Last time

Lemma. If $D, E$ are effective on a curve $C$, then $\dim |D| + \dim |E| \leq \dim |D + E|$

Definition. A line bundle $L$ on $C$ is special if $h^1(L) > 0$.

Clifford's Theorem (i)

minimum of $\dim L$ on $C$ with $\mathcal{O}(D)$ special and $g(C) \geq 2$

Then $\dim |D| \leq \frac{1}{2} \deg D$

Proof. As $D$ is special, $\mathcal{O}_C(D)$ is
effective. By the lemma
\[ \dim |D| + \dim |W_C - D| \leq \dim |W_C| = g - 1 \]
oroH \[ \dim |D| - \dim |W_C - D| = \deg D + 1 - g \]
Hence \( 2 \dim |D| \leq \deg D \) as required.

we wish to understand when equality \( \dim |D| = \frac{1}{2} \deg D \) in Clifford's theorem can occur.

Three cases are obvious

(i) \( D = 0 \)

(ii) \( D = W_C \)

(iii) \( D = a g_1 \) on a hyperelliptic curve (i.e. \( h^0(OC_D) = 2 \), \( \deg D = 2 \))
Then $C$ hyperelliptic, $L \equiv g^1_2$, 
\[ \phi_L : C \to \mathbb{P}^1. \]
Ten (i) $L$ is the only $g^1_2$ on $C$
(ii) $\omega_C = (g-1)L$

\[ \text{pf} \]
Consider $\phi_{gC} : C \to \mathbb{P}^{g-1}$. As $C$ is hyperelliptic, \( \phi_{gC} \) is not a closed immersion.

We claim it is not birational.

Let $p \neq q$ be any $g^1_2$. By the argument for the failure of $gC$ to be $\nu_0$, \( \mathfrak{g} \) is a base point of \( (gC-p) \) and \( \phi_{gC}(p) = \phi_{gC}(q) \). So \( \deg \phi_{gC} \geq 2 \).
Let $\sigma = \deg \phi_{\omega_c}$ and let
\[ d = \deg (\phi_{\omega_c}(C)) \leq \sigma g - 1 \]

$2g - 2 = \deg \omega_c \Rightarrow d \leq g - 1$

If $X = \text{normalization of } \phi_{\omega_c}(C)$

\[ X \xrightarrow{\nu} \phi_{\omega_c}(C) \leq \sigma g - 1 \]

$M := \mathcal{O}(C)$

\[ \deg M = d \leq g - 1 \]

$\dim \mathcal{O}_P \geq \deg M + 1$

By Clifford's Theorem (ii), we must have

$\dim \mathcal{O}_P \leq \frac{1}{2} \deg M + 1$ if $\nu'(C) > 0$

and thus $\nu'(C) = 0$.

Then $h^0(C) = \mathcal{O}(C) = \deg M + 1 - g(x)$

so $g(x) = 0$ and $h^0(C) = \deg M + 1 = g - 1$

$\deg M = g - 1$
\[ M \cong \mathbb{P}^{g-1} \cap P \cap (g-1). \text{ Further } v = \Phi_M \text{ and, since } M \text{ is very ample, } v \text{ is a closed embedding} \]

\[ \Rightarrow \Phi_{\mathbb{P}^1}(C) = \mathbb{P}^1 \text{ and } \Phi_{\mathbb{P}^1}(C) \cong \mathbb{P}^1 \]

\[ \text{induced by } \Theta_{\mathbb{P}^1}(g-1). \]

Hence \( C \xrightarrow{f} \mathbb{P}^1 \xrightarrow{\Theta_{g-1}} \mathbb{P}^1 \) (44)

\[ \Phi_{\mathbb{P}^1}(C) \]

For degree reasons, the first factor has degree 2. \( C \xrightarrow{2:1} \mathbb{P}^1 \). Further, as we saw, it identifies any \( P_1, q \) s.t. \( p \equiv q \mod 2 \).

This forces \( f \to \) to be unique \( \theta_2 \) and \( f = \phi^{g-1}_f \). This implies \( \phi_{\mathbb{P}^1} = f \otimes \theta_{g-1} \).
Clifford's Theorem (Cc)

Suppose $D \not\sim \mathcal{O}/\omega^2$ is an effective, special divisor \( \omega \). Then \( \dim \text{Id} = \frac{3}{2} \deg D \).

