Surfaces

Def: A smooth surface is a smooth projective variety of dim 2 (i.e.,
we usually write $S$ or $X$ for a surface.

The Intersection Pairing

$X$ sm. surface

Let $C \subset X$ be a smooth curve and $D \subset X$
a very ample divisor. By Bertini,

$\exists Y \sim D$ smooth s.t. $C \cap D$ mond

transversally. Define $i(C, D) := \deg_C \Theta_C(D)$

Lemma: In the situation above,

$i(C, D) = \#(C \cap D')$
pf Follows from
\[ 0 \to O_c (C-D) \to O_c \to O_c \to O \to 0 \]

In particular, \( \#(C \cap D) \) does not depend on the choice of \( D \). Further, if \( D \) and \( C \) is also very ample, \( D \) \& \( D \cap D \) meet transversally,
\[
\iota (C \cap D) = \# (C \cap D) = \# (D \cap C) = \iota (D \cap C)
\]
\[
\deg_c (O_c (D)) \quad \deg_D (O_D (C))
\]

**Theorem**

\( \exists \) a pairing \( \iota : \text{Div}_X \times \text{Div}_X \to \mathbb{Z} \)

s.t.
\( \iota \) is \( (C \cap D) = \# (C \cap D) \) for \( C \cap D \) smooth meeting transversally
\( \iota \) is \( C \cap D = \iota (D \cap C) \) (Symmetry)
(iii) \( i(C_1 + C_2, D) = i(C_1, D) + i(C_2, D) \) (additivity)

(iv) \( C_1 \sim C_2 \Rightarrow i(C_1, D) = i(C_2, D) \)

**Proof:** Let \( B \subseteq Div(X) \) be the set of s.a. divisors. We first define \( i : B \times B \to \mathbb{Z} \). If \( c \in B \) is fixed, we define \( CC(D) \) as such: let \( C' \sim C \) s.t. \( C' \) smooth and \( D' \cap D \) s.t. \( D' \) smooth \( \cap D \) meets \( C' \) transversally.

\[ i(C, D) := i(C', D') = \#(C' \cap D') \]

We've already observed that this does not depend on the choice of \( D' \). It is also symmetric and \( i \) does not depend on the choice of \( C' \).
To extend \( \hat{c} \) to all of \( \text{Div}_X \times \text{Div}_X \) we:

**Exercise** Any \( D \in \text{Div}_X \) has \( D \sim A - B \)

for \( A, B \) u.a.

Then for \( D_1, E \in \text{Div}_X \), write

\( D \sim A_1 - B_1, E \sim A_2 - B_2 \), \( A_i, B_i \) u.a.

and define

\[
\hat{c}(D, E) = \hat{c}(A_1, A_2) - \hat{c}(A_1, B_2) - \hat{c}(B_1, A_2) + \hat{c}(B_1, B_2)
\]

This is independent of the choices and has the required properties.

**Remark** If \( C \) smooth, \( D \) arbitrary, the above shows \( \hat{c}(C, D) = \deg(C \cap D) \)
Exercise 2.1 \[ C^2 = \deg \mathcal{O}_C(C) = \deg \mathcal{O}_{C/X}(C) \]

If \( C \subseteq X \) smooth

\[ \omega_C = (\omega_X \otimes \mathcal{O}_{C/X})|_C \]

so

\[ 2g - 2 = \varepsilon(C, \omega_C + C) \]

Exercise 2.3 \( C \subseteq P^2 \) of degree \( d \)

\[ 2g - 2 = (\mathcal{O}(d), \mathcal{O}(d-3)) \]

\[ = d(d-3)(\mathcal{O}(5), \mathcal{O}(1)) \]

\[ = d(d-3) \text{ (two lines meet transversally in 10 points)} \]

Theorem (R-R for Surfaces)

Let \( D \subseteq X \) be a divisor on a surface.

