Curves Cont'd

Prop Let $\text{Pic}(C)$, $C$ a sm. curve

(i) If $\deg L \geq 2g$, then $L$ is globally generated.

(ii) If $\deg L \geq 2g+1$, then $L$ is very ample.

If $M \in \text{Pic}(C)$ such that $\deg M \geq 2g+1$, then

$h^i(M) = h^0(wC \otimes M^*) = 0$ as

$\deg(wC \otimes M^*) < 0$

This if $\deg L \geq 2g$, $h^0(L) = h^0(\mathcal{O}(\varphi)) + 4g + 1$

by R-R

If $\deg L \geq 2g+1$, $h^0(L) = h^0(L - \varphi) + 2$ by R-R

Up to $C$

Let's now investigate the curve $C$ more closely.
If $g = 0$, $\deg w_c = -2$ so $w_c$ is not effective.
If $g = 1$, $w_c = \mathcal{O}_C$ which is $\text{glob. gen}$ but not ample (it defines a $\text{cat. morp}$ $\mathcal{C} \to \text{spec } b$)

What is $g \geq 2$?

**Lemma**

(If $g \geq 2$, we define a finite morphism $\mathcal{C} \to \mathbb{P}^1$)

**Proof**

As $\deg w_c > 0$, in this case, it suffices to show $w_c$ is $\text{glob. gen}$.

I.e., $h^0(w_c(-p)) = g - 1 \forall p \in C$.

By RFA, $h^0(w_c(-p)) = h^i(\mathcal{O}_p)$.

$$h^0(\mathcal{O}(p)) - h^0(\mathcal{O}(p)) = 1 + 1 - g = 2 - g$$

So it suffices to show $h^0(\mathcal{O}(p)) = 1$.

Suppose $h^0(\mathcal{O}(p)) = 2$ ($g - 1 \leq h^0(w_c(-p)) \leq g$)
If \( h^0(\mathcal{O}(P)) = 2 \) then \( \mathcal{O}(P) \) is a line bundle, and we have \( C \to \mathbb{P}^1 \) of degree 1 \( \Rightarrow \mathbb{P}^1 \). 

**Def**: \( C \) is said to be **hyperelliptic** if there is a morphism \( f: C \to \mathbb{P}^1 \) of degree 2.

**E.g.**
- If \( g = 2 \) then \( \deg C = 2 \) so \( \phi_{\mathcal{O}_C} : C \to \mathbb{P}^1 \) has \( \deg 2 \Rightarrow C \) is hyperelliptic.

**Fact**: There are hyperelliptic curves of each genus.

**Thm**: Assume \( g \geq 2 \). Then \( \mathcal{O}_C \) is very ample \( \iff C \) is not hyperelliptic.

**Proof**: Given \( c \geq 2 \) it follows \( h^0(\mathcal{O}_C(-D)) = g - 2 \) for any \( \mathcal{O}_C \) of degree 2.
\( h_0(\Theta(C)) - h_1(\Theta(C)) = 2 - 1 = 1 \)

\( h_0(\omega_C(-D)) = h_1(\omega(D)) = h_0(\Theta(C)) - g - 3 \)

So \( (\Theta(C))_d \) has degree 2 with

\( h_0(\Theta(C)) = 2 \) \( (h_0(\Theta(C)) \leq 2 \Rightarrow h_0(\omega_C(-D)) \leq g - 1) \)

If \( \omega \) has degree 1 \( g(\omega) > 0 \) then

\( h_0(\omega) \leq 1 \)

Thus \( (\Theta(C))_d \) has degree 2 with \( h_0(\Theta(C)) = 2 \)

\( (\Theta(C))_d \) is \( \Theta(C) \) with degree 2.

So \( (\Theta(C))_d \) is plane quartic.

Conversely, suppose \( (\Theta(C))_d \) is plane quartic.

Then \( N_{\omega} = O_{\omega}(4) \), so \( \omega_C \sim \omega_2 \otimes O_{\omega}(4) \)

\( \sim O_{\omega}(4) \)
\( \odot C \subset \mathbb{P}^2 \) is canonically embedded if \( \omega_C \) is very ample \( \implies \) \( C \) not hyperelliptic.

First let us state, without proof, Max Noether's Theorem:

**Max Noether's Theorem**

If \( C \) is not hyperelliptic then the restriction map

\[ h^0(\mathbb{P}^2, \mathcal{O}(n)) \to h^0(C, \omega_C^\otimes n) \]

induced by \( C \hookrightarrow \mathbb{P}^2 \) are surjective.

i.e. \( \bigoplus h^0(C, \omega_C^\otimes n) \)

is the graded homogeneous coordinate ring of the canonically embedded curve \( C \).

