Fundamentals

\[ S = k \times r_1 \ldots \times r_3 \]

\[ 0 \to M_1 \to M_2 \to M_3 \to 0 \quad \text{short exact sequence of graded } S\text{-modules}. \]

Taking the \( i+j \)-th graded piece of the long exact sequence for \( Tor \):

\[ \ldots \to Tor^i_{s} (M, k) \to Tor^i_{s} (M_1, k) \]

\[ Tor^i_{s} (M_2, k) \to Tor^i_{s} (M_3, k) \to \ldots \]

Write

\[ K_{i+1} (M) := Tor^i_{s} (M, k) \]

The above sequence is

\[ \to K_{i+1} (M_3) \to K_{i+1} (M_2) \to K_{i+1} (M_1) \to K_{i+1} (M) \]

\[ \ldots \to K_{i+1} (M_3) \to K_{i+1} (M_2) \to K_{i+1} (M_1) \to K_{i+1} (M) \to \ldots \]
"Long exact sequence of Kodaira cohomology"

Lemma "Semi-continuity"

\[ \pi: X \rightarrow S \text{ flat, projective morphism of } \]
\[ \text{f.t. schemes, } \pi_i S \text{ integral.} \]

Let \( \mathcal{L} \in \text{Pic}(X_i) \).

Assume \( h^0(X_i, \mathcal{L}_i), h^0(X_i, (g-1) \mathcal{L}_i), \)
\( h^0(X_i, (g-2) \mathcal{L}_i), h^0(X_i, (g+1) \mathcal{L}_i) \) are all
constant for \( S \subseteq S_1 \).

Then \( \psi: (S_1 \rightarrow \mathbb{Z} \)
\[ S \rightarrow b \big( X_i, \mathcal{L}_i \big)_{\text{Pic}(X_i)} \]
is upper semi-continuous.

Proof wlog \( S = \text{Spec } R \) affine.

\[ S_i = \pi_i \mathcal{L}_i, \quad F^i = \mathcal{H}^i_\pi (L^g), \quad \mathcal{F}^i_\mathcal{L} = \mathcal{H}^i_\pi (L^g) \]
\[ F^i_\mathcal{L} = \mathcal{H}^i_\pi (L^{g+1}) \text{ all a.b.s.} \]
Consider the Koszul complex
\[ \Lambda E \otimes F \to \Lambda E \otimes \delta_2 \to \Lambda E \otimes \delta_3 \]
of \( R \)-modules.
\[ \delta (s, \ldots, s_p \otimes \tau) \to s(s, \ldots, s_p \otimes \tau \otimes s_i \otimes s_j) \]
where \( \delta \)
\[ \big( \text{Nat} \otimes \text{Nat} \big) \to \text{Nat} \]
\[ \text{Hom} \left( \big( \text{Nat} \otimes \text{Nat} \big), \text{Nat} \right) \]
\[ \cong \text{Hom} \left( \big( \text{Nat} \otimes \text{Nat} \big) \otimes \text{Nat} \otimes \text{Nat}, \mathbb{L} \right) \]
But have \( \text{Nat} \otimes \text{Nat} \to \mathbb{L} \), \( \text{Nat} \otimes \text{Nat} \to \mathbb{L} \)
for \( g \in \text{Hom} \left( \big( \text{Nat} \otimes \text{Nat} \big), \text{Nat} \right) \)
For \( e \in \text{Spec}(R) \), \( \text{Spec}(X_{\text{proj}} \to \mathbb{L}) \) is the middle conv. of
\[ \Lambda \text{Nat} \otimes \text{Nat} \otimes k(p) \rightarrow \Lambda \text{Nat} \otimes \mathbb{L} \otimes k(p) \]
\[ \rightarrow \Lambda \text{Nat} \otimes \mathbb{L} \otimes k(0) \]
\[ \Rightarrow \text{deg}(\varphi_p, L_p) = \dim \ker(\delta_2 \otimes L_p) \\
- \dim \text{Im}(\delta_1 \otimes L_p) \]

\[ = \text{rk}(\Lambda^p \otimes \mathcal{F}_2) - \dim \text{Im}(\delta_2 \otimes L_p) \]

\[ - \dim \text{Im}(\delta_1 \otimes L_p) \]

Suffices to show: \( \varphi : A \rightarrow B \) morph of 

\[ \text{f.g., free graded modules the function} \]

\[ p \rightarrow \text{rk}(\varphi \otimes L_p) \]

is lower semi-continuous. But for \( r \in \mathbb{N} \)

\[ \exists \text{p} \in \text{Spec} R | \text{rk}(\varphi \otimes L_p) < \frac{r}{2} \]

is closed, with ideal given by the 

entire of the matrix \( \Lambda^p \varphi \)

