Last time (Axler 8.4)

$(\lambda, v)$ is a generalized eigenpair if

$(\phi - \lambda \text{id})^m (v) = 0$ for some integer $m$

$G(\lambda \phi) = \bigcup \ker (\phi - \lambda \text{id})^m$

is a generalized eigenspace.

Lemma. $U$ fin. dim., $n = \dim V$

Let $\psi \in \mathcal{L}(V, V)$ satisfy $\psi^m = 0$ for some $m \geq 0$. Then $\psi^n = 0$ for $n = \dim V$

($\Rightarrow$ $\psi$ nilpotent by our definition)

Proof. Suppose $\ker \psi^i = \ker \psi^{i+1}$ for some $i$. Then, suppose $v \in \ker \psi^{i+2}$, so

$\psi^{i+2}(v) = 0 \Rightarrow \psi^{i+1} (\psi(v)) = 0$
\[ \Rightarrow \psi^i(\psi^0) = 0 \quad (\text{ker } \psi^{i+1} = \text{ker } \psi^i) \]
\[ \Rightarrow v \in \text{ker } \psi^{i+1}. \]

So \( \text{ker } \psi^{i+2} \leq \text{ker } \psi^{i+1} \leq \text{ker } \psi^{i+2} \)

And \( \text{ker } \psi^{i+1} = \text{ker } \psi^{i+2} \)

So \( (\text{Obiv}) \Rightarrow \text{ker } \psi^{i} = \text{ker } \psi^{i+1} = \text{ker } \psi^{i+2} = \ldots = \text{ker } \psi^{j} \)

for all \( j \geq 0 \).

So let \( s := \min \{ i : \text{ker } \psi^{i} = \text{ker } \psi^{i+1} \} \)

\[ \leq m \quad (\text{as ker } \psi^{m} = V) \]

So

\[ 0 \neq \text{ker } \psi \neq \text{ker } \psi^{2} \neq \ldots \neq \text{ker } \psi^{s} = \text{ker } \psi^{s+1} \]

\[ \ldots = \text{ker } \psi^{m} = V \]

\[ \dim 2 \geq \dim 2 \]

we have \( \dim (\text{ker } \psi^{i}) \geq 2, \quad 0 \leq i \leq s \)
\[ \Rightarrow s \leq \dim(\ker \psi^s) = \dim(V) = n \]

So \( \ker \psi^s = \ker \psi^n = \emptyset \Rightarrow \psi^n = 0 \]

**Corollary**

Let \( \phi \in L(V, V) \)

Let \( \lambda \) be a generalized eigenvalue

and \( n = \dim V \). Then

\[ G(\lambda, \phi) = \ker(\phi - \lambda \text{id})^n \]

**Proof**

By definition

\[ G(\lambda, \phi) = \bigcup_{m \in \mathbb{N}} \ker(\phi - \lambda \text{id})^m \]

So \( \ker(\phi - \lambda \text{id})^n \subseteq G(\lambda, \phi) \) immediately from the definition.

On the other hand, suppose \( \psi \in G(\lambda, \phi) \) and let

on the other hand, suppose \( \psi \in G(\lambda, \phi) \) and let

let \( W = \ker \psi^m \). Then \( W \) is invariant under \( \psi \) and \( \psi|_W \) is nilpotent.
Previous lemma \( \Rightarrow \psi_{w} \dim w = 0 \)

\( \Rightarrow \psi_{w}^{n} = 0 \Rightarrow \nu \in \ker \psi^{n} \).

**Lemma (Axler, pg 247)**

Let \( \phi \in L(U, U) \). Suppose \((\lambda_{i}, v_{i}) \), \( i = 1, \ldots, m \) are generalized eigenpairs, with \( \lambda_{i} \neq \lambda_{j} \) for \( i \neq j \).

Then \( v_{1}, \ldots, v_{m} \) is linearly independent.

**Proof**

Suppose \( \alpha_{1} v_{1} + \ldots + \alpha_{m} v_{m} = 0 \), \( \alpha_{i} \in k \).

We wish to show \( \alpha_{1} = \ldots = \alpha_{m} = 0 \).