Then

\[ \chi(C \otimes \mathcal{O}(D)) = \frac{1}{2} (D, D - \omega_C) + \chi(\mathcal{O}_X) \]
pf observe that both sides only depend on the linear equivalence class (D).
So we can set \( D = C - E \) (\( C, E \) smooth).
Consider the s.e.s.'s
\[
\begin{align*}
0 & \to \mathcal{O}(C-E) \to \mathcal{O}(C) \to \mathcal{O}_E(x) \to 0 \\
0 & \to \mathcal{O}_x \to \mathcal{O}(C) \to \mathcal{O}_C \to 0
\end{align*}
\]
\[
\chi(\mathcal{O}_x(D)) = \chi(\mathcal{O}_x(0)) - \chi(\mathcal{O}_E(0)) \\
= \chi(\mathcal{O}_x) + \chi(\mathcal{O}_C) - \chi(\mathcal{O}_E) \\
= \chi(0) + \deg(\mathcal{O}_{C-0}) + 1 - g_C - (\deg(\mathcal{O}_B(0)) + 1 - g_E)
\]
By example 2

\[ g_c = \frac{1}{2} (c, c + wc) + 1 \]

\[ g_E = \frac{1}{2} (E, E + wc) + 1 \]

So \[ \chi(\Omega^c_\omega (C)) = \chi(\Omega^c_\omega) + (c, c) \]

\[ -\frac{1}{2} (c, c + wc) - (c, E) \]

\[ + \frac{1}{2} (E, E + wc) \]

\[ = \chi(\Omega^c_\omega) + \frac{1}{2} (c - E, c - E - wc) \]

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Some Basic Hodge Theory

A very small amount of Hodge theory is fairly essential to study surfaces \( \mathbb{C} \).

Let \( X \) be a smooth complex surface. Consider \( X \) with the Euclidean topology, call this \( X^F \). Then let
$H^k(X, \mathbb{C}) := H^k(X^c, \mathbb{C})$

the Čech cohomology of the constant sheaf $\mathbb{C}$.

The Hodge index theorem says

\[
H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)
\]

where $H^{p,q}(X) := H^q(X, \Omega^p_{\mathbb{C}})$.

Here this is the ordinary cohomology for projective varieties that you are used to (by GAGA Theorem). It's also coh. in fine top).

Further there is a "complex conjugation" $f_{c}^{H^{p,q}} : H^{p,q} \to H^{q,p}$.

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\[\Rightarrow \quad h^{p,q}(X) = h^{q,p}(X)\]

\[h^{p,q} = \dim H^{p,q}\]

Next, recall that, in Euclidean top, we have
the Exponential Exact Sequence

\[ 0 \to \mathbb{Z} \to \Omega^{\infty}_{X} \to \Omega^{a}_{X} \to 0 \]

of sheaves of additive groups, where

\[ \Omega^{\infty}_{X} \] is the sheaf of holomorphic \( f_{X} \)

\[ \Omega^{a}_{X} = \text{invertible sheaf} \]

One has

\[ \text{Pic}(X) \to H^{1}(\Omega^{a}_{X}) \]

The boundary map gives

\[ \text{Pic}(X) \xrightarrow{\partial} H^{2}(\mathcal{O}^{\ast}_{X}) \to H^{2}(\Omega^{a}_{X}) = H^{0}(\mathcal{O}_{X}) \]

The map \( H^{2}(\mathcal{O}_{X}) \xrightarrow{\alpha} H^{2}(\mathcal{O}_{X}) \) is the projection.

Set \( H^{\prime \prime}(\mathcal{O}_{X}) := H^{2}(\mathcal{O}_{X}) \cap H^{1}(\mathcal{O}_{X}) \)
The above shows (a) \( H^{2,0} = H^{0,2} \)

\[ \text{Im}(B) \leq H^{1,1}(X, \mathbb{Z}) \]

**Fact** \( \text{Im}(B) \cong H^{1,1}(X, \mathbb{Z}) \)

**Defn** \( H^{1,1}(X, \mathbb{Z}) = \text{Im}(B) \) is called the Neron–Severi group, \( \text{NS}(X) \). It is a finitely generated Abelian group.

