\[ \frac{\text{Hodge Index Thm}}{\times \text{ surface.}} \]

\[ \text{Num} \times \text{Num} \rightarrow \mathbb{Z} \]
\[ c, D \rightarrow \text{sign} \left( \chi_f \right) \]
\[ \text{has signature} \ (1, \chi_f(x) - 1) \]

**PF**

From last time, we are left with proving:

**H** example, \( D \in \text{Pic}(X) \) not numerically trivial with \( D \cdot H = 0 \) \( \Rightarrow \) \( D^2 < 0 \).

We first rule out \( D^2 = 0 \).

If \( D^2 = 0 \), then since \( D \) is not numerically trivial, \( E \in \text{Pic}(X) \) with \( (D \cdot E) = 0 \).

Consider \( E' := (H)^2 E - (E \cdot H)H \).

Then \( E' \cdot H = 0 \) and \( (D \cdot E') = (H)^2(E \cdot H) \neq 0 \).
Consider $D' = hD + E'$

we still have $(D', H) = 0$, but now

$$(D')^2 = 2n (D \cdot E) + h^2$$

so for suitable $n$ we reduce to the case $(D)^2 > 0.$

Set $H' = D + nh$.

Exercise For $n \gg 1$, $H'$ ample.

So $(D \cdot H') = (D)^2 > 0$.

For $n \gg 1$, $(H' \cdot \omega_X - nD) < 0$

$\Rightarrow \mu(C, \omega_X - nD) = 0$ for $C = nh^2$.

$\Rightarrow \mu(C, \omega_X - nD) = h^2 \mu(nD) = 0$

so $\mu = \mu(nD) - \mu(nD) = \frac{1}{2} (nD \ln D - c_X)$
\[ + X(\Theta x) \]
\[ \Rightarrow \text{for } n \gg 0 \]
\[ (\text{as } n^2 (D)^2 \text{-term dominate}) \]
\[ \Rightarrow \text{if } (nD \cdot H) \gg 0 \text{ for } n \gg 0 \]
\[ \Rightarrow \text{using} \quad \text{Beating for } n \gg 0 \]
\[ \Rightarrow (D \cdot H) \gg 0 \]

Contradiction.

FLATNESS

(Until Ch 24)

Motivation: we want to consider "families" of varieties.
Define $A$ a ring, $M \in \text{Mod}_A$.

Then $M$ is flat (as an $A$-module) if the (right exact) functor $M \otimes_A -$ is exact.

Let me list some basic facts from commutative algebra (see e.g. [4], §1.2):

**Prop**

1. Free $A$-modules are flat.
2. $M, N$ flat $\Rightarrow M \otimes_A N$ flat.
3. $M, N$ flat $\Rightarrow M \otimes_A N$ flat.
4. Let $B$ be an $A$-algebra. $M$ flat $A$.
   $\Rightarrow M \otimes_A B$ flat $\Rightarrow B$.
5. Suppose $B$ is a flat $A$ algebra.
6. Suppose $B \in \text{Mod}_B$ flat. Then $M$ is flat as an $A$ module.
Theorem (Ziu, Thm 2.4)

If \( M \in \text{Mod}_A \), then \( M \) is flat \( \iff \)

\( \forall I \subseteq A, \text{ ideal, } I \otimes_A M \rightarrow IM \) is an isomorphism.

E.g.: \( \frac{2}{22} \) is not flat \( \mathbb{Z} \) because

\( \frac{22 \otimes \mathbb{Z}}{2/22} \rightarrow \mathbb{Z} \otimes \mathbb{Z} \frac{2/22}{22} \)

\[ \mathbb{Z} \]

\[ \frac{2}{22} \]

\[ \mathbb{Z} \]

\( \frac{2}{22} \) becomes trivial, so it can be moved over.

**Important Corollary**

Let \( A \) be a principal ideal domain.
Then an $A$-module $M$ is flat if and only if it is torsion-free over $A$.

**Proof**

Let $I = (a)$ be an ideal.

