Theorem (Structure Theorem for Nilpotent Maps)

If $V$ is a vector space and $\Phi \in L(V)$ is nilpotent, $n = \dim V$.

Then there exist invariant subspaces $U_1, \ldots, U_e$ of $V$ such that

$$V = U_1 \oplus \cdots \oplus U_e$$

with $\phi \in L(U_i, U_i)$ principal for all $i$.

For all $i = 1, \ldots, e$.

Preliminary Remarks

By what we saw last time, this theorem is equivalent to saying that $\exists$ a basis $B$ for $V$ s.t.
\[ M(\phi ; B, B) = \begin{pmatrix}
J_{n_1} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & J_{n_k}
\end{pmatrix} \]

where \( J_{n_i} \) are left-shift matrices of size \( n_i \) (\( n_i = 1 \Rightarrow J_1 = 0 \)).

This in turn is the same as saying that there exist \( v_1, \ldots, v_k \in \mathbb{V} \) s.t.

\[ (a) \phi v_1, \phi^2 v_1, \ldots, \phi^{n_1-1} v_1, v_2, \phi v_2, \phi^2 v_2, \ldots, \phi^{n_2-1} v_2, \ldots, v_k, \phi v_k, \ldots, \phi^{n_k-1} v_k \]

is a basis.

\[ (b) \phi v_1 = 0, \phi^2 v_2 = 0, \ldots, \phi^{n_k} v_k = 0. \]

So the claim is equivalent to
\( \phi \) nilpotent \( \Rightarrow \exists n_1, \ldots, n_k \) satisfying (a), (b) (Axler, pg 271)

If we prove the claim by induction on \( n = \dim V \).

If \( n = 1 \), \( \phi : C \to C \) is the zero map. (\( \sim \) its nilpotent) and the claim is obvious. \((M(\phi; B; B) = 0 = G_{3,3})\)

Assume \( n \geq 2 \). If \( \phi = 0 \), the claim is again obvious (take \( n_1 = n_2 = 2 \)).

Otherwise we use the same trick as earlier. Set \( W = \text{Im} \phi \subseteq V \). As \( \phi \) is nilpotent, \( \ker \phi \neq 0 \) (as \( 0 \) is an \( e \)-value) and hence \( \dim W < \dim V \) (by rank-nullity). By induction, we may decompose
\( \omega = \omega_i \oplus \cdots \oplus \omega_e \)

s.t. \( \phi \) is principal. Thus \( \exists \epsilon_i \in \Omega \)

with \( a_i , \phi(a_i), \ldots , \phi^n(a_i) \) a basis of \( \mathcal{A}_i \) for some \( n_i \) and \( \phi(a_i) = 0 \)

for all \( 1 \leq i \leq e \).

As \( a_i \in \mathcal{A}_i \subseteq \omega_i \), \( \exists v_i \in V \) s.t. \( \phi(v_i) = a_i \).

So set \( V_i = \text{Span}(v_i, a_i = \phi(a_i), \ldots , \phi^{n_i}(a_i)) \)

Clearly \( \phi(V_i) = a_i \) and so \( V_i \) is invariant. We want to show that \( \phi_{v_i} \) is principal.

Claim \( v_i, \phi(v_i), \ldots , \phi^{n_i}(v_i) \) give a basis for \( v_i \).
Proof of claim 1

We have to show $\psi, \phi(\psi), \ldots, \phi^n(\psi)$ are linearly independent.

Suppose $\lambda_0 \psi + \lambda_1 \phi(\psi) + \ldots + \lambda_n \phi^n(\psi) = 0$

Applying $\phi$ to $\phi(c_i)$

$\lambda_0 \phi(\psi) + \lambda_1 \phi^2(\psi) + \ldots + \lambda_i \phi^{i+1}(\psi, c_i) = 0$

$\phi^n(c_i)$

$\Rightarrow \lambda_0 = \ldots = \lambda_{n-1} = 0 \Rightarrow c_i, \phi(c_i), \ldots, \phi^{n-1}(c_i)$ give a basis.
of $U_i$.

But then $\lambda n_i \phi_{n_i}^i(U_i) = \lambda n_i \phi_{n_i}^{n_i+1} = 0$

$\Rightarrow \lambda n_i = 0$ as well, so the claim is proven.

So we see $\phi_{U_i}$ is principal. We could be done if

$$V = U_1 \odot \ldots \odot U_e.$$ 

This is not quite true, because there may be extra elements in the kernel to account for.

Claim 2: $V = U_1 \odot \ldots \odot U_e \odot T$

This will finish the proof, as $\phi_{T,0}$ is a sum of principal self-shifts (of size 1).
pf of claim

Let $V = V_1 + \ldots + V_l$. We firstly claim that this is a direct sum.

Suppose $x_i \in V_i$, $1 \leq i \leq l$ and $\sum x_i = 0$. But $\phi(x_i) \in U_i$ and

Then $\phi(x_i) = 0 \forall i \geq 0$ the $u_i$ form a direct sum.

