $V$.

Examples

$D \mathbf{v} = \mathbf{b}^3, \quad \phi : V \rightarrow V$

$\phi(x, y, z) = (2x, y+z, 3y+5z)$

If $e_1, e_2, e_3$ be the standard basis of $V$.

Then $\text{Span}(e_i)$ is an invariant subspace because

$\phi(Ce_1) = \phi(Ce_0) = 2ce_1$

or

$\phi(Ce_2 + ce_3) = \phi(Ce_2) = (0, ce_2 + (3c+5d)e_3)$

$\in \text{Span}(e_2, e_3)$ is an invariant subspace because

$\phi(Ce_2 + ce_3) = \phi(Ce_2 + ce_3)$

$= (0, ce_2 + (3c+5d)e_3)$

$\in \text{Span}(e_2, e_3)$
(iii) If $\phi \in L(U, V)$ then the kernel $\ker \phi \subset U$ is always an invariant subspace.

(ii) $\phi \in L(U, V)$. For $\phi$ is an invariant subspace $\ker \phi \subset \ker \phi$.

(iii) $\phi \in L(U, V)$. For $\phi$ is an invariant subspace $\ker \phi \subset \ker \phi$.

(iv) $\phi \in L(U, V)$.

Define $\text{Fix} \phi := \{ v \in U | \phi(v) = 0 \}$

The subspace of fixed points of $\phi$.

Then $\text{Fix} \phi$ is an invariant subspace.

Eigenvalues

Both $U$ finite dimensional $v, s / k$ for $\phi \in L(U, V)$. If $v \in U$, let $k$ be an s.t.
\[ \phi(x) = \lambda x \]

then we say

- \((\lambda, x)\) is an eigenpair of \(\phi\)
- \(\lambda\) is an eigenvalue of \(\phi\)
- \(x\) is an eigenvector of \(\phi\)

with eigenvalue \(\lambda\)

If \(\lambda\) is an eigenvalue (i.e., \(\exists v \in V\) \(\phi v = \lambda v\)) then the set \(E = \{v \in V \mid \phi v = \lambda v\}\) of eigenvectors with eigenvalue \(\lambda\) is called the \(\lambda\)-eigenspace. It is an invariant subspace of \(\phi\).
Let $V$ be a finite-dimensional vector space, let $\phi \in \mathcal{L}(V)$, and let $\lambda$ be an eigenvalue of $\phi$.

(i) $\lambda$ is an eigenvalue of $\phi$.
(ii) $\phi - \lambda \text{id} \in \mathcal{L}(V)$ is not injective.
(iii) $\det (\phi - \lambda \text{id}) = 0$.

Proof

(iii) $\implies$ (iii):

For a linear operator $T$ on a finite-dimensional vector space, $T$ injective $\implies$ $T$ surjective (Rank-Nullity Theorem).

Thus, $T$ is injective $\implies$ $\det T \neq 0$.

(i) $\implies$ (ii): \[ \phi \nu = \lambda \nu \implies (\phi - \lambda \text{id})\nu = 0 \]

(i) $\implies$ (i) $\iff$ $\nu$ s.t. $\phi \nu = \lambda \nu$.

(iii) $\implies$ (i) $\iff$ $\nu$ s.t. $\nu \in \ker (\phi - \lambda \text{id})$. 
\[ \Rightarrow (\phi - \lambda \text{id}) \nu = 0 \]
\[ \Rightarrow \phi \nu = \lambda \nu. \]

Let \( U \) be a vector space (\( k \))
\( \lambda = \dim \nu \) and \( \phi \in \mathcal{L}(U, U) \).
Suppose \( B \) is a basis of \( U \) and 

\[ M(\phi \cdot B, \nu) = (a_{ij}) \]

\[ M(\phi - \lambda \text{id}, B, \nu) = \begin{pmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{pmatrix} \]

Consider 
\[ p(x) := \det (\phi - x \text{id}) \]
This is a sum of products of \( n \) entries of 
the above matrix.

