The Simplicity of $A_n$

We will prove $A_n$ is simple for $n \geq 5$ by induction.

**Definition.** The conjugation action of a group $G$ on itself is the action $g \cdot h = ghg^{-1}$.

Two elements $g_1, g_2 \in G$ are conjugate if they lie in the same $G$-orbit.

We first deal with the base case $n = 5$.

**Lemma.** Let $N \leq A_5$ be a normal subgroup of $A_5$ and assume $|N| = 1$. Then $N$ contains a 3-cycle $(a, b, c)$.

**Proof.** Let $g \in N$. ...
Let $G = x_1 x_2 \ldots x_k$ be a factorization into disjoint cycles, omitting all 1-cycles. As $g \in S_5$ we have only 2 possibilities.

(i) $k = 1$, $G = (a_1 b_1 c)$ or $G = (a_1 b_1 c d e)$

($G$ is even)

If $k = 1$, $G = (a_1 b_1 c)$ we are done.

If $G = (a_1 b_1 c d e)$

Conjugate by $g = (a_1 b_1) (c_1 d_1) \in A_5$

$g(a_1 b_1 c d e) g^{-1} = (a_1 b_1)(c_1 d_1)(a_1 b_1 c d e)$

$= (a_1 b_1)(c_1 d_1)$

$= (a d c e b)$ \in N$

But then $(a_1 b_1 c d e) (c_1 d_1 c e b) = (a_1 c_1 e_1 c) \in N$ and we are done.

(ii) $k = 2$.

We must have $G = (a_1 b_1)(c_1 d_1)$ ($G$ is even)
Conjugate by $g = (a \ b \ e) \ n$ 

$g^{-1} = (a \ e \ b) \ n$

$g6g^{-1} = (b \ e) (c \ d) \ n$

$(a \ 5) (c \ a) (b \ e) (c \ d) = (a \ 6 \ b \ e) \ n$

Thus $A_5$ is simple.

If $N \triangleleft A_5$, $N+1$, we need to show $N = A_5$.

By the lemma, if a 3 cycle $(a \ 5 \ c) \ n$, we firstly show $N$ contains all 3-cycles.

$(a \ 6 \ c)^{-1} = (a \ c \ b) \ n$.

Thus $N$ contains all those cycles involving any 3 of $a, 5, c$ (there are only 2 at this time).
Now set $x \in \{1, 2, 3, 4, 5\} \setminus \{9, 5, c\}$

$$c, (a_5)(9, 5) (a_5, b_5, c) (a_5) (c) x \in A_5$$

$$\Rightarrow x_5 = (a_5 x_5)$$

$$\Rightarrow (a_5 x_5) \in N$$

So $N$ contains all 3 cycles involving $(a_5, x_5)$

and then all 3 cycles involving $(9, 5, x_5)$

(as can replace $\theta$ with any pair in $\{9, 5, c\}$)

Finally, conjugating any 3 cycle $(a_5 x_5)$ with $x_5 \in \{9, 5, c\}$

by $(a', y) (a_5, x_5) (a', y)^{-1}$

while $y = \{1, 2, 3, 4, 5\} \setminus \{9, 5, 4\} \setminus \{c\}$

and thus $N$ contains all 3 cycles.
Next \( N \) contains any product of 2 transpositions:

\[
(a \ b)^2 = (a \ b)(a \ b) = 1 \in N
\]

\[
(a \ b) (b \ c) = (a \ c)(a \ b) \in N
\]

\[
(a \ b)(c \ d) = (a \ b \ c)(a \ b \ c \ d) \in N.
\]

But \( \sigma \in S_5 \)

\[
\sigma = \tau_1 \ldots \tau_{2k}
\]

a product of an even \# of transpositions.

But \( \tau_i \tau_{i+1} \in N \) in the above.

\[
\Rightarrow \ \sigma \in N \quad \text{so} \quad N = A_5.
\]

We can now prove the following

**Lemma:** Any two 3 cycles are conjugate in \( A_5 \).

**pf:** We will show any permutation
$\begin{pmatrix} a & b & c \end{pmatrix}$ is conjugate to $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$

Let $d, e$ be s.t. $\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} d & e \end{pmatrix} \begin{pmatrix} a & b & c \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$

Let $6 \in S_5$ be the permutation

$6(1) = a, 6(2) = b, 6(3) = c, 6(4) = d, 6(5) = e$

\[ \begin{array}{cccccc}
    & a & b & c & d & e \\
1 & 2 & 3 & 4 & 5 & \\
\end{array} \]

So

$6 \cdot (1\ 2\ 3) \cdot 6^{-1} = (a\ b\ c)$

If $6$ is even, we are done. If $6$ is odd, let $\gamma = (4\ 5)$

$6 \cdot \gamma$

\[ \begin{array}{cccccc}
    & a & b & c & d & e \\
1 & 2 & 3 & 4 & 5 & \\
\end{array} \]

So $6 \cdot \gamma$ is even and

$6 \gamma (1\ 2\ 3) (6 \gamma)^{-1} = (a\ b\ c)$