Lemma

Let $A$ be a Noetherian ring, $I$ injective. Then the sheaf $\tilde{I}$ is flasque on $\text{Spec } A$.

Proof

$x = \text{Spec } A$. Set $\hat{y} = (\text{Supp } I)$. We argue by Noetherian induction on $\hat{y}$. If $\hat{y}$ is a point, $\tilde{I}$ is a skyscraper sheaf, which is trivially flasque.

Enough to show $U \subseteq x \Rightarrow \tilde{I}(x) \rightarrow \tilde{I}(y)$ (if $U \subseteq y$, then $\tilde{I}(x) \rightarrow \tilde{I}(U)$ is surj. $\tilde{I}(y)$.

If $U \cap y = \emptyset$ then the claim is obvious (as $\tilde{I}(y) = 0$). So suppose $U \cap y \neq \emptyset$.

Then there exists $x \in U$ such that $x \notin y$, so $x \notin \hat{y}$. 

With $\tilde{I}$ the sheaf of locally free modules $\tilde{I}$ is flasque.
Consider

\[ I' \quad \text{if} \]
\[ P(C, I) \rightarrow P(C, I) \rightarrow P(X, I) = I \]

Claim 1: If \( I \) is an injective module on a Noetherian ring \( A \) and \( f \in I \), then the localization \( I \rightarrow I_f \) is surjective.

Proof of claim:

Let \( b^c = \text{Ann}_A(f^c) \leq \text{Ann}_A(f^{c+1}) = b^c+f^{c+1} \)

As \( A \) is Noetherian, \( f^r \) s.t.

\[ b^c = b^r \quad \forall j \geq r \quad (a) \]

Let \( x = y^r \in I_f, y \in I \)

Consider the map \( (f^n) \rightarrow I \)

\[ f^m \rightarrow f^r \]
This is well-defined: if \( a \tilde{f} = 0 \) then 
\( a \tilde{f}^* = 0 \) by (1) \( \Rightarrow a \tilde{f} y = 0 \)

As \( I \) is injective, we can extend \( \tilde{f} \) to

\[ \tilde{f} : A \to I \]

Then \( \tilde{f} \tilde{f}^* \tilde{g} (y) = \tilde{f} (f^* \tilde{g}) = \tilde{f} \tilde{g} \)

\[ \Rightarrow \tilde{g} (y) = x \in I \]

So, from this claim, we have

\[ \tilde{n} (x, i) \to \tilde{n} (y, i) \to \tilde{n} (x_{f}, i) \]

Let \( s \in n (y, i) \) and let \( s' \in I \) be the image of \( s \). Suppose \( f \in I \) is sent to \( s' \). Suppose \( t \in n (y, i) \) is restriction of \( f \).
Then $t' - s$ vanishes on $X_f$ since

$$\text{supp}(t' - s) \subsetneq Z(f) = X - X_f$$

If we write $t = 2 \Theta$

this means $t' - s \in \mathbb{P}^2 (\mathcal{O}_1, \mathcal{I})$

Cohomology with support in 8

Suffices to show $\mathbb{P}^2 (X_1, T) \rightarrow \mathbb{P}^2 (\mathcal{O}_1, \mathcal{I})$

and $\cap \mathcal{I}$

Freudent (Ha III, ex 5.6)

$$\mathbb{P}^2 (X_1, T) = \{ \text{some } I \mid f^\wedge \equiv 0 \text{ for some } \mathcal{I} \}$$

Claim 2: $\mathcal{I}_2 = \mathbb{P}^2 (X_1, T)$ is an injective $\mathcal{A}$-module.
Proof of claim
Let \( b \subseteq A \) be an \( A \)-primary ideal.

Need to show \( \mathfrak{p} \) extends to \( A \).

As \( \mathfrak{p} = \mathfrak{p} \cap I \) is \( \mathfrak{p} \)-primary and \( b \) is finitely generated, \( \exists k \) s.t.

\[ f^k \mathfrak{p}(x) = 0 \]

Now Krull's Theorem says precisely that

\[ f^{k'} \mathfrak{p} \supseteq \mathfrak{p} \quad \text{for some } k' \]

so \( \mathfrak{p}(\mathfrak{p}(f^{k'})) = 0 \)

and we have a map

\[ \mathfrak{p} : \mathfrak{D} / \mathfrak{D} \cap (f^{k'}) \rightarrow \mathfrak{p} \]
\[ A \longrightarrow A' \overset{(f^k)}{\longrightarrow} \overset{\psi}{\longrightarrow} J' \quad \text{\(T: 	ext{r'ng} \Rightarrow \text{injektiv}\)} \]

\[ \exists \quad B \overset{\varphi}{\longrightarrow} \overset{\cup (f^k)}{\longrightarrow} J \overset{\exists}{\longrightarrow} I \]

Since \( f^k \), \( \psi'(A) = 0 \) (\( \psi' \) as above)

\[ \psi'(A) \leq J = \exists \inf \{1 + f^k \mid m=0 \} \text{ for some } n \]

so \( \psi' : A \rightarrow J \) is our desired injection.