Then \( D = \mathcal{O}/\omega^m \) for \( m \leq \frac{1}{2} \).

So \( C \) is hyperelliptic and \( n \leq g - 1 \).

**Proof.**

Suppose \( \omega \) is not \( \frac{1}{2} \).

Notice that if \( n \geq g \), \( D = \omega^m \).

\( \Rightarrow \omega(D) \) non-special.

We firstly show \( \omega(f \omega^n) = \omega \).

For \( n \leq g - 1 \) (\( = \) equality in Clifford's inequality).

The claim holds for \( n = g - 1 \) (\( f^\omega \sim \omega^g \)).
\[ 0 \rightarrow f^* \rightarrow \Omega_c \rightarrow \Omega_T \rightarrow 0 \]

for \( T \in \text{Supp} \)

\[ 0 \rightarrow f^{-1} \rightarrow f^* \rightarrow \Omega_T \rightarrow 0. \]

So \( h^0(f^{-1}) + \dim \text{Im} \sigma_T = h^0(f^*) \)

\( f^* \) is glob. \( \Rightarrow \dim \text{Im} \sigma \geq 1 \)

\[ h^0(f^{-1}) \leq h^0(f^*) - 1 \]

\[ \leq n \]

by descending induction.

Overall, \( C \xrightarrow{\phi_T} \mathbb{P}^1 \rightarrow \mathbb{P}^{n-1} \)

\[ h^0(f^{-1}) \geq n \]

So \( h^0(f^{-1}) = n \)

\( \Rightarrow \) not special
Next, suppose $D \neq 0$. We have $\dim |D| = \frac{1}{2}\deg D$. We firstly claim $C$ is hyperelliptic. If $\deg D = 2\dim |D|$ is even, we prove the claim by induction on $n = \dim |D|$. If $n = 1$, $D$ is a $g_2$ and the claim holds.

If $n = \dim |D| \geq 2$, choose $E \in (\omega_C - D)$. Fix $p, q \in C$, $p \in \text{Supp}(E)$, $q \notin \text{Supp}(E)$. As $\dim |D| \geq 2$, there exists $D'$ st. $p, q \in \text{Supp}(D')$.

Let $D'$ be the greatest sub divisor of both $D, E$.

Thus, $p \in D'$, $q \notin D' \Rightarrow \deg D' \geq 2$

$\deg D' < \deg D$
Have

\[ G \Rightarrow O(CD') \Rightarrow O(D) \Rightarrow O(E) \Rightarrow O(D+E-D') \]

\[ \Rightarrow \dim |D'| + \dim |E_1| \leq \dim |D'| \\
\quad + \dim |D+E-D'| \\
\quad \geq \dim |\omega_c - D| \]

\[ = \frac{1}{2} d \]

\[ \dim |D| - \dim |\omega_c - D| \]

\[ = d + 1 - g \]

\[ \dim |\omega_c - D| = \frac{1}{2} d + g - 1 \]

So \ LHS = g - 1 \Rightarrow \ RHS = g - 1 \Rightarrow \omega_c - 1

\[ \dim |D'| + \dim |D+E-D'| = g - 1 \]

\[ \dim |D'| - \dim |\omega_c - D'| = \deg + 1 - g \]

\[ \Rightarrow \dim |D'| = \frac{1}{2} \deg (D') \]
We see $C$ is hyperelliptic by induction.

Lastly, we show that we must have $D$ a multiple of $g_2$. Let $r = \text{dim } D_1$.

Consider $D + (g_{-1-r})g_2$

\[
\text{degree} = \deg (D) + 2(g_{-1-r}) = 2(g_{-1})
\]

\[
\text{dim } 1 \geq (d_1 + (g_{-1-r})g_2)
\]

$D \Rightarrow D + (g_{-1-r})g_2 = kC$

$\Rightarrow D = r g_2 \quad \Box$

\[\text{Def:} \quad \text{L is a c.e. with } h'(L) \geq 1 \text{ c.f.}
\]

\[\text{Cliff } L = \deg L - 2 \dim L + 1\]
\text{Cliff} \,(C) := \min \left\{ \text{Cliff} \, L \mid h(C) \geq 2 \quad \text{and} \quad h(L) \geq 2 \right\}

Clifford's Thm \Rightarrow \text{Cliff} \,(C) \geq 0 \quad \text{if equality occurs \quad if \quad C \text{ is hyperelliptic}}

\underline{Surfaces}

\text{Defn} \quad \text{A smooth surface is a smooth projective variety of dim} \ 2 \quad \text{or} \quad h = k \quad \text{.}

The intersection pairing \quad X \text{m. surface}

Let \( C \subset X \) be a smooth curve and \( \text{DSX a very ample divisor} \) by Berkini \( f \) \( D \sim D' \) \text{ in } \text{D smooth st.}
C and D meet transversally.

