Nilpotent Operators

Very far from diagonalizable.

\( V \text{ fin. dim } v.s. \ k, \ \phi \in L(V, V), n = \text{dim} V \)

**Notation** \( \phi^m := \underbrace{\phi \cdots \phi}_{\text{m times}} \)

**Proposition**

\( \phi \in L(V, V) \) as above, \( k = \mathbb{C} \).

T.f.a.e. ("the following are equivalent")

(i) The only eigenvalue of \( \phi \) is 0 (i.e., 0 is an eigenvalue)

(ii) \( \exists m \leq n = \text{dim} V \) s.t. \( \phi^m = 0 \)

Proof (i) \( \Rightarrow \) (i) Let \( v \neq 0 \) be an eigenvector. So \( \phi(v) = \lambda v \)
\[ 0 = \phi^m = \phi \cdots \phi (v) = \phi^{m-1} (v) = \cdots = \phi (v) = v \]

(i) \implies (ii)

We do this by induction on \( n = \dim V \).

If \( n = 1 \), \( \phi = [a] \) for some \( a \). Then:

\[ X_\phi (a) = \text{det} [a - \lambda] = a - \lambda \]

So, if 0 is an eigenvalue,

\[ X_\phi (0) = 0 \implies a = 0 \implies \phi = 0 \]

so (ii) holds in this case.

Now let \( V \) be a vector space of \( \dim n \) and \( \phi \in \mathcal{L}(V, V) \) such that 0 is the sole eigenvalue.
Then $\ker \phi \neq 0$ since $\mathbb{C}$ is an $\ell$-tale.
and $W := \text{Im } \phi \leq V$ has $\dim \leq n$ by
the rank-nullity theorem.
$W$ is an invariant subspace (as we have
seen) so consider $\phi \in L(W, W)$. If
$\dim W = 0$, $\phi = 0$ and we are done.
Otherwise, $X \phi_W$ is a polynomial of
degree $\geq 1$ over $\mathbb{C}$, which $\phi$ has a
root.
So $\phi_W$ has an $\ell$-value.
$\ell$-values of $\phi_W$ are also $\ell$-values of $\phi$
$\ell$-values of $\phi_W$ are also $\ell$-values of $\phi$ (as
$\phi(s) = \lambda s$, $s \in W \Rightarrow \phi(s) = \lambda s$),
$t$ as an element in $V$, so $\phi$ is the
sole $\ell$-value of $\phi_W$. 

Thus \( 1 \leq m \leq n-1 \) s.t. \( \phi^m_w = 0 \) by induction.

Now \( W = \text{Im} \phi \), so, \( \forall \omega \in U \), \( \phi(\omega) \in W \)

\( \Rightarrow \phi^m \cdot \phi(\omega) = 0 \). Thus \( \phi^{m+1} = 0 \).

\( \phi^m (\phi(\omega)) \)

This gives the claim.

**Definition.** We say the linear operator \( \phi \) is nilpotent if \( \phi^m = 0 \).

**Example.**

\[ \phi : \mathbb{C}^4 \to \mathbb{C}^4 \]

\[ \phi(x_1, x_2, x_3, x_4) = (y_1, y_2, y_3, 0) \]

"Leftward shift"
Then \( \phi^2(x_1 y_1 z_1 w) = \phi(y_1 z_1 w, 0) \)

\[
= (y_1 z_1 w, 0, 0)
\]

\( \phi^3(x_1 y_1 z_1 w) = (w, 0, 0, 0) \)

\( \phi^4(x_1 y_1 z_1 w) = (0, 0, 0, 0) \)

So \( \phi \) is nilpotent.

You can also see this from the characteristic polynomial.

W.r.t. standard basis \( B \)

\[
M(\phi; B, B) = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

So \( \chi_\phi(x) = \det \begin{pmatrix}
-x & 1 & 0 & 0 \\
-x & 0 & 1 & 0 \\
0 & -x & 0 & 0 \\
0 & 0 & -x & 0
\end{pmatrix} \)
This is upper $\Lambda$.

Exercise (a homework & Q)

A upper $\Lambda \Rightarrow \text{det} A = q_{11} \ldots q_{nn}$

(the diagonal term)

so $\chi_p(A) = (\lambda - 1)^4 \lambda = \lambda^5$

Thus 0 is the only eigenvalue

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**Jordan Normal Form**

This is a very important topic. It is also covered in ch 3.D of Axler.

We have seen in the example above that left-shift operators give examples of nilpotent operators.