There is further an Abelian group

\[ \text{Num}(X) := \text{Pic}(X) / \text{numerical equivalence} \]

where \( L \sim L' \) \( \text{num} \) if \( \langle L, M \rangle = \langle L', M \rangle \) for all \( M \in \text{Pic}(X) \)

It is a Theorem of Neron that

\[ \text{Num}(X) \cong \text{NS}(X) / \text{Torsion} \]

So \( \text{Num}(X) \) is free Abelian group
The rank \( p(x) := \text{rk} \text{ Num}(x) \) is called the Picard rank of \( x \).

\[ \text{Thm (Hodge Index Thm)} \]

Let \( H \) be ample on the surface \( X \).

Let \( D \in \text{Pic}(X) \) s.t. \( D \) is not numerically trivial (i.e. \( D \not\sim 0_X \)) and \( D \cdot H = 0 \). Then \( D^2 < 0 \).

\[ \text{Proof} \]

Suppose \( D^2 \geq 0 \).

If \( D^2 = 0 \), then since \( D \) is not numerically trivial, \( \exists E \in \text{Pic}(X) \) with \( (D \cdot E) \neq 0 \).

Consider \( E' := (H^2)E - (E \cdot H)H \)

Then \( E' \cdot H = 0 \) and \( (D \cdot E') = (H^2)(E \cdot H) \)

(\( \text{note } H \text{ ample } \implies (H^2) \neq 0 \) )
Consider $D' = nD + E'$. We still have $(D', H) = 0$.

But $(D')^2 = 2n(D, E) + E^2 > 0$ for suitable $n$.

Thus we reduce to the case $(D)^2 \geq 0$.

Consider $H' = D + nH$.

Exercise. For $n > 0$, for ample $H'$, ample, $(D, H') = (D)^2 > 0$.

So $(D, H') = (D)^2 > 0$ for $n$ sufficiently large.

$(H', K_C - nD) < 0$

As $H'$ ample this $\Rightarrow h^0(K_C - nD) = 0$.

So $R^1 \Rightarrow h^0(nD) - h^1(nD) = \frac{1}{2} (nD \cdot K_C - K_C) + \gamma(\mathcal{O}_X)$.

So $R^2 \Rightarrow h^0(nD) - h^1(nD) = \frac{1}{2} (nD \cdot K_C - K_C) + \gamma(\mathcal{O}_X)$.

(As $n^2(D)^2$ term dominates.)
\( \Rightarrow \quad h^0(n \mathcal{D}) > 0 \quad \text{for} \quad n \gg 0 \)

\( \Rightarrow \quad (n \mathcal{D} \cdot H) > 0 \quad \text{for} \quad n \gg 0 \)

(As \( nH \approx a_2 \) using Bertini)

\( \Rightarrow \quad (D \cdot H) > 0 \)

Contradiction.

**Corollary**

Consider the non-degenerate bilinear form

\[ \text{Num}(X) \times \text{Num}(X) \rightarrow \mathbb{R} \]

\[ (C_1, C_2) \rightarrow i(C_1, C_2) \]

Then \( \mathcal{O}_R \) has index \( \sum_1 \mathcal{p}(C_1 - 1) \)

\( \mathcal{P} \) is a bilinear form on the real vector space \( \text{Num}(X) \otimes \mathbb{R} \). As it is non-degenerate, by standard facts, it
can be diagonalized to a form which is diagonal with \( \pm 1 \) on the diagonal.

Index \( = (\#(+1's), \#(-1's)) \) on diag.

The form is not negative definite, since

3 \( H \in \text{Pic}(K) \) with \( H \) u.a. and so

\( (H)^2 > 0 \).

Choose an orthogonal basis extending \( H \). By the H.I.T all elements of the basis other than \( H \) must have the same.