Let \( T_{i} = (\phi - \lambda_{i} \text{id}) \), \( i = 1, \ldots, m \).

Suppose \( \alpha_{j} \neq 0 \) for some \( j \).

Let \( m \) be the least integer s.t.

\( T_{j}^{m} (v_{j}) = 0 \) (such an \( m \) exists as \( v_{j} \) is a generalized eigenvector).
\[ w = T_j^{m-1}(w) \] so \[ T_j w = 0. \]

Notice \[ T_j w = (\phi - \lambda_j \text{id}) w \]
\[ = (\phi - \lambda_j \text{id} + \lambda_j \text{id} - \lambda_j \text{id}) w \]
\[ = (T_j + (\lambda_j - \lambda_j) \text{id}) w \]
\[ = (\lambda_j - \lambda_j) w \text{ for all } j. \]

Set \[ S = T_j^{m-1} \circ T_2 \circ \ldots \circ T_{j-1} \circ T_j \circ T_{j+1} \circ \ldots \circ T_m \]
for \[ n = \dim V, \]

Notice for any \[ v \in V \]
\[ T_a \circ T_b = (\phi - \lambda_a \text{id}) (\phi - \lambda_b \text{id}) \]
\[ = P_a(\phi) P_b(\phi) \]
\[ = P_b(\phi) P_a(\phi) = T_b \circ T_a \]

for \[ P_a = x - \lambda_a \]
So $S\nu_i$ for $i \neq j$

(3) $T_i \nu_i = 0$ as $\nu_i \in G(\mathcal{A_i}, \emptyset)$

Applying $S$ to (3) then yields

$\alpha_j S\nu_j = 0$

or $\alpha_j \Pi (\lambda_j - \lambda_i) \omega = 0$ by (4)

$i \neq j$

As $\delta_i \neq \lambda_i$ for $i \neq j$ this $\Rightarrow \alpha_j = 0$

which is a contradiction.

Suppose $(\chi_i, \nu_i)$ is a generalized eigenpair. Then $\nu_i \in \ker (\phi - \lambda_i \cdot \text{id})$

$\Rightarrow (\phi - \lambda_i \cdot \text{id})$ not injective $\Rightarrow (\phi - \lambda_i \cdot \text{id}) \nu_i = 0$

injective $\Rightarrow \exists \nu_i$ s.t. $(\phi - \lambda_i \cdot \text{id}) \nu_i = 0$

$\Rightarrow \lambda_i$ is an eigenvalue (in the ordinary sense).
Lemma

Let $V$ be a vector space, $n = \dim V$, and $\phi \in \mathcal{L}(V, V)$. Then

$$V = \ker \phi \oplus \im \phi^n.$$

Proof

Let $v \in \ker \phi^n \cap \im \phi^n$. Then $\phi^n v = 0$ and further $\phi^n w = v$ for some $w \in V \implies \phi^n w = 0 \implies \phi^w = 0$ (as $\phi$ nilpotent, $\lambda \leq \dim V$)

$\implies v = \phi^n w = 0$.

So $\ker \phi^n \cap \im \phi^n$ form a direct sum.

Hence $\dim (\ker \phi^n) + \dim (\im \phi^n) = n$.

By rank-nullity, this, so, together with the above, we must have

$$V = \ker \phi^n \oplus \im \phi^n.$$
The following theorem is **very important**

**Thm**

Let \( V \) be a finite dimensional vector space of dimension \( n \) over \( \mathbb{C} \) and \( \Phi \in \mathcal{L}(V, V) \). Assume \( \lambda_1, \ldots, \lambda_m \) are the distinct eigenvalues of \( \Phi \).

Then \( \mathcal{C}(\lambda_1, \Phi) \oplus \cdots \oplus \mathcal{C}(\lambda_m, \Phi) = V \).

Hence \( V \) has a basis consisting of generalized eigenvectors.

\[
\mathcal{C}(\lambda_i, \Phi) = \ker (\Phi - \lambda_i \cdot \text{id})^n
\]

**Pf**

We will prove the result by induction on \( n \). If \( n = 1 \), then \( V = \text{Span}(v_1) \) has a basis consisting of the eigenvector \( v_1 \).