Define \( i(C, D) := \deg_c (\mathcal{O}_C(D)) \)

**Lemma**

In the situation above,

\[ i(C, D) = \# (C \cap D) \]

**Proof**

Follows from \( 0 \to \mathcal{O}(D) \to \mathcal{O} \to \mathcal{O}_C \to 0 \)

In particular, \( \# (C \cap D) \) does not depend on the choice of \( D \). Further, if \( C \) is also very ample, \( C \) and \( D \) meet transversally.

\[ i(C, D) = \# (C \cap D) = \# (C \cap C) = i(C, C) \]

\[ \deg_c (\mathcal{O}_C(D)) \]
Theorem

For a pairing \( \mathfrak{c} : \mathbb{D} \mathfrak{v} \times \mathbb{D} \mathfrak{v} \rightarrow \mathbb{Z} \)

s.t.

(i) \( \mathfrak{c}(C_1, D) = \#(C \cap D) \) for \( C, D \)

Smooth meeting transversally

(ii) \( \mathfrak{c}(C_1, D) = \mathfrak{c}(D, C) \) (symmetric)

(iii) \( \mathfrak{c}(C_1, C_2, D) = \mathfrak{c}(C_1, D) + \mathfrak{c}(C_2, D) \) (additive)

(iv) \( C_1 \sim C_2 \Rightarrow \mathfrak{c}(C_1, D) = \mathfrak{c}(C_2, D) \).

Proof Let \( B \subseteq \mathbb{D} \mathfrak{v} \) be the set of u.a. divisors. We first define \( \mathfrak{c} : B \times B \rightarrow \mathbb{Z} \).

If \( C \in B \) is fixed, then define \( \mathfrak{c}(C, D) \) as the number of divisors \( D' \subseteq \mathbb{D} \mathfrak{v} \) such that \( D' \sim D \) s.t. \( D' \) smooth, \( C' \) smooth, and \( D' \sim D \) s.t. \( D' \) smooth, \( C' \) smooth, and \( D' \) meets \( C' \) transversally.
\( \iota(C_{YD}) := \iota(C'_{YD}) = \#(C_{YD}) \)

we've already observed that this does not depend on the choice \( D' \).
It also does not depend on the choice \( C' \) as
\[ \#(C'_{YD}) = \#(D'_{YC}) = \#(D'_{YC'}) \]
Further it is obviously symmetric, additive and independent of \( C \in \text{CL} \).
To extend \( \iota \) to all of \( \text{Div}(X) \times \text{Div}(X) \)
Use: Exercise Any Div(X) is a

\( D \succeq A - B \) for \( A, B \) very ample.

Then for \( D | B \cdot (D | X) \) write
\( D \succeq A'_1 - B'_1, A'_2 - B'_2, \ldots \)
and define

$$
\hat{c}(D \cdot E) = \hat{c}(A_1 \cdot A_2) - \hat{c}(A_1 \cdot B_2) - \hat{c}(B_1 \cdot A_2) + \hat{c}(B_1 \cdot B_2)
$$

This is clearly independent of the chosen expressions $D \sim A_1 - B_1$ etc.

Remark: if $c$ is smooth, $D$ antidiv.,

the above shows $\hat{c}(C(D)) = \deg \circ c(D)$

(D $\sim D_+ - D_-$ u.a. $D_+ \cap D_- = \text{mft+g}$, $C$ trans.)

E.g. $c^2 = \deg \circ c(C) = \deg (Nc(x))$

E.g. 2: if $c \leq x$ is smooth,

$$
\omega_c = (\omega_x \otimes Nc(x))_c
$$

so $2g - 2 = \hat{c}(C, K_x + C)$
Eg. 3. \( C \subseteq \mathbb{P}^2 \) of degree \( d \)

\[ 2g - 2 = (\Theta(d), \Theta(d-3)) \]

\[ = d \cdot (d-3) \cdot (\Theta(1) + \Theta(1)) \]

\[ = d \cdot (d-3) \] (two curves meet in one pt).