Return to our problem, we need to show

\[ \Pi_2 (X, \overline{z}) \Rightarrow \Pi_2 (U, \overline{z}) \]

\[ J \text{ injective} \quad \overline{\varphi}(u) \]

\[ 1 \text{ injective} \quad \overline{\varphi}(u) \]
Thus if quasi -open = quasi-seperated = U \cap V = V \cap U \implies \langle U \rangle = \langle V \rangle.

Two points are quasi-open if and only if their complements intersect.

Thus the points of X all pull back onto the points of Y = \text{Spec} \ A.

This finishes the proof as we are done by Noetherian induction.

\[ \sum_{i} (\mathbb{Z}_2 \otimes \mathbb{Z}_2) = \mathbb{Z}_4 \]
Ext and Ext

$(X, \mathcal{O}_X)$ a ringed space, $\tau$ an $\mathcal{O}_X$-module

have a functor

$\text{Hom}(F, -): \text{Mod}_{\mathcal{O}_X} \rightarrow \text{Ab}$

$G \rightarrow \text{Hom}_{\mathcal{O}_X}(F, G)$

we also have

$\text{Hom}(\mathcal{F}, -): \text{Mod}_{\mathcal{O}_X} \rightarrow \text{Mod}_{\mathcal{O}_X}$

$G \rightarrow \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, G)$

so $\text{Hom} = \text{RHom}$
These are left exact functors and so we define $\text{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ resp. $\text{Ext}^i_{\mathcal{O}_X}(\mathcal{G}, \mathcal{H})$ for the $i$th right derived functors of $\text{Hom}_X$ resp. $\text{Hom}_X$.

Lemma \text{A.G. in Mod}_{\mathcal{O}_X}$

(i) $\text{Ext}^i_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{G}) = 0$ for $i \geq 0$

(ii) $\text{Ext}^i_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{G}) = \mathcal{G}(\mathcal{O}_X)$

Proof:

(i) $\text{Hom}_X(\mathcal{O}_X, \mathcal{G}) = \mathcal{G}$ so $\text{Hom}_X(\mathcal{O}_X, \mathcal{G})$

is the identity functor and (i) is obvious.

(ii) $\text{Ext}^i(\mathcal{O}_X, \mathcal{G}) = \text{Hom}(\mathcal{O}_X, \mathcal{G}) \otimes \mathcal{G}$

(iii) $\text{Hom}_X(\mathcal{O}_X, \mathcal{G}) = \mathcal{G}(\mathcal{O}_X)$ so

$\text{Ext}^i_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{G}) = R^i \mathcal{G}(-) = k^i(\mathcal{G}(-)) \otimes \mathcal{G}$
Prop

\[ 0 \to \mathcal{E} \to \mathcal{H} \to 0 \] in Modbox, Modbox

i.e.,

\[ 0 \to \text{Hom}(H, M) \to \text{Hom}(G, M) \]

\[ 0 \to \text{Hom}(F, M) \to \text{Ext}(H, M) \to \cdots \]

and likewise for Ext.

PF: \[ 0 \to \mathcal{E} \to \mathcal{I} \to 0 \] is res.

\[ \text{Hom}(-, I^i) \] is exact for each \( i \),

so get

\[ 0 \to \text{Hom}(X, I^i) \to \text{Hom}(G, I^i) \]

\[ \to \text{Hom}(F, I^i) \to 0 \]

ses. of complexes.
we get the claim from the
\[ \text{E.g.s. of colorology.} \]
for \( \text{Ext}^i \) you need to use:

Exercise (Valalp, p. 749, Ex 30.2.4)

\[ \xrightarrow{\text{I injective } \mathcal{O}_X \text{-module}} \]

\[ \Rightarrow \text{Hom}(\cdot, I) \text{ is an exact functor} \]

\[ \mathcal{O}_X \]
Prop Let $L$ be locally free.