The above is the first step towards understanding the structure of the canonical
We next want to prove Clifford's Theorem.

Let $d_1, d_2$ be effective divisors on a curve $C$.

Then $\dim (D_1 + d_2) \leq \dim (D_1 E) + 1$

**Proof**

Consider $H^0(C, \mathcal{O}(D_1)) \times H^0(C, \mathcal{O}(E))$

$$\rightarrow H^0(C, \mathcal{O}(D_1 E))$$

$(s_1, f) \mapsto s_1 f$

By diagonalizing we get a morphism

$$\phi : |D_1| \times |E| \rightarrow |D_1 E|$$

$(s_1, f) \mapsto s_1 f$

This map is obviously finite to one, a
there are only finitely many ways to decompose an divisor into 2 eff. divs.

Thus \( \dim(T_m E) = \dim(D) + \dim(E) \)
\( \leq \dim(D + E). \)

**Clifford's Thm**

**Let** \( D \) **be an effective divisor on** \( C \)
**with** \( h^0(D|D) > 0, \) \( g(C) \geq 1. \)

Then \( \dim(D|D) \leq \frac{1}{2} \deg(D), \) **with equality occurring** if and only if \( D = qD = cC \) **or** \( C \) **is hyperelliptic,** \( D = L^2 \) **for** \( L \) **a** \( 2 \)-**gon** **on** \( C. \)

\( \text{C. i.e. } \deg f = 2, (q(C) \geq 2) \)

**Proof**

If \( D \) is effective \& \( h^0(D|D) > 0, \) \( cC - D \)

is effective. By the lemma
\[ \dim(D_1 + \dim(W_c - D_1) \leq \dim(W_c) = g - 1 \quad (1) \]

\( \text{OtoH} \)

\[ \dim(D_1 - \dim(W_c - D_1) = \deg G + g \quad (RR) \]

(1) + (2) gives

\[ 2 \dim(D_1) \leq \deg D \]

\[ \dim(D_1) \leq \frac{\deg D}{2} \quad \text{and} \]

Next, suppose \( g \) is a gld \( D = \emptyset \).

We proceed by induction on \( n \). If \( n \neq 1 \),

\[ \dim(L_1) = 1 = \frac{\deg(L)}{2} \quad \text{we claim} \quad \dim(L_1)^{\perp} = \Lambda \]

Next, \( L \) is bpf \( \ell_0(L) \geq 2 \) and we have

\[ \ell_0(L) \cap L^2 = \ell_0(L) \]

Taking determinants \( \Rightarrow N = L^{-1} \Rightarrow \)

\[ N \Rightarrow H^0(L) \otimes \mathbb{Q} \Rightarrow L \Rightarrow \]
\[ 0 = \mathcal{L}^{-1} \rightarrow \mathcal{O}_C \rightarrow \mathcal{L} \rightarrow 0 \]  
\[ 0 = \mathcal{L}^{-1} \rightarrow \mathfrak{S}^2 \rightarrow \mathcal{L}^+ \rightarrow 0 \]

\[ \log(\mathcal{L}^+) \leq 2(n+1) - n = n + 2 \]

\[ \phi_+ \phi_\infty \rightarrow \mathcal{O}_C \rightarrow h(\mathcal{L}^+) \geq n+1 \]  
\[ \phi_+ \phi_\infty \rightarrow \mathcal{O}_C \rightarrow h(n) \]

Next suppose \( D \neq 0 \). We claim \( C \) is hyperelliptic.

\[ \deg D = 2 \dim \mathcal{O}_C \] is even. We have to reduce to \( n \neq \dim \mathcal{O}_C \). If \( n \leq 1 \) and \( D \) is a \( g^1_2 \) $\Rightarrow$ $C$ hyperelliptic.
If \( n = \dim \{D\} \geq 2 \), choose \( E \in \langle \omega_c - D \rangle \)

Fix \( p, q \in C, \ p \in \text{Supp}(E), \ q \notin \text{Supp}(E) \).

As \( \dim \{D\} \geq 2 \), \( 3 \notin \{D\} \) st.

\( p, q \in \text{Supp} \ D \).

Let \( D' = \text{DNE} \) (largest div contained in both \( D, E \)).

Since \( q \notin \text{Supp}(D') \), \( \deg D' < \deg D \).

\( p \in \text{Supp}(D') \Rightarrow \deg D' > 0 \).

