1. (Worth two questions) We are going to prove the Zassenhaus Lemma, giving a generalization of the Second Isomorphism Theorem. Let $G$ be a group. Let $A \unlhd A^*$, $B \unlhd B^*$ be four subgroups of $G$.

(a) Show that, if $N, M$ are two normal subgroups of any group then $NM$ is a normal subgroup. Using this, show $D := (A \cap B^*)(A^* \cap B)$ is a normal subgroup of $A^* \cap B^*$.

(b) Prove:
\[ \frac{A(A^* \cap B^*)}{A(A^* \cap B)} \cong \frac{A^* \cap B}{D}. \]

(c) Use the symmetry between $A$ and $B$ to conclude
\[ \frac{A(A^* \cap B^*)}{A(A^* \cap B)} \cong \frac{B(B^* \cap A^*)}{B(B^* \cap A)}. \]

(This result is due to Zassenhaus (1934), who used it to simplify the proof of the Schreier Refinement Theorem.)

2. (Worth two questions) Let $G$ be a finite $p$-group.

(a) Let $H$ be a subgroup which is not normal and let $X$ be the set of conjugates of $H$, under the $G$ action. Consider the conjugation action of $H$ on $X$ (obtained by restricting the $G$ action). Show there are at least $p$ orbits of size 1.

(b) From the previous part, there is some $g \in G$ such that $gHg^{-1} \neq H$ and $\{gHg^{-1}\}$ forms its own orbit. Using this, show that the normalizer $N_G(H)$ is strictly larger than $H$.

(c) A subgroup $H$ is maximal if $H \leq H'$ implies the subgroup $H' \in \{H, G\}$. Show that every maximal subgroup of $G$ is normal.

3. Prove that the center $Z(S_n)$ is trivial (i.e. $Z(S_n) = \{1\}$) for $n \geq 3$.

4. A Dihedral Group $D_{2n}$ for $n \geq 2$ is a group of order $2n$ generated by two elements $s, t$ such that
\[ s^n = 1, \quad t^2 = 1, \quad tst = s^{-1}, \]
see §1.2 of D&F. Show that every group of order $2n$ is either cyclic or dihedral, for $p$ a prime.

5. Prove there is no simple group of order 36.

6. The quaternions is a group $Q$ with order eight and two generators $a, b$ such that
\[ a^4 = 1, \quad b^2 = a^2, \quad bab^{-1} = a^{-1}. \]

(a) Show that $Q$ has a unique element of order two, and that this element generates the centre $Z(Q)$.

(b) Show that every subgroup of $Q$ is normal.

Write up solutions to the following exercises from Dummit and Foote.

- Section 4.5: 6, 8, 19.