More generally, you can construct nilpotent operators by combining left-shifts:
Example

$\phi : C^5 \to C^5$

$(x, y, z, w, u) \mapsto (y, z, 0, 0, 0)$

Looks like two left-shifts (on $C^3 \oplus C^2$) combined.

It has matrix (in standard bases)

$$M(\phi, B, B) = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

We have $\phi^3 \neq 0$.

We will show any nilpotent operator is, in a suitable basis ("the Jordan basis"), a combination of left-shifts, as in the above example.
Definition
definition we say \( \phi \in \mathcal{L}(U, U) \) is principal if there exists \( v \in V \) such that
\[ \phi^n(v) = 0 \]
and further \( \phi, \phi(v), \ldots, \phi^{n-1}(v) \) form a basis of \( V \), where \( n = \dim V \).

Prop \( \phi \in \mathcal{L}(U, U) \) is principal if and only if \( \exists \) a basis \( B \) of \( V \) s.t.
\[ M(\phi, B, B) = J_n \text{ value} \]

Left-shift matrix of \( \sigma \times n \)
\[ J_n := \begin{pmatrix}
0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 1
\end{pmatrix} \]
i.e. $\mathfrak{m} = (a_i)_{i \leq 1}$ where $\sum a_{i-1} i = 1$, $2 < i \leq n$

I.e. principal maps are left-shift in a suitable basis $\phi(b_1 \ldots b_n) = (b_2 \ldots b_{n-1} 0)$

pf

Assume $\phi$ is principal, let $B = \phi^{-1}(1), \phi^{-2}(1) \ldots, \phi^{-n}(1)$

Then $\phi(b_1 \phi^{-1}(1) + b_2 \phi^{-2}(1) + \ldots + b_n \phi^{-n}(1) + 0)$

$= b_2 \phi^{-1}(1) + b_3 \phi^{-2}(1) + \ldots + b_n \phi^{-n}(1) + 0$

Since $\phi^{-n}(1) = 0$

i.e. $\phi$ is a "left-shift" as req'd.

Conversely, suppose we have a basis $B = \phi_1, \ldots, \phi_n$ s.t.
\[\phi(b_1v_1 + \ldots + b_nv_n) = b_2v_1 + b_3v_2 + \ldots + b_{n-1}v_{n-1}\]

Then
\[
\begin{align*}
\phi(v_n) &= v_{n-1} \\
\phi^2(v_n) &= v_{n-2} \\
& \vdots \\
\phi^{n-1}(v_n) &= v_1 \\
\phi^n(v_n) &= 0
\end{align*}
\]

Hence \(\phi\) is principal with \(\alpha = v_n\).

**Theorem (Structure Theorem for Nilpotent Maps)** Let \(\phi \in \mathcal{L}(V)\) be nilpotent, \(n = \dim V\). Then there exist invariant subspaces \(U_1, \ldots, U_k\) of \(V\) such that \(V = U_1 \oplus \ldots \oplus U_k\) and
\( \varphi_{i,u_i} \in \mathcal{L}(u_i; u_i) \) is principal for all \( i = 1, \ldots, k \).

**Preliminary remarks**

Note that, by the previous proposition, the theorem is equivalent to saying that there exists a basis \( B \) for \( U \) such that

\[
M(\varphi, B, B) = \begin{pmatrix}
J_n & 0 & \cdots & 0 \\
0 & J_n & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & J_n_k
\end{pmatrix}
\]

where \( J_n \) are left-shift matrices of size \( n \times n \).

This in turn is the same as saying that there exist \( x_1, \ldots, x_k \in U \) such that
(a) $\varphi_1, \varphi_2, \ldots, \varphi^{n-1} \varphi_1, v_1, \varphi v_2, \ldots, \varphi^{n-1} v_2$

\[ \vdots \quad \varphi v_k, \ldots, \varphi^{n-1} (v_k) \]

is a basis.

(b) $\varphi^n v_1 = 0$, $\varphi^n v_2 = 0$, \ldots, $\varphi^n (v_k) = 0$.

So the claim is equivalent to

$\varphi$ nilpotent $\Rightarrow$ $\exists \, v_1, \ldots, v_k \in V$ satisfying

$\varphi$ nilpotent $\Rightarrow$ $\exists \, v_1, \ldots, v_k \in V$

(a-b) (This is Axler's version, pg 221)

(a) (b) (This is Axler's version, pg 221)

We prove the claim by induction on $n = \dim V$.

If $n = 1$, $\varphi : V \to V$ is the zero map (as it is nilpotent) and the claim is obvious.

