Today!

Why should you care about syzygies?

(no proof today!)

Syzygies = relations amongst eq's of a variety. (Scheuer, ca 1850)

E.g. (Twisted cubic)

\[ \mathbb{V} : \mathbb{P}^1 \to \mathbb{P}^3 \]

\[ [u:v] \mapsto [u^3 : u^2v : uv^2 : v^3] \]

\[ X = \mathbb{V}(\mathbb{P}^1) \subseteq \mathbb{P}^3 \]

\( X \) is the intersection of 3 quadric surfaces

\( \mathbf{f}(x, y, z, w) = yw - z^2 \)

\( g = yz - xw \)

\( h = xz - y^2 \)

Two independent syzygies

\[ xf + yg + zh = 0 \]

\[ yf + zg + ch = 0. \]
Hilbert's setup

\[ S = \mathbb{C}[x_1, \ldots, x_n] \]

Graded ring: \( S_d = \mathbb{C} \) hom. polys of degree \( d \)

Let \( M \) be a f.g. graded \( S \) module

E.g. \( S(-n) \) is the graded \( S \) module which, as an ungraded module, is just \( S \), but as a graded module has

\[ S_d(-n) = S_{d-n} = \mathbb{C} \) hom. polys of degree \( d-n \)

A free graded module is, by definition, a sum of twisted modules \( S(m) \).

If \( M \) is an f.g. graded \( S \) module, have

Hilbert function

\[ f_M : \mathbb{Z} \to \mathbb{Z} \]

\[ d \mapsto \dim_s M_d \]

The Hilbert polynomial is written \( P_M \)

\[ (P_M(x) = \frac{x}{(x-1)} \) for \( d \geq 0) \]
Let's define some invariants of $M$.

Hilbert Syzygy Theorem (1890)

If $M$, f.g. graded, $S = \mathbb{C}[x_1, \ldots, x_n]$ module

Then $M$ has a minimal free resolution

$$0 \rightarrow M \rightarrow F_0 \rightarrow F_i \rightarrow \cdots \rightarrow F_n \rightarrow 0$$

of length at most $n+1$.

*Free* means each $F_i$ can be written

$$F_i = \oplus S(-i-j)$$

Minimal means each $S_i : F_i \rightarrow F_{i-1}$ take a basis of $F_i$ to a minimal set of generators for $\text{Im}(S_i)$.

It guarantees $F_0$ is unique up to iso.

Each map $S_i$ in the resolution $F_i \rightarrow M \rightarrow 0$ is required to be homogeneous of degree zero.
Thus, the twistsings $S(n)$ appearing keep track of the degrees of the polynomials defining $S$.

E.g. $X \subseteq \mathbb{P}^3$ the twisted cubic as before.

The homogeneous coordinate ring $S/I_X$ ($S = \mathbb{C}[x,y,z]$) has minimal resolution

$$0 \to S \to S \to S(-2) \oplus S(-3) \to 0$$

$$A = \begin{bmatrix} yz - z^2, & y^2 - xy, & x^2 - y^2 \end{bmatrix}$$

$$B = \begin{bmatrix} x & y \\ y & z \\ z & w \end{bmatrix}$$

The Betti table is the table with $(i,j)_{th}$ entry

E.g. For twisted cubic

$$\begin{array}{ccccc}
  & 0 & 1 & 2 & 3 \\
0 & 1 = b_9 & 0 = b_10 & 0 & 0 \\
1 & 0 = b_{91} & 3 = b_{11} & 2 & 0
\end{array}$$
we will see next week how to generalize the above vastly. E.g. a rational normal curve of degree $d$ has the following Betti table

<table>
<thead>
<tr>
<th>i</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>...</th>
<th>d-1</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>...</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>(d)</td>
<td>2(d)</td>
<td>...</td>
<td>(d-1)(d) = (d-1)</td>
<td></td>
</tr>
</tbody>
</table>

Hilbert's original motivation in defining the Betti numbers $b_{ij}$ was that $f^n_M$ (and also $P_i$) be written in terms of them.

Namely, set

\[ \Delta_{ij} := \sum (-1)^i b_{j-i, i} \]

Hilbert proved

\[ f_M(d) = \sum (-1)^{i+1} \binom{n+\ell - i - j}{n} \Delta_{ij} \]
The quantities \( \binom{n-d-i}{n} \) are polynomial in \( d \) in the range \( n-d-j \geq 0 \). But we set \( \binom{a}{n} \) if \( a < n \).

**Objective of the Course**

Study conjectures relating Betti nos. of curves to their Brill-Noether theory, with application to the moduli space of curves.

Recall a curve. The Brill-Noether loci and

\[ W_d^r(C) := \bigcap \{ L \text{ of degree } d, h^0(L) \geq r+1 \} \]

If \( C \) is general, these are all smooth of dimension

\[ \delta_d(C, \subseteq) := g - (r+1)(g-d+r) 
    = h^0(\mathcal{O}_C) - h^0(C) h'(C) \]
For special cases, $\omega^d_+(\mathbb{C})$ may have higher dim than usual.

Some invariants:

**Genus** \( \text{Gen}(C) := \min \{ d \mid \omega^1_+ = 0 \} \)

\[ = \min \{ d \mid \exists \text{ map } C \to \mathbb{P}^3 \} \]

**Clifford Index**

\[ \text{Cliff}(C) = \min \{ \deg(A) - 2r(A) \mid A \in \text{Pic } C, \deg A \leq g-1, \lambda(A) \geq 2 \} \]

**let** \( C \) be a curve, \( L \) a l.b.

\[ \Psi_C(C) := \bigoplus_n \text{H}^0(C, nL) \]

is a graded \( S := \text{Sym}(\text{H}^0(C, L)) \) module

\[ \mathfrak{p}_i q(C_1, L) := \mathfrak{p}_i q(C, L) \text{ (out } (C_1, L) \text{ if understood)} \]
Here are some sample results (for L u.a.)

- Castelnuovo–Mumford

\[ \text{deg}(C) \geq 2g+1 \implies C \subseteq \mathbb{P}^r \text{ projective normal} \]

\[ (C \implies b_{0,j} = 0 \text{ for } j \geq 2) \]

- Green (84)

\[ \text{deg}(C) \geq 2g+1+p \implies F^{i,j} = 0 \text{ for } i \leq p, j \geq 2 \]

For \( L = w_C \)

- Noether If \( C \) is not hyperelliptic (i.e. cliff 2)

\[ \phi_{w_C} : C \to \mathbb{P}^{g-1} \text{ is proj. normal} \]

- Enriques–Petri–Babbage

If further \( \text{cliff}(C) \geq 2 \), \( L \) is gen. by quadrics (i.e. \( b_{1,j} = 0 \) for \( j \geq 2 \))
Green's Conjecture (84)

If \( p < \text{cliff}(C) \) then \( \Phi_{p,2}(C, \omega) = 0 \)