The next proposition will be used in the 

proof of the Hirschowitz–Ramanan theorem.
Notation

\( X \) projective variety, \( L \) p.6.
\( F \) coherent sheaf.
\( \Gamma_c(F; L) \) the graded \( \text{Sym}^r h^0(L) \)

\[ \Gamma_c(F; L) \] is the graded \( \text{Sym}^r h^0(L) \)

module \( \Gamma_c(F; L) := \bigoplus q \quad h^0(X; qL \otimes F) \)

\[ K_{p, q}(X, F; L) := K_{p, q} \left( \Gamma_c(F; L) \right) \]

Proof

Let \( X \) be a projective variety, \( L \) very ample and assume \( X \) is normally generated

\[ ( h^0(\mathbb{P}^r, \mathcal{O}(n)) \to h^0(X, L^n), \quad n \geq 1 ) \]

\[ r = h^0(L) - 1 \]

Then

\[ K_{p, q}(X, F; L) = h^0 \left( \mathbb{P}^r, \mathcal{I}_X^{p+1} (p+1)^2 \mathcal{I}_X \right) \]

\[ 0 \to \mathcal{I}_X \to \mathcal{O} \to \mathcal{O}_{\mathbb{P}^r} 
\to 0 \to 0 \]
inclue
\[ 0 \to \bigoplus_{q \geq 0} H^0(\mathbb{P}^n, I_X \otimes \mathcal{O}(q)) \to \bigoplus_{q \geq 0} H^0(\mathbb{P}^n, \mathcal{O}(q)) \to \bigoplus_{q \geq 0} H^0(X, qI) \to 0 \]

by the assumption that \( X \) is normally generated.

Let
\[ 0 \to K_{p,1}(\mathbb{P}^n, \mathcal{O}(1)) \to K_{p,1}(X, I) \]
\[ 0 \to K_{p-1,2}(\mathbb{P}^n, I_X \otimes \mathcal{O}(1)) \to K_{p-1,2}(X, I) \]

so the claim follows from the next lemma.
Lemma \[ k_{p,q}(\mathbb{P}^r, \mathcal{O}(1)) = 0 \text{ if } (p,q) \neq (0,0) \]

\textbf{Proof:} Set \( V = H^0(\mathbb{P}^r, \mathcal{O}(1)) = \mathbb{C}^{r+1} \)

We need to show:

\[ V \otimes H^0(\mathbb{P}^r, \mathcal{O}(q-1)) \rightarrow V \otimes H^0(\mathbb{P}^r, \mathcal{O}(q)) \]

\[ \rightarrow V \otimes H^0(\mathbb{P}^r, \mathcal{O}(q+1)) \]

is exact. (for \((p,q) \neq (0,0)\))

But we know that the Koszul complex

\[ 0 \rightarrow S' \rightarrow V \otimes S(-1) \rightarrow V \otimes S(-2) \]

\[ S' := \text{Sym} V \]

is exact.

\[ H^0(\mathbb{P}^r, \mathcal{O}(q)) = S_q \]

So the claim follows by considering the degree \(p+q\) strand of the Koszul complex.
Recall

Green's Conj.

For genus g curve

\[ \text{Conj.}: \quad \text{deg} \mathcal{C}_{(g,0)} = 0 \implies p < \text{Cliff}(g) \]

\text{(or } p \geq g) \]

Equivalently,

\[ \text{deg} \mathcal{C}_{(g,0)} = 0 \]

For \( p > g - \text{Cliff}(g) - 2 \) by S-D.

This is a deep conjecture and many people have tried to prove it (Eisenbud, Schreyer, Cossin, etc.).

Hirschowitz-Ramanan. Test it for \( g=2k-1 \).
Have seen: Kur et al.

\[ \mathbb{E} C \in \text{gen}(C) \leq \mathbb{E} \]

Green's Conjecture predicts:

\[ b \leq \lambda_k \]

\[ \rightarrow \text{gen} \mathbb{C} \leq k \]

Idea: construct a divisor:

\[ K_0 \leq M \text{ by } \text{parametrising} \]

\[ \mathbb{E} C \in \text{gen}(C) \leq \mathbb{E} \]

and compare it to Kur.

\[ C \leq \text{universal curve} \]

\[ \mathbb{C} \]
Let $M$ be the kernel of
\[ 0 \to M \to P \to O_{C(2)} \to O_C \to 0. \]
This relaxes $\varnothing_{p_{g-1}(2)}^2$.

We wish to study the locus of $C_{[\varnothing]}$ and $\varnothing_{p_{g-1}(2)}$.

For $b : C_{(\varnothing_{2C})}$ to \text{i.e.} $h^0(C_{(\varnothing_{2C})}) = 0$ $k-2$

\[ h^0(P_{g-1}^k, \overline{\mathcal{M}}_{g-1}(2)) \to h^0(C \cap \overline{\mathcal{M}}_{g-1}(2)). \]

is not injective.

Study degeneracy locus of
\[ R^1 \mathcal{F}_{\varnothing} \mathcal{M}_{g-1}(2) \to \mathcal{F}_{\varnothing} \mathcal{M}_{g-1}(2) \]