Assume $n \geq 2$. If $\varphi = 0$ the claim is again obvious (take $v_1 = \mathbb{v} = 2$)
Omitting we use the same trick as before. Set \( W = \text{Im} \varphi \subseteq V \). As \( \varphi \) is nilpotent, \( \ker \varphi \to (a) \) (e-value) and hence \( \dim W \leq \dim V \) (by rank-nullity).

By induction, we may decompose

\[ W = U_1 \oplus \ldots \oplus U_r \]

s.t. \( \varphi |_{U_i} \) is principal.

Thus \( \exists u_i \in U_i \) with

\[ \varphi^{ni}(u_i) = 0 \quad \text{for some } ni \]

and \( \varphi^{ni}(u_i) = a_i \) for all \( 1 \leq i \leq n \).

As \( u_i \in U_i \subseteq V \) s.t.

\[ \varphi(u_i) = a_i u_i \]
So set $V_i = \text{Span} \{ u_i, \phi(u_i), \ldots, \phi^{n_i-1}(u_i) \}$

Clearly $\phi(V_i) = U_i$.

Claim 1. $\phi(u_i), \phi^2(u_i), \ldots, \phi^{n_i}(u_i)$ give a basis for $U_i$.

Proof of claim 1. We have to show that $\phi(u_i), \phi^2(u_i), \ldots, \phi^{n_i}(u_i)$ are linearly independent.

Suppose $\lambda_0 u_i + \lambda_1 \phi(u_i) + \cdots + \lambda_{n_i} \phi^{n_i}(u_i) = 0$.

Applying $\phi$

$\lambda_0 \phi(u_i) + \lambda_1 \phi^2(u_i) + \cdots + \lambda_{n_i} \phi^{n_i+1}(u_i) = 0$

$\phi(u_i), \phi^2(u_i), \ldots, \phi^{n_i}(u_i)$ a basis of $U_i$.
But then \( \lambda n; \phi^*(u_i) = \lambda n; \phi^{i-1}(u_i) = 0 \)

\[ \Rightarrow \lambda n; = 0 \text{ as well, so the claim is proven.} \]

Since \( \phi(u_i) = u_i \), \( u_i \) are invariant and further \( \phi \mid_{u_i} \) is principal, since

\[ \phi^{*i+1}(u_i) = \phi^{*i}(u_i) = 0, \]

Claim 2: \( \exists T \in \ker \phi \) s.t.

\[ V_1 \oplus \ldots \oplus V_c \oplus T = V \]

This will finish the proof \((\phi \mid_{T} = 0 \text{ is trivial})\).

Let \( V' = V_1 + \ldots + V_c \). We firstly claim that this is a direct sum.

Suppose \( x_i \in V_i, 1 \leq i \leq c \) and \( \sum x_i = 0 \)
$\Rightarrow \sum \phi(x_i) = 0$. But $\phi(x_i) \in U_i$ and then $\phi(x_i) = 0 \forall i$, as the $U_i$ form a direct sum.

we can write

\[ x_i = q_0 u_i + q_1 \phi(v_i) + \ldots + q_n \phi^n(v_i) \]

and then $\phi(x_i) = 0$

$\Rightarrow a_0 \phi(v_i) + a_1 \phi^2(v_i) + \ldots + a_n \phi^n(v_i)$

$\Rightarrow q_0 = a_1 = \ldots = a_{n-1} = 0$

so $x_i = a_n \phi^n(v_i)$ for some scalars $a_n \in U_i$

$\Rightarrow \sum x_i = 0 \Rightarrow \sum a_n \phi^n(v_i) = 0$

$\Rightarrow a_n = 0 \forall i$ (since $U_i$ form a direct sum)
\[ X_i = 0. \]

So \[ V' = \bigoplus_{i=1}^e V_i. \]

To finish the proof, any basis for \( V' \)
can be extended to a basis for \( V \).

In particular, extend the basis
\[ \phi: V \rightarrow \mathbb{F} / \mathbb{F} \]
by adding in the basis elements
\[ t_i, \ldots, t_k. \]

Since \( \phi(V') = \mathbb{F} = \text{Im} \phi \), for each \( t_i \)
exists \( w_i \in V' \) s.t.
\[ \phi(t_i) = \phi(w_i). \]

Let \( t_i' = t_i - w_i \in \ker \phi. \)
Then \( \text{Span } \mathcal{E} \setminus \mathcal{V} \setminus \{e_i \mid 1 \leq i \leq l, 1 \leq j \leq \nu_i \} \setminus \{e_j \mid 1 \leq i \leq k \} = V \)

And this list must remain a basis (it was correct length), so set

\[ T = \text{Span } (t_1, \ldots, t_{k-1}) \]

Then \( V = U' \otimes T \) and \( \phi \big|_T = 0 \), as required.