1. **(1 point)** Suppose \( \phi : V \to W \) is a linear map, and \( v_1, \ldots, v_k \in V \) are vectors such that \( \phi(v_1), \ldots, \phi(v_k) \) are linearly independent. Show that \( v_1, \ldots, v_k \) are linearly independent.

2. **(1 point)** Suppose \( V, W \) are finite dimensional vector spaces over \( F \) and \( \phi : V \to W \) is a linear map.
   
   (a) Show that \( \phi \) is injective if and only if there exists a linear map \( \psi : W \to V \) such that \( \psi \circ \phi = \text{id}_V \) (where \( \text{id}_V \) is the identity map \( V \to V \)).
   
   (b) Show that \( \phi \) is surjective if and only if there exists a linear map \( \theta : W \to V \) such that \( \phi \circ \theta = \text{id}_W \).

3. **(2 points)** Suppose \( U, V, W \) are finite dimensional vector spaces over \( F \) and \( \phi : U \to V, \psi : V \to W \) are linear maps.
   
   (a) Show that \( \dim \ker(\psi \circ \phi) \leq \dim \ker \phi + \dim \ker \psi \).
   
   (b) Show that \( \dim \im(\psi \circ \phi) \leq \min\{\dim \im \phi, \dim \im \psi\} \).

The next two questions will require you to read about direct sums (Axler, page 21).

4. **(2 points)** Consider \( V = \{ p : \mathbb{R} \to \mathbb{R} \mid \exists a_0, \ldots, a_3 \in \mathbb{R} : p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \forall x \in \mathbb{R} \} \), the vector space of polynomials over \( \mathbb{R} \) of degree at most 3. Let \( W \) be the subspace \( \{ p \in V \mid p(2) = 0 \} \).
   
   (a) Find a basis for \( W \).
   
   (b) Extend this to a basis for \( V \).
   
   (c) Find a subspace \( U \) of \( V \) such that \( V = W \oplus U \) is a direct sum.

5. **(2 points)** Suppose \( V \) is a vector space and \( U_1, \ldots, U_m \) are subspaces of \( V \) such that each \( U_i \) is finite-dimensional (however, you may not assume that \( V \) is finite-dimensional).
   
   (a) Prove that the sum \( U_1 + \cdots + U_m \) is finite-dimensional, and moreover that
   
   \[
   \dim(U_1 + \cdots + U_m) \leq \dim U_1 + \cdots + \dim U_m .
   \] (1)

   (b) Show that equality holds in (1) if, and only if, \( U_1 + \cdots + U_m \) is a direct sum.

6. **(2 points)** If \( f : U \to V \) is a linear map, and \( B_1, B_2 \) are bases for \( U \) and \( V \), we write \( M(f; B_1, B_2) \) for the matrix of \( f \) with respect to the bases \( B_1, B_2 \). Our goal in this question is to compute matrices with respect to different bases.
   
   Consider \( V = \mathbb{R}^3 \) and \( \phi : V \to V \) given by
   
   \[
   \phi(x, y, z) = (x + 2y + 3z, 4x + 5y + 6z, 7x + 8y + 9z) .
   \]
   
   Let \( B_1 = e_1, e_2, e_3 \) be the standard basis for \( \mathbb{R}^3 \).
   
   (a) Write down \( M(\phi; B_1, B_1) \).
   
   (b) Write \( u_1 = \phi(e_1) = (1, 4, 7) \) and let \( B_2 = u_1, e_2, e_3 \). Compute the matrices \( M(\text{id}_V; B_1, B_2) \) and \( M(\phi, B_1, B_2) \).
(c) Write \( u_2 = \phi(e_2) = (2, 5, 8) \) and let \( B_3 = u_1, u_2, e_3 \). Compute the matrices \( M(\text{id}_V; B_2, B_3) \) and \( M(\phi, B_1, B_3) \).

(d) Find bases for \( \text{Im}\phi \) and \( \text{Ker}\phi \).

1. (Bonus Question, for extra credit) Axler, 3.C, Question 5