Have
\[
0 \rightarrow \mathcal{O}(D') \rightarrow \mathcal{O}(D) + \mathcal{O}(E) \rightarrow \mathcal{O}(D + E - D') \rightarrow 0
\]

Taking global sections,
\[
\dim \{D\} + \dim \{E\} \leq \dim \{D'\} + \dim \{D + E - D'\}
\]

\( \omega_c - D \)

\( \omega_{c-D'} \)

\( \leq \dim \{\omega_c\} \) by lemma
\[ \dim(D') - \dim(D') = d + 1 - g \]
\[ \frac{1}{2} d \quad \dim(D') = -\frac{1}{2} d + g - 1 \]
\[ \dim(D') + \dim(D') = g - 1 = \dim(D') \]

Thus we must have
\[ \dim(D') + \dim(W_c - D') = d + g - 1 \]
\[ \dim(D') - \dim(W_c - D') = d + g - 1 \]
\[ \Rightarrow \dim(D') = \frac{1}{2} \deg(D') \]

Thus we see \( C \) is hyperelliptic by induction.

To see \( D \) is a multiple of \( g' \), let \( r = \dim(D') \). Consider
\[ \dim(D') + (g - 1 - r) g' \]
\[ \text{deg} D = \deg D - 2r + 2(g-1) = g-2 \]
\[ \dim \geq \dim(D1 + (g-1-r)) = g-1 \]
\[ \Rightarrow \dim(D1 + (g-1-r)g2 = K_c \]
\[ \text{Fact C hyperelliptic} \Rightarrow K_c = (g-1)g2 \quad \text{(soon)} \]

\[ \Rightarrow \dim = g2 \]

Ref:
1. If \( f \) is a b.i. with \( h^1(f) \geq 1 \), define
2. \[ \text{Cliff}(L) = \deg L - 2\dim L - 1 \]
3. \[ \text{Cliff}(C) = \min \{ \text{Cliff}(C) \mid h^1(C) \geq 2 \geq h^0(C) \geq 2, g \geq 3 \} \]
Clifford's Theorem: \( \operatorname{Cliff}(C) \geq 0 \) and equality occurs iff \( C \) is hyperelliptic. The Clifford genus is one of the most important invariants of a curve.

If \( C \) is hyperelliptic, \( L = g \frac{1}{2} \)

1. \( \Phi: C \to \mathbb{P}^1 \). Then
2. \( \Phi \) is the only \( g \frac{1}{2} \) on \( C \).
3. \( \omega_C = (g-1)L \).

Proof: Consider \( \Phi: C \to \mathbb{P}^1 \). \( \Phi \) is not a closed immersion, we claim it is also not birational.

Let \( p \neq q \in L \). We have that \( q \) is a base point of \( L \omega_C - p \) \( \Phi \) \( \omega_C(p) = \Phi \omega_C(p) \).

(Show that it follows by some arguments.)
so \( \deg \omega_c \geq 2 \). Let \( u = \deg \omega_c \) and 
let \( d = \deg (\omega_c(c)) \leq p g^{-1} \).

\[ 2g - 2 = d u \implies d \leq g - 1. \]

If \( X = \text{norm. of } \omega_c(c) \)

\[ f \]

\[ X \implies \omega_c(c) \leq p g^{-1} \]

\[ \text{norm.} \]

\[ \deg M = d, \quad h^0(M) \geq g \geq \deg M + 1 \]

\[ \leq g - 1 \]

\[ \text{which did not use the claim!} \]

The first part of Cliff's Theorem:

\[ h^0(M) \leq \frac{1}{2} \deg M + 1 \] if \( h^1(M) > 0 \)

so need \( h^1(M) = 0 \). Then

\[ h^0(M) = h^0(cM) = \deg M + 1 - g(X). \]

So \( g(X) = 0 \implies X \cong \mathbb{P}^1 \) and then need

\[ h^0(M) = \deg M + 1, \]
so \( M \cong \Theta \pi_i (g^{-1}) \)

Further, \( f = \phi_m \) is a closed end

\((M \circ \alpha) \implies \phi_m(\alpha) = \pi^1 \) and \( \phi \)

fades

\[ \begin{array}{c}
C \to \pi^1 (g^{-1}) \to \pi^1 \left( \phi^{-1} \right) \to C \end{array} \]

\((g^{-1})\)-uple embedding

and must have \( n = \deg \phi_{m} = 2 \).

As \( g \) identifies \( p_1 \) for \( p+q \) any \( g_2 \)

there must be a unique \( g_2 \) \& with

we must see \( \phi \circ g^{-1} = g \)

we further see \( g \circ \phi^{-1} = g \)

from \( \text{cat} \).