Math 52: Homework 6

Due 10 am, Tuesday November 27 (via Gradescope)

1. Find which of the following vector fields are conservative. If they are conservative, find a potential function $f$ such that $\vec{F} = \nabla f$.

   (a) $\vec{F}(x, y) = \langle 2y + \sin(2x), 3x + \cos(3y) \rangle$

   (b) $\vec{F}(x, y) = \langle 4x^2y - 5y^4, x^3 - 20xy^3 \rangle$

   (c) $\vec{F}(x, y) = \langle x^3 + \frac{y}{x}, y^2 + \ln(x) \rangle$

   (d) $\vec{F}(x, y) = \langle 1 + ye^{xy}, 2y + xe^{xy} \rangle$

   (e) $\vec{F}(x, y) = \langle \cos(x) + \ln(y), \frac{x}{y} + e^y \rangle$

2. A 150 pound person slides down a frictionless slide from a point 100 ft in the air to a point on the ground under the influence of the gravitational force $\vec{F} = \langle 0, 0, -150 \rangle$.

   For each of the following slide paths, choose a parametrization of the path the person will follow starting at time $t = 0$ at the top of the slide.

   Then find the amount of work done by the gravitational force $\vec{F} = \langle 0, 0, -150 \rangle$.

   (a) The slide follows the straight line segment from $(100, 0, 100)$ to $(0, 0, 0)$.

   (b) The slide follows the parabolic arc $z = y^2/100$ and $x = 0$ from $(0, 100, 100)$ to $(0, 0, 0)$.

   (c) The slide follows a spiral path which from the top view, circles in the $(x, y)$ plane around a circle of radius 5, so that starting 100 feet up, the slide circles around five full times.

3. Let $C$ be the curve $x(t) = t, y(t) = 2t$ for $0 \leq t \leq 1$. Let $D$ be the curve $x(t) = \sin(t), y(t) = 2 - 2\cos(t)$ for $0 \leq t \leq \pi/2$. If an electric force acts on a particle by $\vec{F} = \langle y^2 + 2xy, x^2 + 2xy \rangle$, how much work is done by the electric force to send the particle along path $C$? How much work is done to send it along path $D$?
4. The curl of a vector field \( \vec{F} = (F_1(x, y, z), F_2(x, y, z), F_3(x, y, z)) \) is defined to be

\[
\text{curl}(\vec{F}) = \nabla \times \vec{F} = \left| \begin{array}{ccc}
\vec{i} & \vec{j} & \vec{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
F_1(x, y, z) & F_2(x, y, z) & F_3(x, y, z)
\end{array} \right|
\]

\[
= \left( \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)
\]

Calculate the curl of the following vector fields:

(a) \( \vec{F} = (x^2, y^2, z^2) \)

(b) \( \vec{F} = (xy^2, yz^2, zx^2) \)

(c) \( \vec{F} = (x + \sin(yz), y + \sin(xz), z + \sin(xy)) \)

5. Let \( \vec{F} = (y^2 + 2xy, x^2 + 2xy) \). Calculate the work done by \( \vec{F} \) along the curve \( C \) parametrized by

\[
x(t) = t + \sin \left( t^2(t - 1) \right); \quad y(t) = \cos \left( \frac{\pi t}{2} \right) \cdot \ln \cos t
\]

for \( 0 \leq t \leq 1 \).

6. Find the surface area of the part of the cone \( z = 9 - \sqrt{x^2 + y^2} \) that lies above the plane \( z = 0 \).

7. Compute the surface area cut from the cylinder \( y^2 + z^2 = a^2 \) by the cylinder \( x^2 + y^2 = a^2 \), i.e. the surface where \( y^2 + z^2 = a^2 \) and \( x^2 + y^2 \leq a^2 \).

8. Let \( S \) be the first octant part of the sphere \( x^2 + y^2 + z^2 = a^2 \) with normal vector pointing away from the origin and \( \vec{F} = (xz, zy, z^2) \).

Find the flux

\[
\int_S \vec{F} \cdot \vec{n} \, dS
\]

by parameterizing the surface, finding the normal vector field and then computing the surface integral.
9. Compute the work of the force \( \vec{F} = (x + z) \hat{i} + (x - y + 2z) \hat{j} + (y - 4x) \hat{k} \) acting along the perimeter of the triangle \( \Delta ABC \) with vertices \( A (1, 0, 0) \), \( B (0, 1, 0) \) and \( C (0, 0, 1) \). Assume that the path starts at \( A \), then goes to \( B \), \( C \) and comes back to \( A \). Solve the problem in three ways:

(a) directly: by parametrizing each side of the triangle and computing the corresponding line integrals.
(b) by using Stokes theorem and integrating over the interior of the triangle \( \Delta ABC \). Make sure that you correctly orient the solid triangle.
(c) By using Stokes and integrating over three triangles meeting at \( (0, 0, 0) \) and contained in coordinate planes.

10. Use Stokes theorem to compute

\[
\int_C y \, dx + z \, dy + x \, dz
\]

where \( C \) is the circle in which the plane \( x + y + z = 0 \) intersects the sphere \( x^2 + y^2 + z^2 = a^2 \) oriented counter clockwise when viewed from above.