We can write

$$x_i = a_0 v_i + a_1 \phi(v_i) + \ldots + a_{n_i} \phi^{n_i}(v_i)$$

But $\phi(x_i) = 0$

$\implies a_0 \phi(v_i) + a_1 \phi^2(v_i) + \ldots + a_{n_i} \phi^{n_i}(v_i) = 0$

$\implies a_0 = a_1 = \ldots = a_{n_i-1} = 0$
So \( x_i = a_i \phi_i^n(v_i) \) for some scalars \( e_i \).

As \( \sum x_i = 0 \) \( \Rightarrow \sum a_i \phi_i^n(v_i) = 0 \)

\( e_i \).

\( \Rightarrow a_i = 0 \) \( e_i \) (Since \( e_i \) form a direct sum).

\( \Rightarrow x_i = 0. \)

So \( V = \bigoplus_{i=1}^e V_i \).

To finish the proof, any basis for \( V \)

can be extended to a basis for \( V \).

In particular, extend the basis \( \phi_i^n \) to a

\( \sum_{i=1}^e \phi_i^n v_i \), \( 1 \leq i \leq e \), \( 1 \leq i \leq n \) to a
basis V by adding in the basis elements $t_1, \ldots, t_k$.

Since $\phi(v') = \omega = \text{Im} \phi$ for each $t_i$

exists $v_{i}' \in V$ s.t. $\phi(t_i) = \phi(v_{i}')$

let $t_i' = t_i - v_{i}' \in \ker \phi$

then $\text{Span} \subseteq \phi(v_{i}') \mid i = 0, 3, 0 \leq i \leq \ell$

so $\text{Span} = \phi(v_{i}')$

and this list must remain a basis

if it has correct length

set $T = \text{Span} (t_1', \ldots, t_k')$

then $V = V' \oplus T$ and $\phi|_T = 0$, as requested.
Jordan Normal Form

We now return to the study of general linear operators $\phi$.

Our first task is to break a space $V$ down into $\phi$-invariant subspaces. We want our pieces to have a simple behaviour. We want our pieces to generalize both eigenspaces and invariant subspaces.

Spaces $U_i$ st. $\phi(U_i) \subseteq U_i$.

Define $V$ a vector space over $\mathbb{K}$, $\phi \in L(V, V)$.

Suppose $x \in \mathbb{K}$, $v \in V$ is st.

$$(\phi - \lambda \text{id})^m (v) = 0$$

for some $m$. Then we call $(\lambda, v)$ a generalized eigenpair. $\lambda$ is the generalized
eigenvalue, \( \lambda \) is the generalized eigen vector with gen. e-value \( \lambda \).

If \( (\lambda, v) \) is a gen. eigenpair, then

\[
G(\lambda, \phi) := \{ v \in V : (\phi - \lambda I_d)^m v = 0 \text{ for some } m \in \mathbb{Z}_+ \}
\]

\[
= \bigcup_{k \in \mathbb{N}} \ker (\phi - \lambda I_d)^k
\]

is a generalized eigenspace.

**Defn**

Let \( p(x) \in \mathbb{P}(\mathbb{R}) \) be a polynomial,

and \( \phi \in \mathcal{L}(V, V) \) (\( V \) vs. \( k \)).

Say \( p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n \)

we define \( p(\phi) := a_0 + a_1 \phi + a_2 \phi^2 + \ldots + a_n \phi^n \in \mathcal{L}(V, V) \)
Exercise (will be on Hand)

If \( p(x), q(x) \in \mathbb{P}(k) \)
then \( p(\phi) q(\phi) = (pq)(\phi) = q(\phi)p(\phi) \)

Lemma

Suppose \( \dim V=n \)
Then \( GC(\phi, \phi) = \ker(\phi-\text{id})^n \)
(i.e. we only need the power \( n=\dim V \))

Proof

By definition,
\[
GC(\phi, \phi) = \bigcup_{k \in \mathbb{N}} \ker(\phi-\text{id})^k
\]
so \( \ker(\phi-\text{id})^n \leq GC(\phi, \phi) \) from the def.

On the other hand, suppose \( \psi \in GC(\phi, \phi) \).
Let $\Psi = \phi - \lambda \text{id}$. We have

$$\Psi^m v = 0$$

for some $m$. So let $W = \text{Ker} \Psi^m$. Then $W$ is invariant for $W$. If $w \in W$ then $\Psi^{m-1}(\Psi(w)) = 0$

$$\Rightarrow \Psi^m(\Psi(w)) = 0$$

$$\Rightarrow \Psi(w) \in W.$$

Further, $\Psi|_W$ is nilpotent as $\Psi^m = 0$.

It follows that we can write $\Psi|_W$ in the form

$$\begin{bmatrix} \mathbb{E} & \mathbb{E} \\ \mathbb{E} & \mathbb{E} \end{bmatrix} \Rightarrow \Psi|_W = 0$$

$$\Rightarrow \dim W = 0$$

$$\Rightarrow \Psi^m = 0$$

$$\Rightarrow v \in \text{Ker} \Psi^m.$$