Subspaces

Recall If \( U \) is a vector space \( \mathbb{F} \) and \( \mathcal{S} \) is a subset, then we say \( U \) is a subspace if:

(i) \( 0 \in U \)

(ii) \( U \) is closed under addition.

(iii) \( U \) is closed under scalar multiplication.

Example

\( V = \mathbb{R}^2 \), \( \mathcal{S} = (a, b) \in \mathbb{V} \)

Let \( \text{span}(\mathcal{S}) \) denote \( \{ \sum k_i \mathbf{v}_i : k_i \in \mathbb{R}, 1 \leq i \leq 3 \} \)

\( \bigoplus \mathbf{v} \)

This is the line spanned by \( \mathcal{S} \)
(also written \( <v> \))

Then \( kv \subset k^2 \) is a subspace:
\[
0 = 0 \in kv \text{ so axiom (i) holds.}
\]

For any \( \lambda, \mu \in k \)
\[
\lambda v + \mu v = (\lambda \mu) v \in kv
\]
so axiom (iii) holds.

Further, \( \mu (\lambda v) = (\mu \lambda) v \in kv \)
so axiom (iv) holds.
Here is a very important example of a subspace.

**Definition**

Let $V, W$ be vector spaces over $k$ and let $T : V \rightarrow W$ be a linear map. The kernel (or nullspace) of $T$ is defined as

$$\text{Ker}(T) := \{ v \in V \mid T(v) = 0 \}.$$

**Prop.** $\text{Ker}(T) \subseteq V$ is a subspace.

**Proof.** We have to show (i), (ii), (iii) of the subspace definition hold.

(i) holds since $T(0) = 0$ (we saw this last time).
Suppose \( u, v \in \ker T \), i.e. \( Tu = Tv = 0 \)
Then \( T(u + v) = Tu + Tv \) by "linearity"
\[ = 0 + 0 \] of \( T \)
\[ = 0 \]
and so \( u + v \in \ker T \). Thus (iii) holds.

Lastly, suppose \( u \in \ker T \), \( \lambda \in \mathbb{K} \)
\[ T(\lambda u) = \lambda T(u) = \lambda 0 = 0 \]
\[ \lambda \]
As \( T \) is a linear map
(pasr (ii) of def or "homogeneity")
\[ \Rightarrow \lambda u \in \ker T \] and (iii) holds.

**Example**

Let \( V = \{ \text{functions } f : \mathbb{R} \to \mathbb{R} \} \)
Let \( U = \{ \text{infinitely differentiable} \} \)
functions $f: \mathbb{R} \to \mathbb{R}$

(Inf. differentiable means $f'(x)$ exists and
is continuous for all $x \geq 0$)

Then $U \subseteq V$ is a subspace. Further, there is a linear map

$$T: U \to U$$

$$T(f(x)) := f'(x)$$

(check $T$ is linear).

$$\ker T = \left\{ f(x) \mid f'(x) = 0 \right\}$$

$$= \left\{ \text{constant function } f(x) = c \right\}$$

$$c \in \mathbb{R}$$

**Important Remarks**

Let $U \subseteq V$ be a subspace. Then $U$
is a vector space in its own right (with operations as defined for $W$)

Recall

Let $f : S_1 \rightarrow S_2$ be a function between two sets $S_1, S_2$. Then $f$ is said to be injective if, for any $x, y \in S_1$,

$$f(x) = f(y) \Rightarrow x = y.$$

Lemma

Let $T : V \rightarrow W$ be a linear map between $k$-vector spaces. Then $T$ is injective if and only if

$$\ker T = \{0\}.$$
Proof

Suppose $T$ is injective and let $v \in \ker T$. Then $Tv = 0$. But, we also have seen that $To = 0$. Since $T$ is injective we must have $v = 0$.

Hence, $\ker T = \{0\}$.

Now suppose $\ker T = \{0\}$. Let $u, v \in V$ with $Tu = Tv$.

Then $T(u - v) = Tu - Tv = 0$

$\Rightarrow u - v \in \ker T = \{0\}$

$\Rightarrow u - v = 0$ and so

$u = v$.

Hence $T$ is injective. \(\Box\)
Definition/Recollection

Let \( f: S_1 \to S_2 \) be a function between two sets.

The image of \( f \) (or "range") is

The subset \( \text{Im} \ f \subseteq S_2 \) (also written \( f(S_1) \) or, in Axler, range \( f \))

is the set

\[
\text{Im} \ f = \{ x \in S_2 \mid \exists y \in S_1 \text{ with } f(y) = x \}
\]
Prop
Let $T: V \rightarrow W$ be a linear map between two $k$-vector spaces. Then $\text{Im}(T) \leq W$ is a subspace.