Consider 
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the above matrix.
\[ p(x) = \sum \text{sgn}(6) \prod_{b=1}^{n} b_{6b+1} \]

Each term \( \text{sgn}(6) \prod_{b=1}^{n} b_{6b+1} \) is

degree \( \leq n \) in \( x \) so \( p(x) \in P_n(\mathbb{K}) \)

Further \( \deg(\text{sgn}(6) \prod_{b=1}^{n} b_{6b+1}) \leq n \)

Unless \( 6 = \text{id} \) in which case

\[ \text{sgn}(6) \prod_{b=1}^{n} b \]

\[ = (a_{01} - x)(a_{02} - x) \ldots (a_{0n} - x) \]

\[ = (-1)^{n} x^{n} + \text{lower order terms} \]

Define let \( V \) be a vector space of
we define
\[ X_\psi := \det (\psi - \lambda \text{id}) \in \mathbb{F}_n (\mathbb{C}). \]
the "characteristic polynomial" of \( \psi \).

we have
\[ X_\psi (\lambda) = (-1)^n + a_{n-1} \lambda^{n-1} + \ldots + a_0 \lambda^0. \]

Furthermore, \( \lambda \in \mathbb{C} \) is an eigenvalue of \( \psi \)
\( \Leftrightarrow \) \( \det (\psi - \lambda \text{id}) = 0 \)
\( \Leftrightarrow \) \( \lambda \) is a root of the
characteristic polynomial, i.e.
\[ X_\psi (\lambda) = 0. \]

Remark: if \( \mathbb{C} \) is "algebraically
closed" meaning every polynomial of degree \( \geq 1 \)
has a root, so
\[ \phi(\lambda) = (\lambda - \lambda_1) \cdots (\lambda - \lambda_n) \]
where \( \lambda_1, \ldots, \lambda_n \in \mathbb{C} \) are the roots of \( \phi \). The roots need not be distinct.

E.g. \( \lambda_i \) may equal \( \lambda_j \).

\[ E.g. \quad U = k^2 \]
\[ \phi : U \to U, \quad \phi(x,y) = (xty, -xty) \]
\[ \beta = (e_1, e_2) \]
\[ M(\phi, \beta, \beta) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \]
\[ \det (\phi - \lambda \text{id}) = \det \begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix} \]
Now \( \det \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \sum_{i,j} (\text{sgn} \theta) x_{6i,11} x_{6(i+1)2} \)
\[ 6v S_2 = \sum_i (1, 2)^3 \left| \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \right| = x_1 x_2^2 - x_2 x_1 x_2 \]

So \[ \det (\phi - \lambda I) = (1-\lambda)^2 - (-1) \]
\[ = 4\lambda^2 - 2\lambda + 1 + 1 \]
\[ = \lambda^2 - 2\lambda + 2 \]

This has no roots over \( \mathbb{R} \)
\[ \lambda^2 - 2\lambda + 2 \geq 1 \quad \text{over} \quad \mathbb{R} \]

\( \mathbb{C} \) its roots are
\[ 2 \pm \sqrt{4 - 8} = 1 \pm \sqrt{-4} = (\pm i) \]

So \( \phi \) has eigenvalues \( 1 + i, 1 - i \) over \( \mathbb{C} \).

E.g. \( \phi : L(k^2, k^2) \)
\[ \phi(x+y) = (x, x + y), \quad xy \in k \]
\[
\mu(\phi; B, B) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}
\]

\[
\det(\phi - \lambda \text{id}) = \det \begin{pmatrix} 1 - \lambda & 0 \\ 1 - \lambda & 1 - \lambda \end{pmatrix}
\]

\[
= (1 - \lambda)^2
\]

so \( \phi(\lambda) \) has just one root, \( \lambda = 1 \),

which occurs with multiplicity 2.