$L^u := \text{Hom}_X (L, O_X)$

Then $\tilde{F} : G \in \text{Mod} O_X$

(i) $\text{Ext}_X^i (\tilde{F} \otimes X, G) \cong \text{Ext}_X^i (F \otimes O_X, G)$

(ii) $\text{Ext}_X^i (\tilde{F} \otimes O_X, G) \cong \text{Ext}_X^i (F \otimes O_X, G)$

**Proof:**

If $I$ is an injective $O_X$ module, then

$\text{Hom}_X (-, I \otimes \mathbb{L}) \cong \text{Hom}_X (- \otimes \mathbb{L}, I)$

is injective, so $I \otimes \mathbb{L}$ is injective.

Thus if $G \rightarrow I^0$ is an injective resolution,

then so is $G \otimes \mathbb{L} \rightarrow I^0 \otimes \mathbb{L}$.

(exact)
\[ \text{Hom}_X(\mathcal{E}_X, \mathcal{I}) \cong \text{Hom}_X(\mathcal{F}, \mathcal{I} \otimes \mathcal{U}) \]

as complexes.

Taking can yield (i).

(ii) is the same.

\[ \text{Prop } X \text{ projective scheme over a noth. alg. } \]
\[ A, \text{very ample line bundle, } \]
\[ \mathcal{O}_X(1) \in \text{Gh}(X). \]

Then
\[ \text{Ext}_X^i(\mathcal{F}, \mathcal{G}(n)) \simeq \prod_{i \in \mathbb{N}} \text{Ext}^i_X(\mathcal{F}, \mathcal{G}(n)) \]

for \( n > 0 \).

If \( i = 0 \) this is obvious so assume \( i > 0 \)

Let \( \mathcal{F} \) be locally free.

\[ \text{Ext}_X^i(\mathcal{F}, \mathcal{G}(n)) \cong \text{Ext}_X^i(\mathcal{O}_X, \mathcal{F} \otimes \mathcal{G}(n)) \]
\[ \mathcal{H}^i(X, \mathcal{F} \otimes \mathcal{G}(n)) = 0 \text{ for } i > 0 \]

(Serre Vanishing, Thom (8.1.4 of)

So the claim holds for \( F \) locally free.

Now let \( F \in \operatorname{Coh}(X) \).

Then we can write \( \mathcal{F} \) as coherent, \( X \) projective.

\[ 0 \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow 0 \quad \text{for } \mathcal{E} \text{ locally free} \]

\( ( \text{Ho, II, 5.18} \) \)

(\( \text{Even } \mathcal{E} \sim \bigoplus \mathcal{O}(n) \))

Get

\[ 0 \rightarrow \operatorname{Hom}_X(\mathcal{F}, \mathcal{G}(n)) \rightarrow \operatorname{Hom}_X(\mathcal{E}, \mathcal{G}(n)) \rightarrow \operatorname{Ext}^1_X(\mathcal{F}, \mathcal{G}(n)) \rightarrow 0 \]

\( \text{as } \mathcal{E} \text{ loc. free} \)
\[
\text{and } \quad \Ext X i (F, G_i(n)) \cong \Ext X i -1 (F, G_i(n)) \quad (A)
\]
for \( i \geq 2 \).

Hence,
\[
0 \Rightarrow \Hom O (F, G_i(n)) \Rightarrow \Hom O (F, G_i(n)) \quad (A)
\]
\[
\Rightarrow \Hom O (F, G_i(n)) \Rightarrow \Ext X 1 (F, G_i(n)) \Rightarrow 0
\]
\[
\text{and } \quad \Ext X i (F, G_i(n)) \cong \Ext X i -1 (F, G_i(n)) \quad (B)
\]
for \( i \geq 2 \).

Now, given any sequence of sheaves on
\[
F_0 \rightarrow F_1 \rightarrow \cdots \rightarrow F_r
\]
a page sequence
\[
\text{H}_q \text{III} \quad \text{ex 5.10}
\]
then, for \( n \gg 0 \)
\[
\text{H}(F_0(n)) \rightarrow \text{H}(F_1(n)) \rightarrow \cdots \rightarrow \text{H}(F_r(n)) \quad \text{exact}
\]

Thus, for \( n \gg 0 \), \( \text{H} \) is exact on \((\mathcal{M}, \mathcal{G})\).
and company (a) give the claim for i.e.
(A) + (B) then give the claim by including