Consider $\tau_a: A \to I$

$x \rightarrow ax$

We have $M = A \otimes_A M \xrightarrow{\tau_a \otimes id} I \otimes_A M$

$g \downarrow$

$\tau_a \otimes id$

$\downarrow$

$f$

$IM$

So $f$ is an iso $\implies$ $g$ is an iso

$\implies$ $g$ is injective (as it is always surj)

$\implies$ $a \cdot m = 0 \implies M = 0$

$\implies a$ is not a torsion elt.

$\implies a$ is not a torsion elt.

So this holds if $I$ is torsion free over $A$.
Prop (Flatness is Local)

An $A$ module $M$ is flat $\iff$ $M_p$ is a flat $A_p$ module

$A_p$ is a prime.

Pf: Suppose $M$ is a flat $A$-module.

Let $0 \to N' \to N \to N'' \to A_p$ module

\[ \Rightarrow 0 \to M \otimes A N' \to M \otimes A N \to M \otimes A N'' \to \]
\[ M \otimes A N_p \to M \otimes A N_p \to M \otimes A N_p \]

\[ M \otimes A N_p \]

\[ M \otimes A N_p \]

\[ M \otimes A N_p \]

\[ M \otimes A N_p \]

\[ \Rightarrow \text{extra } \]

\[ \Rightarrow M_p \text{ flat } A_p \text{ prime.} \]

Conversely, suppose $M_p$ is a flat $A_p$ module $U_p$. 
\[
\begin{align*}
&0 \to N' \to N \to N'' \to 0 \text{ be exact} \\
&\text{sequence of } A \text{ modules} \\
&0 \to K \to \bigoplus_{A} N' \to \bigoplus_{A} N \to \bigoplus_{A} N'' \to 0 \\
&\text{Want } K = 0. \\
&\text{Enough } K_{p} = 0 \text{ if } p \in \mathfrak{p} \text{ prime.} \\
&\text{Enough localization exact, } \\ &\text{which gives the claim.} \\
\end{align*}
\]

We can now deliver flatness on schemes.

\textit{Definition: } \text{If } F \in \text{QCoh}(X) \text{ is flat at } p \in X \text{ if } \tilde{F}_{p} \text{ is a flat } \mathcal{O}_{X, p} \text{ module.} \\
\text{If } \tilde{F} \text{ is flat at } p \text{ and } \tilde{F}_{p} \text{ is a flat at all points.} \\
\tilde{F} \text{ is flat if it is flat at all points.} \\
\text{If } \pi : X \to Y \text{ (} X, Y \text{ scheme) is} \\
\text{flat at } p \in X \text{ if } \mathcal{O}_{Y, p} \text{ is a flat} \\
\mathcal{O}_{X, p} \text{ module.}
It is flat if it is flat at all \( p \in X \).

More generally, with \( \mathcal{F} \) above, \( \mathcal{F} \in \mathbf{QG} \text{h}(A) \) is flat over \( Y \) at \( p \in X \)
if \( \mathcal{F}_p \) is a flat \( \mathcal{O}_{Y,p} \)-module.

\textbf{Vacir 24.2.4 Exercise}

\( B, A \) alg. Ten \( B \to A \) flat \( \implies \)
\( \text{Spec } A \to \text{Spec } B \) flat.

More generally, \( B \to A \) alg. map, \( M \) an \( A \)-mod, ten \( M \) is \( B \)-flat \( \implies \)
\( M \) is flat over \( \text{Spec } B \).

Flatness is preserved by base change, has
a transitivity property etc (Ex 24.2.4)
(Ex 24.2.5)
0. THM (Takii) Ex. 24.2.8)

Cohomology commutes with flat base change.

\[
\begin{array}{cccc}
X' & \to & X \\
\downarrow \pi & & \downarrow \pi \\
Y' & \to & Y \\
\downarrow \rho & & \\
\text{flat} & & \\
\end{array}
\]

\(F \subset H^0(X)\).