So, by induction assume \( n > 1 \) and the desired result holds on all vector spaces of smaller dimension.
we use the trick that we have already used twice before.

Set \( W = \text{Im}(\phi - \lambda, \text{id}) \).

Then \( W \) is invariant under \( \phi \), because

\[
\phi((\phi - \lambda, \text{id})w) = (\phi \phi)q(\phi)w = (q(\phi)\phi)w = q(\phi)p(\phi)w \in W
\]

for \( p(x) = x, \ q(x) = (x - \lambda) \).

Also \( v \in \ker(\phi - \lambda, \text{id}) \).

\( \Rightarrow \dim W < n \) by rank-nullity.

So the induction hypothesis applies to \( \phi|_W \).

By the previous lemma

\[
V = G(\chi, \phi) \oplus W \quad (\text{C})
\]

The eigenvalues of \( \phi|_W \) are all
elements of $\mathcal{M} \ldots \mathcal{M}$.

Then, $W = G(CX_1, \phi(w)) \oplus \cdots \oplus G(CX_m, \phi(w))$.

By induction.

To complete the proof, we need to show

$G(CX_i, \phi(w)) = G(CX_i, \phi)$

for $i = 2, \ldots, m$.

The inclusion

$G(CX_i, \phi(w)) \subseteq G(CX_i, \phi)$

is obvious. Suppose $u \in G(CX_i, \phi)$, $i \neq 1$.

So $u = v + w$ for $v \in G(CX_1, \phi)$.

Now, by (8),

$u = v + w_2 + \cdots + w_m$

for $w_i \in G(CX_i, \phi) \subseteq G(CX_i, \phi)$ by (9).

I.e., $v + w_2 + \cdots + (w_i - u) + \cdots + w_m = 0$. 
Since each e-vector corresponding to distinct e-values are linearly independent, we have:

\[ v = w_2 = w_{i-1} = w_{i+1} = \ldots = w_n = 0 \]

and \[ u = w \in \mathbb{C}^{n \times n} \psi(w) \] as required.

**Defn (Jordan Block):** Let \( \lambda \) be an eigenvalue of \( A \) and we let the Jordan block \( J_n(\lambda) \) of order \( n \) be the \( n \times n \) matrix

\[
J_n(\lambda) = \begin{pmatrix}
\lambda & 1 & 0 & \cdots & 0 \\
0 & \lambda & 1 & \cdots & 0 \\
0 & 0 & \lambda & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \lambda
\end{pmatrix}
\]

**THM (Jordan Normal Form):** Let \( \psi \) be a linear transformation of \( V \) over \( \mathbb{C} \) with dimension \( n \). Let \( \phi \in \mathbb{C}^{n \times n} \psi \). Then \( \exists \) a basis \( B \) of \( V \) s.t.
\[ M(\phi; B, B) = \begin{pmatrix} \mathcal{J}_n(\lambda) & 0 \\ 0 & \mathcal{J}_p(\lambda_d) \end{pmatrix} \]

for suitable \( n_1, \ldots, n_p \).

\# For any eigenvalue \( \lambda \)

\[ \phi - \lambda \text{id} \quad \in C(\lambda, \phi) \]

Hence a basis \( B_i \) of \( C(\lambda, \phi) \) s.t.

\[ M(\phi - \lambda \text{id}; B, B) \]

\[
\begin{bmatrix}
\mathcal{J}_{m_1}(0) & & \\
& \ddots & \\
& & \mathcal{J}_{m_p}(0)
\end{bmatrix}
\]

where \( \mathcal{J}_{m_j}(0) \) is the left-shift matrix of size \( m_j \).

By thm on canonical form of nilpotent matrices
\[ M(\phi; B_x, B_z) = \begin{bmatrix} \mathcal{J}_{m_1}(\phi) & 0 \\ 0 & \mathcal{J}_{m_3}(\phi) \end{bmatrix} \]

Since \( V = \Theta \mathcal{G}(\gamma, \theta) \) the result follows.