Proof
$T0 = 0 \Rightarrow 0 \in \text{Im}(T)$.
Now suppose $u, v \in \text{Im}(T)$. Thus, there exist $a, b \in V$ such that $T(a) = u$ and $T(b) = v$.

$\text{Then } u + v = T(a) + T(b) = T(a + b)$.
$\Rightarrow u + v \in \text{Im}(T)$.

For any $\lambda \in k$.,
$\lambda u = AT(a) = T(\lambda a)$

$\Rightarrow \lambda u \in \text{Im}(T)$

Hence $\text{Im}(T) \subseteq W$ is a subspace.

Recall we say $f : S_1 \to S_2$ is surjective if $\text{Im} f = S_2$.

Con A linear map $T : V \to W$ is an isomorphism if and only if

(C(i)) $\ker T = \{0\}$

(C(ii)) $\text{Im } T = W$

PF Recall $T$ is an $\Leftrightarrow T$ injective

$\Leftrightarrow$ (i) $T$ injective $\Leftrightarrow$ $\ker T = \{0\}$ and $\text{Im } T = W$.
Here is one more example of a subspace. Let \( U \) be a vector space and let \( \mathbf{v}_1, \ldots, \mathbf{v}_m \) be vectors in \( U \). A linear combination of \( \mathbf{v}_1, \ldots, \mathbf{v}_m \) is a vector of the form
\[
\lambda_1 \mathbf{v}_1 + \cdots + \lambda_m \mathbf{v}_m,
\]
for \( \lambda_i \in \mathbb{R} \).

**Definition**

Let \( \mathbf{v}_1, \ldots, \mathbf{v}_m \in U \) as above.

The subset
\[
\text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_m) = \left\{ \sum_{i=1}^{m} \lambda_i \mathbf{v}_i : \lambda_i \in \mathbb{R} \right\}
\]

of \( U \) is called the span of \( \mathbf{v}_1, \ldots, \mathbf{v}_m \).
Prop. $\text{span}(v_1, \ldots, v_m)$ is the smallest subspace of $V$ containing $v_1, \ldots, v_m$.

pf

We will first show that

$$\text{span}(v_1, \ldots, v_m) \subseteq V$$

is a subspace.

We have $0 = 0 v_1 \in \text{span}(v_1, \ldots, v_m)$.

Next, suppose $u = \lambda_1 v_1 + \ldots + \lambda_m v_m$

and $v = \mu_1 v_1 + \ldots + \mu_m v_m$

are elements of $\text{span}(v_1, \ldots, v_m)$.

Then $u + v = (\lambda_1 + \mu_1) v_1 + \ldots + (\lambda_m + \mu_m) v_m$

$\in \text{span}(v_1, \ldots, v_m)$.
and \( x \cdot \mathbf{u} = (x \cdot v_1) \mathbf{v}_1 + \ldots + (x \cdot v_m) \mathbf{v}_m \in \text{Span} (\mathbf{v}_1, \ldots, \mathbf{v}_m) \text{ for all } x \in \mathbb{F} \).

Hence \( \text{Span} (\mathbf{v}_1, \ldots, \mathbf{v}_m) \) is a subspace.

Next, we need to show \( \text{Span} (\mathbf{v}_1, \ldots, \mathbf{v}_m) \) is the smallest subspace of \( V \) containing \( \mathbf{v}_1, \ldots, \mathbf{v}_m \). Clearly \( \mathbf{v}_i \in \text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_m) \) for \( 1 \leq i \leq m \).

Suppose \( W \subseteq V \) is a subspace containing \( \mathbf{v}_1, \ldots, \mathbf{v}_m \) and let \( x_1, \ldots, x_m \in \mathbb{F} \).

As \( W \) is a subspace, 
\[ x_i \mathbf{v}_i \in W \text{ for } 1 \leq i \leq m. \]

(Cartan (iii) of a subspace) and then \( x_1 \mathbf{v}_1 + \ldots + x_m \mathbf{v}_m \in W \) (Cartan (ii) of subspace).
The span \( \langle \mathbf{v}_1, \ldots, \mathbf{v}_n \rangle \) is defined as
\[
\mathbf{x} = \sum_{k=1}^{n} c_k \mathbf{v}_k
\]
in \( \mathbb{F}^m \), where \( \mathbb{F} \) is a field.

\[ \subseteq \mathbb{W} \]

**Finite dimensional Vector space**

**Definition**

A vector space \( V \) over \( \mathbb{F} \) is called finite dimensional if there exist a finite set \( \mathbf{v}_1, \ldots, \mathbf{v}_n \) of vectors such that

\[
\text{Span} \left( \mathbf{v}_1, \ldots, \mathbf{v}_n \right) = V.
\]