Then \( \rho^* \) a natural iso

\[
\rho^* (R^i \pi_* F) \cong R^i \pi'_* \rho'^* F
\]

Here are some more useful results.

Prop: \( f: X \to Y \) flat, \( Y \) irreducible.

Then every non-empty, open subset \( U \subset X \) dominates \( Y \) (i.e. \( f(U) \subset Y \) is dense).
PF wlog \( \eta \) affine \( \mathcal{U} = \text{Spec} A \to \text{Spec} B \) flat.

\( B = A \) flat.

Let \( \eta \) be the generic point of \( X \).

\( \mathcal{U} = \{ \text{nilpotents} \} \) the nilradical of \( A \).

\[ B/\mathcal{U}B = B \otimes A/\mathcal{U}A \leq B \otimes A \text{Frac}(A/\mathcal{U}A) \]

by flatness \[ B \otimes A k(\eta) \]

\[ \to \mathcal{O}(\mathcal{U} \cap \eta) \]

We wish to show \( \mathcal{U} \cap \eta \neq \emptyset \).

But otherwise \( B = \mathcal{U}B \) so \( B \) is nilpotent,

which \( \Rightarrow A \) nilpotent \( \Rightarrow \mathcal{U} = \emptyset \).
Flatness + Tor

Let $M$ be an $A$-module. Then free resolutions exist, so, for any other module $N$

$$
\cdots \to F_2 \to F_1 \to F_0 \to N \to 0
$$
a free resolution.

Then $\text{Tor}_i^A(M,N)$ is defined to be the homology of $F_0 \otimes_A M$ at the $i$th stage.

Some basic properties (Vakil Ch. 23):

- $\text{Tor}_0^A(M,N) \cong M \otimes_A N$
- $\text{Tor}_1^A(M,N) \cong \text{Tor}_1^A(N,M)$
- $\text{Tor}$ is symmetric
- It follows from the derived functor formalism that, if
$0 \to N' \to N \to N'' \to 0$ s.e.s.

Then we have

$\Rightarrow \text{Tor}_i^A(M, N') \to \text{Tor}_i^A(M, N)$

$\Rightarrow \text{Tor}_i^A(M, N'') \to M \otimes_A N' \to M \otimes_A N'' \to 0^{(4)}$

**Corollary (Ex 24.3.C, Vakil)**

If $N''$ as above is flat, Marshall's, we have:

$0 \to M \otimes_A N' \to M \otimes_A N \to M \otimes_A N'' \to 0$

If $M$ flat $\Rightarrow \text{Tor}_i^A(M, -) = 0$ for

$c > 0$ by defn.

But $\text{Tor}_i^A(M, N'') = \text{Tor}_i^A(N'', M) = 0$
(1) $0 \to \mathfrak{m} \otimes_A N' \to \mathfrak{m} \otimes_A N \to \mathfrak{m} \otimes_A N^n \to 0$

Exact from ( atl).

**Exercise 23.1.0**

Show $\text{Tor}_1^A (M, N) = 0 \Rightarrow M \text{ flat}$

($\text{Tor}_1^A (M, N) = 0 \Rightarrow M \text{ flat}$)

**Theorem**

$M \in \text{Mod}_A$ is flat

$\Rightarrow \text{Tor}_1^A (M, A/I) = 0 \forall I \subseteq A$ ideal

**Proof**

By above exercise it suffices to show $\text{Tor}_1^A (M, N) = 0$ finitely generated.

We prove the claim by induction on the number of generators of $N$. If $n = 1$, $A \otimes M \to N \Rightarrow N = A/(0)$, so this is our assumption.
Else, we have
\[ 0 \to A \xrightarrow{f} N \to Q \to 0 \]
\[ \xrightarrow{\text{Ann}(an)} \]

Further, \( Q \) is generated by \( \mathbb{F}(a_i), i \leq n-1 \)
\[ \to \text{Tor}^1(A, M, N) \]
\[ \to \text{Tor}^1(M, Q) \]
\[ \to 0 \]